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STRONG CONVERGENCE THEOREMS FOR FIXED POINTS OF GENERALIZED ASYMPTOTICALLY QUASI-ϕ-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, Hybrid Halpern iteration is investigated for approximating a common fixed point of a family of generalized asymptotically quasi-ϕ-nonexpansive mappings. A strong convergence theorem is established in a Banach space.

Keywords: asymptotically quasi-ϕ-nonexpansive mapping; asymptotically nonexpansive mapping; relatively nonexpansive mapping; generalized projection.

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1. Introduction

Fixed points of nonlinear operators as an important research branch of nonlinear analysis and optimization has been applied in the study of nonlinear phenomena. During the six decades, many famous existence theorems of fixed points were established. However, from the standpoint of real world applications it is not only to know the existence of fixed points of nonlinear mappings, but also to be able to construct an iterative process to approximate their fixed points.

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The computation of fixed points is important in the study of many real world problems, including inverse problems; for instance, it is not hard to show that the split feasibility problem and the convex feasibility problem in signal processing and image reconstruction can both be formulated as a problem of finding fixed points of certain operators, respectively; see [1-6] and the references therein.

Recently, the study of the convergence of various iterative processes for solving various nonlinear mathematical models forms the major part of numerical mathematics. Among these iterative processes, Krasnoselski-Mann iterative process and Ishikwa iterative process are popular and hot. Let C be a nonempty, closed, and convex subset of a underlying space X, and $T : C \to C$ a mapping. Krasnoselski-Mann iterative process generates a sequence $\{x_n\}$ in the following manner:

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \geq 0,$$

where x_0 is an initial. Ishikawa iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n T x_n + (1-\alpha_n)x_n, \\ x_{n+1} = \alpha_n T y_n + (1-\alpha_n)x_n, \quad \forall n \geq 0, \end{cases}$$

where x_0 is an initial.

Ishikawa iterative process is indeed more general thanKrasnoselski-Mann iterative process. But research has been concentrated on the latter due probably to the reasons that the formulation of Krasnoselski-Mann iterative process is simpler than that of Ishikawa iterative process and that a convergence theorem for Krasnoselski-Mann iterative process may possibly lead to a convergence theorem for Ishikawa iterative process provided the sequence $\{\beta_n\}$ satisfies certain appropriate conditions. However, the introduction of the process Ishikawa iterative process has its own right. As a matter of fact, process Krasnoselski-Mann iterative process may fail to converge while Ishikawa iterative process can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space; see [7]. Both Krasnoselski-Mann iterative process and Ishikawa iterative process have only weak convergence, in general; see [8] and [9].

In many disciplines, including economics [10], quantum physics [11], and control theory [12], problems arises in infinite dimension spaces. In such problems, strong convergence (norm convergence) is often much more desirable than weak convergence, for it translates the physically tangible property that the energy $\|x_n - x\|$ of the error between the iterate x_n and the solution x eventually becomes arbitrarily small. The important of strong convergence is also underlined in [13], where a convex function f is minimized via the proximal-point algorithm: it is shown that the rate of convergence of the value sequence $\{f(x_n)\}$ is better when $\{x_n\}$ converges strongly that it converges weakly. Such properties have a direct impact when the process is executed directly in the underlying infinite dimensional space. To improve the weak convergence of Krasnoselski-Mann iterative process and Ishikawa iterative process, so called hybrid projections have been considered; see [14-20] for more details and the references therein. Halpern iterative process was initially introduced in [21]; see [21] for more details and the references therein. Halpern iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \end{cases}$$

where x_0 is an initial and u is a fixed element in C.

Halpern showed that the following conditions

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

are necessary in the sense that if Halpern iterative process is strongly convergent for all nonempty, closed, and convex subsets of a Hilbert space H and all nonexpansive mappings on C, then the sequence $\{x_n\}$ must satisfy conditions (C1), and (C2). Due to the restriction of (C2), Halpern iterative process is widely believed to have slow convergence though the rate of convergence has not be determined. Thus to improve the rate of convergence of Halpern iterative process, one can not rely only on the process itself; instead, some additional step of iteration should be taken.

One of the purposes of this paper is to show (HIP) is strong convergence under (C1) only with the help of projections.

The purposes of this paper is to study Halpern iterative process with the help of additional projections. We prove Halpern iterative process is strong convergence under (C1) only with the help of projections. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, Halpern iterative process is studied with the help of projections. A strong convergence theorem is established in a Banach space.

2. Preliminaries

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for all $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

Let $\rightharpoonup$ and $\rightarrow$ denote the weak and strong convergence, respectively. Recall that a Banach space E has the Kadec-Klee property if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach spaces, then E enjoys the Kadec-Klee property.

Let C be a nonempty closed and convex subset of a Banach space E, and $T : C \to C$ a mapping. The mapping T is said to be asymptotically regular on C if

$$\lim_{n\to\infty} \sup_{x\in C} \|T^{n+1}x - T^n x\| = 0.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H, and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [22] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x,y \in E.$$

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem $\phi(\bar{x},x) = \min_{y\in C} \phi(y,x)$. Existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y,x) \le (\|y\| + \|x\|)^2, \quad \forall x,y \in E.$$

Indeed, if E is a reflexive, strictly convex and smooth Banach space, then for $x,y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. A point p in C is said to be an asymptotic fixed point of T [23] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x,y \in C.$$

It is well known that if C is a nonempty, bounded, closed and convex subset of a uniformly convex Banach space E, then every nonexpansive self-mapping T on C has a fixed point. Further, the fixed point set of T is closed and convex.

T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|p - Tx\| \le \|p - x\|, \quad \forall p \in F(T), x \in C.$$

T is said to be relatively nonexpansive [24] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall p \in F(T), x \in C.$$

T is said to be ϕ-nonexpansive [25] if

$$\phi(Tx, Ty) \leq \phi(x, y), \quad \forall x, y \in C.$$

Remark 2.1. In Hilbert spaces, the class of ϕ-nonexpansive mappings is reduced to the class of nonexpansive mappings.

T is said to be quasi-ϕ-nonexpansive [25] if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall p \in F(T), x \in C.$$

Remark 2.2. The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings which requires the restriction: $F(T) = \widetilde{F}(T)$.

Remark 2.3. In Hilbert spaces, the class of quasi-ϕ-nonexpansive mappings is reduced to the class of quasi-nonexpansive mappings.

T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, \forall n \geq 1.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [26] in 1972. Since 1972, a host of authors have studied the weak and strong convergence of iterative processes for such a class of mappings.

T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|p - T^n x\| \leq k_n \|p - x\|, \quad \forall p \in F(T), x \in C, \forall n \geq 1.$$

T is said to be relatively asymptotically nonexpansive [27] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \quad \forall p \in F(T), x \in C, \forall n \geq 1.$$

T is said to be asymptotically ϕ-nonexpansive [28] if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(T^n x, T^n y) \le k_n \phi(x,y), \quad \forall x,y \in C, \forall n \ge 1.$$

Remark 2.4. In Hilbert spaces, the class of asymptotically ϕ-nonexpansive mappings is reduced to the class of asymptotically nonexpansive mappings.

T is said to be asymptotically quasi-ϕ-nonexpansive [28] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n x) \le k_n \phi(p,x), \quad \forall p \in F(T), x \in C, \forall n \ge 1.$$

Remark 2.5. The class of asymptotically quasi-ϕ-nonexpansive mappings is more general than the class of relatively asymptotically nonexpansive mappings which requires the restriction: $F(T) = \widetilde{F}(T)$.

Remark 2.6. In Hilbert spaces, the class of asymptotically quasi-ϕ-nonexpansive mappings is reduced to the class of asymptotically quasi-nonexpansive mappings.

T is said to be an generalized asymptotically quasi-ϕ-nonexpansive mapping if $F(T) \neq \emptyset$, and there exist two nonnegative sequences $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$, and $\{\xi_n\} \subset [0,\infty)$ with $\xi_n \to 0$ as $n \to \infty$ such that

$$\phi(p, T^n x) \le (1+\mu_n)\phi(p,x) + \xi_n, \quad \forall x \in C, \forall p \in F(T), \forall n \ge 1.$$

Next, we provide two examples.

Let $C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and define a mapping T by

$$Tx = \begin{cases} \frac{x}{2}\sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $T^n x \to 0$ uniformly but T is not Lipschitz. On the other hand, it is easy to see that $F(T) = \{0\}$. For each fixed n, define

$$f_n(x) = \|T^n x\|^2 - \|x\|^2, \quad \forall x \in C,$$

and set $\xi_n = \sup_{x\in C}\{f_n(x),0\}$. Then $\lim_{n\to\infty}\xi_n = 0$, and

$$\phi(0,T^n x) = \|T^n x\|^2 \leq \|x\|^2 + \xi_n = \phi(0,x) + \xi_n$$

This show that T is generalized asymptotically quasi-ϕ-nonexpansive but it is not asymptotically quasi-ϕ-nonexpansive.

Let E be any smooth Banach space, and x_0 a non-zero point in E. We define a mapping $T : E \to E$ as follows

$$Tx = \begin{cases} -x, & x \neq \big(\frac{1}{2}+\frac{1}{2^n}\big)x_0, \\ \big(\frac{1}{2}+\frac{1}{2^{n+1}}\big)x_0, & x = \big(\frac{1}{2}+\frac{1}{2^n}\big)x_0, \end{cases}$$

for all $n \geq 1$. Then T is generalized asymptotically quasi-ϕ-nonexpansive with the constant sequence $\mu_n = 0$. But it is not relatively nonexpansive mapping. Let

$$x_n = \big(\frac{1}{2}+\frac{1}{2^n}\big)x_0, \quad \forall n \geq 1.$$

We see from the the definition of T that

$$Tx_n = \big(\frac{1}{2}+\frac{1}{2^{n+1}}\big)x_0, \quad \forall n \geq 1.$$

This implies that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0, \quad \lim_{n\to\infty} x_n = x_0.$$

This shows that $x_0 \in \widetilde{F}(T)$. However, $F(T) = \{0\}$. That is, $x_0 \notin F(T)$.

Remark 2.7. The class of generalized asymptotically quasi-ϕ-nonexpansive mappings is a generalization of the class of generalized asymptotically quasi-nonexpansive mappings in the framework of Banach spaces.

In 2007, Qin and Su [29] considered modifying Halpern iteration for a relatively nonexpansive mapping. To be more precise, they obtained the following results.

Theorem 2.1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a relatively nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that* $\lim_{n\to\infty}\alpha_n = 0$. *Define a sequence $\{x_n\}$ in C*

by the following algorithm:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n), \\ Q_n = \{v \in C : \langle Jx_0 - Jx_n, x_n - v \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \ge 0, \end{cases}$$

where J is the single-valued duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges to $\Pi_{F(T)}x_0$.

Recently, Qin, Cho, Kang and Zhou [30] further improved Theorem QS by considering quasi-ϕ-nonexpansive mappings. To be more precise, they proved the following.

Theorem 2.2. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and $T : C \to C$ a closed and quasi-ϕ-nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)JTx_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1. \end{cases}$$

Assume that the control sequence satisfies the restriction: $\lim_{n\to\infty} \alpha_n = 0$. *Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.*

Very recently, Cho, Qin and Kang [31] reconsidered Halpern iteration for an asymptotically quasi-ϕ-nonexpansive mapping. To be more precise, they proved the following.

Theorem 2.3. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and $T : C \to C$ a closed asymptotically quasi-ϕ-nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Assume that T is asymptotically* *regular on* C*,* $F(T) \neq \emptyset$ *and* $F(T)$ *is bounded. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}[\alpha_n J x_1 + (1-\alpha_n) J T^n x_n], \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + \alpha_n M\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 0, \end{cases}$$

where M *is an appropriate constant such that* $M \geq \phi(w, x_1)$ *for all* $w \in F(T)$*. Assume that the control sequence* $\{\alpha_n\}$ *in* $(0,1)$ *satisfies the following restrictions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$,

(b) $(1-\alpha_n)k_n \leq 1$ *for all* $n \geq 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_{F(T)} x_1$.

In this paper, motivated by the above research, we modify Halpern iterative process on based on hybrid projection methods for a family of generalized asymptotically quasi-ϕ-nonexpansive mappings. A Strong convergence theorem is established in a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property.

In order to our main results, we need the following lemmas.

Lemma 2.1. [22] *Let* E *be a reflexive, strictly convex and smooth Banach space,* C *a nonempty closed convex subset of* E *and* $x \in E$*. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \forall y \in C.$$

Lemma 2.2 [22] *Let* C *be a nonempty closed convex subset of a smooth Banach space* E *and* $x \in E$*. Then,* $x_0 = \Pi_C x$ *if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \quad \forall y \in C.$$

Lemma 2.3. [32] *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, C a nonempty closed convex subset of E and $T : C \to C$ a closed and asymptotically quasi-ϕ-nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

3. Main results

Theorem 3.1. *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and C a nonempty closed convex subset of E. Let $T_i : C \to C$ be an asymptotically regular, closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequences $\{k_{n,i}\} \subset [1, \frac{1}{1-\alpha_{n,i}}]$ for each $i \geq 1$. Assume that $F(T_i)$ is bounded for each $i \geq 1$, and $\mathscr{F} = \cap_{i=1}^{\infty} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad chosen\ arbitrarily, \\ C_{1,i} = C, \\ C_1 = \cap_{i=1}^{\infty} C_{1,i}, \\ x_1 = \Pi_{C_1} x_0, \\ y_{n,i} = J^{-1}\big(\alpha_{n,i} J x_1 + (1-\alpha_{n,i}) J T_i^n x_n\big), \quad n \geq 1, \\ C_{n+1,i} = \{z \in C_{n,i} : \phi(z, y_{n,i}) \leq \phi(z, x_n) + \alpha_{n,i} M + \xi_{n,i}\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \forall n \geq 0, \end{cases}$$

where $M = \sup\{\phi(z, x_1) : z \in \mathscr{F}\}$. Assume that the control sequence $\{\alpha_{n,i}\}$ is chosen such that $\lim_{n\to\infty} \alpha_{n,i} = 0$ for each $i \geq 1$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathscr{F}} x_1$.

Proof. First, we show that C_n is closed and convex so that the projection on C_n is well defined. It suffices to claim that, $\forall i \geq 1$, $C_{n,i}$ is closed and convex for every $n \geq 1$. This can be proved by induction on n.

In fact, for $n = 1$, $C_{1,i} = C$ is closed and convex. Assume that $C_{h,i}$ is closed and convex for some h. For $z \in C_{h,i}$, we see that $\phi(z, y_{h,i}) \leq \phi(z, x_h) + \alpha_{h,i} M$ is equivalent to

$$2\langle z, Jx_h - Jy_{h,i}\rangle \leq \|x_h\|^2 - \|y_{h,i}\|^2 + \alpha_{h,i} M.$$

Hence $C_{h+1,i}$ is closed and convex for each i. Hence, for $\forall i \geq 1$ $C_{n,i}$ is closed and convex. This proves that C_n is closed and convex for each $n \geq 1$.

Next, we prove that $\mathscr{F} \subset C_n$ for each $n \geq 1$. It suffices to claim that $\mathscr{F} \subset C_{n,i}$ for each $n \geq 1$ and for each $i \geq 1$. Note that $\mathscr{F} \subset C_{1,i} = C$. Suppose that $\mathscr{F} \subset C_{h,i}$ for some h and for all i. Then, for $\forall w \in \mathscr{F} \subset C_{h,i}$, we have

$$
\begin{aligned}
&\phi(w, y_{h,i}) \\
&= \phi\Big(w, J^{-1}\big(\alpha_{h,i}Jx_1 + (1-\alpha_{h,i})JT_i^h x_h\big)\Big) \\
&= \|w\|^2 - 2\langle w, \alpha_{h,i}Jx_1 + (1-\alpha_{h,i})JT_i^h x_h\rangle + \|\alpha_{h,i}Jx_1 + (1-\alpha_{h,i})JT_i^h x_h\|^2 \\
&\leq \|w\|^2 - 2\alpha_{h,i}\langle w, Jx_1\rangle - 2(1-\alpha_{h,i})\langle w, JT_i^h x_h\rangle + \alpha_{h,i}\|x_1\|^2 + (1-\alpha_{h,i})\|T_i^h x_h\|^2 \\
&= \alpha_{h,i}\phi(w, x_1) + (1-\alpha_{h,i})\phi(w, T_i^h x_h) \\
&\leq \alpha_{h,i}\phi(w, x_1) + (1-\alpha_{h,i})k_{h,i}\phi(w, x_h) + \xi_{h,i} \\
&= \phi(w, x_h) - \big(1-(1-\alpha_{h,i})k_{h,i}\big)\phi(w, x_h) + \alpha_{h,i}\phi(w, x_1) + \xi_{h,i} \\
&\leq \phi(w, x_h) + \alpha_{h,i}\phi(w, x_1) + \xi_{h,i} \\
&\leq \phi(w, x_h) + \alpha_{h,i}M + \xi_{h,i},
\end{aligned}
$$

which shows that $w \in C_{h+1,i}$. This implies that $\mathscr{F} \subset C_{n,i}$ for each $n \geq 1$ and each $i \geq 1$. This proves that $\mathscr{F} \subset C_n$ for each $n \geq 1$. It follows from Lemma 2.1 that

$$\phi(x_n, x_1) = \phi(\Pi_{C_n}x_1, x_1) \leq \phi(w, x_1) - \phi(w, x_n) \leq \phi(w, x_1),$$

for each $w \in \mathscr{F} \subset C_n$. This shows that the sequence $\phi(x_n, x_1)$ is bounded. Hence the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may, without loss of generality, assume that $x_n \rightharpoonup p$. Since $C_m \subset C_n$ for all $m \geq n$, we have $x_m \in C_n$ for all $m \geq n$. Since C_n is closed and convex, we see that $p \in C_n$ for all $n \geq 1$. It follows that $p \in \cap_{n=1}^{\infty} C_n$. In view of

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1) \leq \phi(p, x_1),$$

we see that

$$\begin{aligned}\phi(p,x_1) &\le \liminf_{n\to\infty}\phi(x_n,x_1)\\ &\le \limsup_{n\to\infty}\phi(x_n,x_1)\\ &\le \phi(p,x_1).\end{aligned}$$

This gets that $\lim_{n\to\infty}\phi(x_n,x_1)=\phi(p,x_1)$. Hence, we have $\|x_n\|\to\|p\|$ as $n\to\infty$. In view of the Kadec-Klee property of E, we obtain that $\lim_{n\to\infty}x_n=p$. By the construction of C_n, we have that $C_{n+1}\subset C_n$ and $x_{n+1}=\Pi_{C_{n+1}}x_1\in C_n$. It follows that

$$\begin{aligned}\phi(x_{n+1},x_n) &= \phi(x_{n+1},\Pi_{C_n}x_1)\\ &\le \phi(x_{n+1},x_1)-\phi(\Pi_{C_n}x_1,x_1)\\ &= \phi(x_{n+1},x_1)-\phi(x_n,x_1).\end{aligned}$$

Letting $n\to\infty$, we obtain that $\phi(x_{n+1},x_n)\to 0$. In view of $x_{n+1}\in C_{n+1}$, we obtain that

$$\phi(x_{n+1},y_{n,i})\le\phi(x_{n+1},x_n)+\alpha_{n,i}M,\quad \forall i\ge 1.$$

Hence, we have $\lim_{n\to\infty}\phi(x_{n+1},y_{n,i})=0,\forall i\ge 1$. This implies that $\lim_{n\to\infty}\|y_{n,i}\|=\|p\|,\quad \forall i\ge 1$ and $\lim_{n\to\infty}\|Jy_{n,i}\|=\|Jp\|,\quad \forall i\ge 1$. This implies that $\{Jy_{n,i}\}$ is bounded. Note that E is reflexive and E^* is also reflexive. We may assume that $Jy_{n,i}\rightharpoonup x^{i,*}\in E^*$ for each $i\ge 1$. It follows that there exists an $x^i\in E$ such that $Jx^i=x^{i,*}$. It follows that

$$\begin{aligned}\phi(x_{n+1},y_{n,i}) &= \|x_{n+1}\|^2-2\langle x_{n+1},Jy_{n,i}\rangle+\|y_{n,i}\|^2\\ &= \|x_{n+1}\|^2-2\langle x_{n+1},Jy_{n,i}\rangle+\|Jy_{n,i}\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ the both sides of equality above yields that $p=x^i$, which in turn implies that $x^{i,*}=Jp$ for each $i\ge 1$. It follows that $Jy_{n,i}\rightharpoonup Jp\in E^*$ for each $i\ge 1$. Since E^* enjoys the Kadec-Klee property, we obtain that $\lim_{n\to\infty}Jy_{n,i}=Jp,\forall i\ge 1$. Note that $J^{-1}:E^*\to E$ is demi-continuous. It follows that $y_{n,i}\rightharpoonup p$ for each $i\ge 1$. Since E enjoys the Kadec-Klee property, we obtain that

$$\lim_{n\to\infty}y_{n,i}=p,\quad \forall i\ge 1.$$

It follows that $\lim_{n\to\infty}\|x_n - y_{n,i}\| = 0, \forall i \geq 1$. Since J is uniformly norm-to-norm continuous on any bounded sets, we have $\lim_{n\to\infty}\|Jx_n - Jy_{n,i}\| = 0, \forall i \geq 1$. Since

$$Jx_n - Jy_{n,i} = \alpha_{n,i}(Jx_n - Jx_1) + (1-\alpha_{n,i})(Jx_n - JT_i^n x_n),$$

we find that $\lim_{n\to\infty}\|Jx_n - JT_i^n x_n\| = 0, \forall i \geq 1$. Since J is uniformly norm-to-norm continuous on any bounded sets, we see that

$$\lim_{n\to\infty}\|Jx_n - Jp\| = 0.$$

Hence

$$\lim_{n\to\infty}\|JT_i^n x_n - Jp\| = 0.$$

The demi-continuity of $J^{-1} : E^* \to E$ implies that $T_i^n x_n \rightharpoonup p$ for each i. Note that

$$|\|T_i^n x_n\| - \|p\|| = |\|JT_i^n x_n\| - \|Jp\|| \leq \|JT_i^n x_n - Jp\|, \quad \forall i \geq 1.$$

It follows that $\|T_i^n x_n\| \to \|p\|$, for each $i \geq 1$, as $n \to \infty$. Since E has the Kadec-Klee property, we obtain that $\lim_{n\to\infty}\|T_i^n x_n - p\| = 0, \forall i \geq 1$. Note that

$$\|T_i^{n+1} x_n - p\| \leq \|T_i^{n+1} x_n - T_i^n x_n\| + \|T_i^n x_n - p\|, \quad \forall i \geq 1.$$

It follows from the asymptotic regularity of T_i that $\lim_{n\to\infty}\|T_i^{n+1} x_n - p\| = 0$, that is, $T_i T_i^n x_n - p \to 0$ as $n \to \infty$. It follows from the closedness of T_i that $T_i p = p$ for each $i \geq 1$. This proves that $p \in \mathscr{F}$.

Finally, we show that $p = \Pi_{\mathscr{F}} x_1$. From $x_n = \Pi_{C_n} x_1$, we have

$$\langle x_n - w, Jx_1 - Jx_n \rangle \geq 0, \quad \forall w \in \mathscr{F} \subset C_n.$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\langle p - w, Jx_1 - Jp \rangle \geq 0, \quad \forall w \in \mathscr{F}.$$

In view of Lemma 2.2, we see that $p = \Pi_{\mathscr{F}} x_1$. This completes the proof.

For a single mapping, we have the following.

Corollary 3.2. *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and C a nonempty closed convex subset of E. Let $T : C \to C$ be* *an asymptotically regular, closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequences* $\{k_n\} \subset [1, \frac{1}{1-\alpha_n}]$. *Assume that* $F(T)$ *is bounded, and* $F(T)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}\big(\alpha_n J x_1 + (1-\alpha_n) J T^n x_n\big), \quad n \geq 1, \\ C_{n+1,i} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + \alpha_n M + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \forall n \geq 0, \end{cases}$$

where $M = \sup\{\phi(z, x_1) : z \in F(T)\}$. *Assume that the control sequence* $\{\alpha_n\}$ *is chosen such that* $\lim_{n\to\infty} \alpha_n = 0$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)} x_1$.

If $T_i : C \to C$ is an asymptotically regular, closed and asymptotically quasi-ϕ-nonexpansive mapping, then we have the following.

Corollary 3.3. *Let* E *be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and* C *a nonempty closed convex subset of* E. *Let* $T_i : C \to C$ *be an asymptotically regular, closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequences* $\{k_{n,i}\} \subset [1, \frac{1}{1-\alpha_{n,i}}]$ *for each* $i \geq 1$. *Assume that* $F(T_i)$ *is bounded for each* $i \geq 1$, *and*

$\mathscr{F} = \cap_{i=1}^{\infty} F(T_i)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i=1}^{\infty} C_{1,i}, \\ x_1 = \Pi_{C_1} x_0, \\ y_{n,i} = J^{-1}\big(\alpha_{n,i} J x_1 + (1-\alpha_{n,i}) J T_i^n x_n\big), \quad n \geq 1, \\ C_{n+1,i} = \{z \in C_{n,i} : \phi(z, y_{n,i}) \leq \phi(z, x_n) + \alpha_{n,i} M\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \forall n \geq 0, \end{cases}$$

where $M = \sup\{\phi(z, x_1) : z \in \mathscr{F}\}$. *Assume that the control sequence* $\{\alpha_{n,i}\}$ *is chosen such that* $\lim_{n\to\infty} \alpha_{n,i} = 0$ *for each* $i \geq 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\mathscr{F}} x_1$.

For the class of quasi-ϕ-nonexpansive mappings, we have from Theorem 3.1 the following immediately.

Corollary 3.4. *Let* E *be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property and* C *a nonempty closed convex subset of* E. *Let* $T_i : C \to C$ *be a closed quasi-*ϕ*-nonexpansive mapping such that* $F(T_i) \neq \emptyset$ *for each* $i \geq 1$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{1,i} = C, \quad C_1 = \cap_{i=1}^{\infty} C_{1,i}, \\ x_1 = \Pi_{C_1} x_0, \\ y_{n,i} = J^{-1}\big(\alpha_{n,i} J x_1 + (1-\alpha_{n,i}) J T_i x_n\big), \quad n \geq 1, \\ C_{n+1,i} = \{z \in C_{n,i} : \phi(z, y_{n,i}) \leq \alpha_{n,i}\phi(z, x_1) + (1-\alpha_{n,i})\phi(z, x_n)\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \forall n \geq 0. \end{cases}$$

Assume that the control sequence $\{\alpha_{n,i}\}$ *is chosen such that* $\lim_{n\to\infty}\alpha_{n,i}=0$ *for each* $i\geq 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\mathscr{F}}x_1$.

In Hilbert spaces, Corollary 3.4 is reduced to the following immediately.

Corollary 3.5. *Let* H *be a Hilbert space and* C *a nonempty closed convex subset of* H. *Let* $T_i:C\to C$ *be a closed quasi-nonexpansive mapping such that* $F(T_i)\neq\emptyset$ *for each* $i\geq 1$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0\in H \quad \textit{chosen arbitrarily},\\ C_{1,i}=C, \quad C_1=\cap_{i=1}^{\infty}C_{1,i},\\ x_1=\Pi_{C_1}x_0,\\ y_{n,i}=\alpha_{n,i}x_1+(1-\alpha_{n,i})T_ix_n, \quad n\geq 1,\\ C_{n+1,i}=\{z\in C_{n,i}:\|z-y_{n,i}\|^2\leq\alpha_{n,i}\|z-x_1\|^2+(1-\alpha_{n,i})\|z-x_n\|^2\},\\ C_{n+1}=\cap_{i=1}^{\infty}C_{n+1,i},\\ x_{n+1}=P_{C_{n+1}}x_1, \forall n\geq 0, \end{cases}$$

where P *is the metric projection. Assume that the control sequence* $\{\alpha_{n,i}\}$ *is chosen such that* $\lim_{n\to\infty}\alpha_{n,i}=0$ *for each* $i\geq 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $P_{\mathscr{F}}x_1$.

Next, we give some result in Hilbert spaces.

Corollary 3.6. *Let* E *be a Hilbert space, and* C *a nonempty closed convex subset of* E. *Let* $T_i:C\to C$ *be an asymptotically regular, closed and asymptotically quasi-nonexpansive mapping with the sequences* $\{k_{n,i}\}\subset[1,\frac{1}{1-\alpha_{n,i}}]$ *for each* $i\geq 1$. *Assume that* $F(T_i)$ *is bounded for each* $i\geq 1$, *and* $\mathscr{F}=\cap_{i=1}^{\infty}F(T_i)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated in the following* *manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i=1}^{\infty} C_{1,i}, \\ x_1 = P_{C_1} x_0, \\ y_{n,i} = \alpha_{n,i} x_1 + (1-\alpha_{n,i}) T_i^n x_n, \quad n \geq 1, \\ C_{n+1,i} = \{z \in C_{n,i} : \|z - y_{n,i}\|^2 \leq \|z - x_n\|^2 + \alpha_{n,i} M + \xi_{n,i}\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 0, \end{cases}$$

where $M = \sup\{\|z - x_1\|^2 : z \in \mathscr{F}\}$. *Assume that the control sequence* $\{\alpha_{n,i}\}$ *is chosen such that* $\lim_{n\to\infty} \alpha_{n,i} = 0$ *for each* $i \geq 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\mathscr{F}} x_1$.

For a single mapping, we have the following.

Corollary 3.7. *Let* E *be a Hilbert space, and* C *a nonempty closed convex subset of* E. *Let* $T : C \to C$ *be an asymptotically regular, closed and asymptotically quasi-nonexpansive mapping with the sequences* $\{k_n\} \subset [1, \frac{1}{1-\alpha_n}]$. *Assume that* $F(T)$ *is bounded, and* $F(T)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = \alpha_n x_1 + (1-\alpha_n) T^n x_n, \quad n \geq 1, \\ C_{n+1} = \{z \in C_n : \|z - y_n\|^2 \leq \|z - x_n\|^2 + \alpha_n M + \xi_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 0, \end{cases}$$

where $M = \sup\{\|z - x_1\|^2 : z \in F(T)\}$. *Assume that the control sequence* $\{\alpha_n\}$ *is chosen such that* $\lim_{n\to\infty} \alpha_n = 0$ *for each* $i \geq 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $P_{F(T)} x_1$.

For the class of quasi-nonexpansive mappings, we have the following.

Corollary 3.8. *Let E be a Hilbert space, and C a nonempty closed convex subset of E. Let $T_i : C \to C$ be a closed and quasi-nonexpansive mapping for each $i \geq 1$. Assume that $\mathscr{F} = \cap_{i=1}^{\infty} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, \quad C_1 = \cap_{i=1}^{\infty} C_{1,i}, \\ x_1 = P_{C_1} x_0, \\ y_{n,i} = \alpha_{n,i} x_1 + (1-\alpha_{n,i}) T_i x_n, \quad n \geq 1, \\ C_{n+1,i} = \{z \in C_{n,i} : \|z - y_{n,i}\|^2 \leq \alpha_{n,i} \|z - x_1\|^2 + (1-\alpha_{n,i}) \|z - x_n\|^2\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 0, \end{cases}$$

where $M = \sup\{\|z - x_1\|^2 : z \in \mathscr{F}\}$. Assume that the control sequence $\{\alpha_{n,i}\}$ is chosen such that $\lim_{n\to\infty} \alpha_{n,i} = 0$ for each $i \geq 1$. Then the sequence $\{x_n\}$ converges strongly to $P_{\mathscr{F}} x_1$.

For a single mapping, we have the following.

Corollary 3.9. *Let E be a Hilbert space, and C a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and quasi-nonexpansive mapping. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = \alpha_n x_1 + (1-\alpha_n) T x_n, \quad n \geq 1, \\ C_{n+1} = \{z \in C_n : \|z - y_n\|^2 \leq \alpha_n \|z - x_1\|^2 + (1-\alpha_n) \|z - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 0, \end{cases}$$

where $M = \sup\{\|z - x_1\|^2 : z \in F(T)\}$. Assume that the control sequence $\{\alpha_n\}$ is chosen such that $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ converges strongly to $P_{F(T)} x_1$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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