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COMMON FIXED POINTS IN COMPLEX S-METRIC SPACE

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Abstract. In this paper, we introduce the complex valued S-metric space, and we show the existence and the uniqueness of a common fixed point of two self mappings in such space.

Keywords: functional analysis; complex S-metric space; common fixed point.

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1. Introduction

Showing the existence and the uniqueness of a fixed point for a self mapping in deferent metric spaces is a very famous problem, which was inspired by the work of Banach [1]. Since then till present time, many results on finding a fixed point in different metric spaces and under many different contraction principles were proved; see, for example, [3], [5], [6], [8] and [10] and the references therein. Also, as an extension of the fixed point problem there are many results in finding a common fixed point for two self mappings on different types of metric spaces; see, for example, [9], [11] and the references therein. But, all of these results were found in real valued metric spaces. In 2011, a complex valued metric space was introduced in

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[2]. Complex valued metric spaces form a special class of cone metric space, but our contraction which has a product and quotient of metrics cannot be extended to cone metric.

In this paper, we introduce a complex valued S-metric space, and we investigate the existence and uniqueness of a common fixed point of two self mappings in such space.

First, we define the partial order $\precsim$ on the set of complex numbers $\mathbb{C}$ by for all z_1 and z_2 in $\mathbb{C}$ we have:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \le Re(z_2), Im(z_1) \le Im(z_2)$$

and

$$z_1 \prec z_2 \text{ if and only if } Re(z_1) < Re(z_2), Im(z_1) < Im(z_2).$$

Also, we write $z_1 1 \precsim z_2$ if one of the following conditions hold:

(i) $Re(z_1) = Re(z_2)$, and $Im(z_1) < Im(z_2)$,

(ii) $Re(z_1) < Re(z_2)$, and $Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) < Re(z_2)$, and $Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) = Re(z_2)$, and $Im(z_1) = Im(z_2)$.

We write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|,$$

and

$$z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

Definition 1.1. Let X be a nonempty set and $\mathbb{C}$ the set of all complex numbers. A complex valued S-metric space on X is a function $S : X^3 \to \mathbb{C}$ that satisfies the following conditions, for all $x, y, z, t \in X$:

(i) $0 \precsim S(x,y,z)$,

(ii) $S(x,y,z) = 0$ if and only if $x = y = z$,

(iii) $S(x,y,z) \precsim S(x,x,t) + S(y,y,t) + S(z,z,t)$.

The pair (X,S) is called a complex valued S-metric space.

Example 1.1. Let $X = \mathbb{C}$ be the set of complex numbers. Define $S : \mathbb{C}^3 \to \mathbb{C}$ by:

$$S(z_1,z_2,z_3) = |max\{Re(z_1), Re(z_2)\} - Re(z_2)| + i|max\{Im(z_1), Im(z_2)\} - Im(z_2)|.$$

It is not difficult to see that $(\mathbb{C}, S)$ is a complex valued S-metric space.

Definition 1.2. If (X,S) is called a complex valued S-metric space, then

1) A sequence $\{x_n\}$ in X converges to x if and only if for all ε such that $0 \prec \varepsilon \in \mathbb{C}$, there exists a natural number n_0 such that for all $n \geq n_0$, we have $S(x_n, x_n, x) \prec \varepsilon$ and we donate this by $\lim_{n\to\infty} x_n = x$.

2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for all ε such that $0 \prec \varepsilon \in \mathbb{C}$, there exists a natural number n_0 such that for all $n, m \geq n_0$, we have $S(x_n, x_n, x_m) \prec \varepsilon$.

3) An S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.

Definition 1.3. Two families of self mappings $\{f_i\}_{i=1}^m$ and $\{g_i\}_{i=1}^n$ are said to be pairwise commuting if the following three conditions hold:

(i) $f_i f_j = f_j f_i$ for all $i, j \in \{1, 2, \cdots, m\}$;

(ii) $g_k g_l = g_l g_k$ for all $k, l \in \{1, 2, \cdots, n\}$;

(iii) $f_i g_k = g_k f_i$ for all $i \in \{1, 2, \cdots, m\}$ and $k \in \{1, 2, \cdots, n\}$.

Next, we prove the following three lemmas for our purposes.

Lemma 1.1. *Let (X,S) be a complex valued S-metric space and $\{x_n\}$ be a sequence in X. Then*

$$\{x_n\} \text{ converges to } x \text{ if and only if } |S(x_n, x_n, x)| \to 0 \text{ as } n \to \infty.$$

Proof. Assume that $\{x_n\}$ converges to x. For $\varepsilon > 0$ let

$$c = \frac{\varepsilon}{\sqrt{2}} + i\frac{\varepsilon}{\sqrt{2}}.$$

Thus, $0 \prec c \in \mathbb{C}$ and there is a natural number n_0, such that

$$S(x_n, x_n, x) \prec c \quad \text{for all} \quad n \geq n_0.$$

Hence,

$$|S(x_n, x_n, x)| < |c| = \sqrt{(\frac{\varepsilon}{\sqrt{2}})^2 + i(\frac{\varepsilon}{\sqrt{2}})^2} = \varepsilon \text{ for all } n \geq n_0.$$

Therefore, we deduce that

$$|S(x_n, x_n, x)| \to 0 \text{ as } n \to \infty.$$

Now, assume that $|S(x_n,x_n,x)| \to 0$ as $n \to \infty$. Hence, given a $c \in \mathbb{C}$, where $0 \prec c$, there exists a natural number $\eta > 0$, such that for $z \in \mathbb{C}$

$$|z| < \eta \text{ implies } z \prec c.$$

Thus, there exists a natural number n_0 such that

$$|S(x_n,x_n,x)| < \eta \text{ for all } n > n_0.$$

Which implies that $S(x_n,x_n,x) \prec c$ for all $n > n_0$. Therefore, $\{x_n\}$ converges to x as desired.

Lemma 1.2. *Let (X,S) be a complex valued S-metric space and $\{x_n\}$ be a sequence in X. Then*

$$\{x_n\} \textit{ is a cauchy sequence if and only if } |S(x_n,x_n,x_{n+m})| \to 0 \textit{ as } n \to \infty.$$

Proof. Assume that $\{x_n\}$ is a cauchy sequence. For $\varepsilon > 0$ let

$$c = \frac{\varepsilon}{\sqrt{2}} + i\frac{\varepsilon}{\sqrt{2}}.$$

Thus, $0 \prec c \in \mathbb{C}$ and there is a natural number n_0, such that

$$S(x_n,x_n,x_{n+m}) \prec c \quad \text{for all} \quad n \geq n_0.$$

Hence,

$$|S(x_n,x_n,x_{n+m})| < |c| = \sqrt{(\frac{\varepsilon}{\sqrt{2}})^2 + i(\frac{\varepsilon}{\sqrt{2}})^2} = \varepsilon \text{ for all } \quad n \geq n_0.$$

Therefore, we deduce that

$$|S(x_n,x_n,x_{n+m})| \to 0 \text{ as } n \to \infty.$$

Now, assume that $|S(x_n,x_n,x_{n+m})| \to 0$ as $n \to \infty$. Hence, given a $c \in \mathbb{C}$ where $0 \prec c$, there exists a natural number $\eta > 0$, such that for $z \in \mathbb{C}$

$$|z| < \eta \text{ implies } z \prec c.$$

Thus, there exists a natural number n_0 such that

$$|S(x_n,x_n,x_{n+m})| < \eta \text{ for all } n > n_0.$$

Which implies that $S(x_n,x_n,x) \prec c$ for all $n > n_0$. Therefore, $\{x_n\}$ is a cauchy sequence as required.

Lemma 1.3. *If (X,S) be a complex valued S-metric space, then*

$$S(x,x,y) = S(y,y,x) \quad \textit{for all} \quad x,y \in X.$$

Proof. Let $x,y \in X$ by condition (iii) of Definition 1.1 we have

$$S(x,x,y) \precsim 2S(x,x,x) + S(y,y,x).$$

In view of $S(x,x,x) = 0$, we find that $S(x,x,y) \precsim S(y,y,x)$. Similarly, we find $S(y,y,x) \precsim S(x,x,y)$. It follows that $S(x,x,y) = S(y,y,x)$.

2. Common fixed points

In this section, we prove the existence and the uniqueness of a common fixed point for two self mapping on a complex valued S-metric space.

Theorem 2.1. *Let (X,S) be a complete complex valued S-metric space and f,g be two self mappings on X satisfying the following contraction condition:*

$$S(fx,fx,gy) \precsim \alpha S(x,x,y) + \frac{\beta S(x,x,fx)S(y,y,gy)}{2S(x,x,gy)+S(y,y,fx)+S(x,x,y)} \quad (\star)$$

for all $x,y \in X$ such that $x \neq y$, $S(x,x,gy)+S(y,y,fx)+S(x,x,y) \neq 0$, where α,β are two non-negative real numbers with $\alpha+\beta<1$ or $S(fx,fx,gy)=0$ if $S(x,x,gy)+S(y,y,fx)+S(x,x,y)=0$. Then f,g have a unique common fixed point.

Proof. Let $x_0 \in X$ and let $x_{2k+1} = fx_{2k}$, $x_{2k+2} = gx_{2k+1}$, $k \in \{0,1,2,\cdots\}$. It follows that

$$\begin{aligned}
S(x_{2k+1},x_{2k+1},x_{2k+2}) &= S(fx_{2k},fx_{2k},gx_{2k+1}) \\
&\precsim \alpha S(x_{2k},x_{2k},x_{2k+1}) \\
&\quad + \frac{\beta S(x_{2k},x_{2k},fx_{2k})S(x_{2k+1},x_{2k+1},gx_{2k+1})}{2S(x_{2k},x_{2k},gx_{2k+1})+S(x_{2k+1},x_{2k+1},fx_{2k})+S(x_{2k},x_{2k},x_{2k+1})} \\
&\precsim \alpha S(x_{2k},x_{2k},x_{2k+1}) \\
&\quad + \frac{\beta S(x_{2k},x_{2k},x_{2k+1})S(x_{2k+1},x_{2k+1},x_{2k+2})}{2S(x_{2k},x_{2k},x_{2k+2})+S(x_{2k+1},x_{2k+1},x_{2k+1})+S(x_{2k},x_{2k},x_{2k+1})}.
\end{aligned}$$

Hence,

$$|S(x_{2k+1},x_{2k+1},x_{2k+2})| \leq \alpha|S(x_{2k},x_{2k},x_{2k+1})| + \frac{\beta|S(x_{2k},x_{2k},x_{2k+1})||S(x_{2k+1},x_{2k+1},x_{2k+2})|}{|2S(x_{2k},x_{2k},x_{2k+2})+S(x_{2k},x_{2k},x_{2k+1})|}.$$

By condition (iii) of Definition 1.1 and Lemma 1.3, we see that

$$\begin{aligned} |S(x_{2k+1},x_{2k+1},x_{2k+2})| &= |S(x_{2k+2},x_{2k+2},x_{2k+1})| \\ &\leq |2S(x_{2k+2},x_{2k+2},x_{2k})+S(x_{2k+1},x_{2k+1},x_{2k})| \\ &= |2S(x_{2k},x_{2k},x_{2k+2})+S(x_{2k},x_{2k},x_{2k+1})|. \end{aligned} \tag{1}$$

Thus,

$$\begin{aligned} |S(x_{2k+1},x_{2k+1},x_{2k+2})| &\leq \alpha|S(x_{2k},x_{2k},x_{2k+1})|+\beta|S(x_{2k},x_{2k},x_{2k+1})| \\ &= (\alpha+\beta)|S(x_{2k},x_{2k},x_{2k+1})|. \end{aligned}$$

Similarly, we get

$$|S(x_{2k+2},x_{2k+2},x_{2k+3})| = (\alpha+\beta)|S(x_{2k+1},x_{2k+1},x_{2k+2})|.$$

If $\eta = \alpha+\beta < 1$, then

$$|S(x_{n+1},x_{n+1},x_{n+2})| \leq \eta|S(x_n,x_n,x_{n+1}| \leq \cdots \leq \eta^{n+1}|S(x_0,x_0,x_1)|.$$

Hence, for any $m > n$ we have:

$$\begin{aligned} |S(x_n,x_n,x_m)| &\leq 2(|S(x_n,x_n,x_{n+1})|+|S(x_{n+1},x_{n+1},x_{n+2})|+\cdots+|S(x_{m-1},x_{m-1},x_m)|) \\ &\leq 2(\eta^n+\eta^{n+1}+\cdots+\eta^{m-1})|S(x_0,x_0,x_1)| \\ &\leq 2\frac{\eta^n}{1-\eta}|S(x_0,x_0,x_1)|. \end{aligned}$$

Therefore, $|S(x_n,x_n,x_m)| \leq 2\frac{\eta^n}{1-\eta}|S(x_0,x_0,x_1)| \to 0$, as $m,n \to \infty$ and hence $\{x_n\}$ is a cauchy sequence. Since, X is complete, we find that $\{x_n\}$ converge to some $v \in X$. We claim that v

is the unique fixed common point of f and g. Assume that $fv \neq v$. Thus, $0 \prec z = S(v,v,fv)$. Therefore,

$$\begin{aligned} z &\precsim S(v,v,x_{2k+2}) + S(x_{2k+2},x_{2k+2},fv) \\ &\precsim S(v,v,x_{2k+2}) + S(gx_{2k+1},gx_{2k+1},fv) \\ &\precsim S(v,v,x_{2k+2}) + \alpha S(x_{2k+1},x_{2k+1},v) \\ &+ \frac{\beta S(v,v,fv)S(x_{2k+1},x_{2k+1},gx_{2k+1})}{2S(v,v,gx_{2k+1}) + S(x_{2k+1},x_{2k+1},fv) + S(v,v,x_{2k+1})} \\ &\precsim S(v,v,x_{2k+2}) + \alpha S(x_{2k+1},x_{2k+1},v) \\ &+ \frac{\beta z S(x_{2k+1},x_{2k+1},gx_{2k+1})}{2S(v,v,x_{2k+2}) + S(x_{2k+1},x_{2k+1},fv) + S(v,v,x_{2k+1})}. \end{aligned}$$

Hence,

$$\begin{aligned} |z| &\leq |S(v,v,x_{2k+2})| + \alpha |S(x_{2k+1},x_{2k+1},v)| \\ &+ \frac{\beta |z||S(x_{2k+1},x_{2k+1},gx_{2k+1})|}{|2S(v,v,x_{2k+2}) + S(x_{2k+1},x_{2k+1},fv) + S(v,v,x_{2k+1})|}. \end{aligned}$$

It is easy to see that as $n \to \infty$, $S(v,v,fv) \to 0$ which contradict our assumption about z. Thus, $fv = v$ and similarly one can show that $gv = v$. Therefore, f and g have a common fixed point. Now, to show uniqueness assume there exist another common fixed point of f and g say w. Hence,

$$S(v,v,w) = S(fv,fv,gw) \precsim \alpha S(v,v,w) + \frac{\beta S(v,v,fv)S(w,w,gw)}{2S(v,v,gw) + S(w,w,fv) + S(v,v,w)} = \alpha S(v,v,w),$$

which implies that $|S(v,v,w)| = \alpha |S(v,v,w)|$, but given the fact that $\alpha < 1$ we deduce that $S(v,v,w) = 0$ and thus $v = w$ as desired. Next, we assume that for all natural numbers k if we have:

$$S(x_{2k},x_{2k},gx_{2k+1}) + S(x_{2k+1},x_{2k+1},fx_{2k}) + S(x_{2k},x_{2k},x_{2k+1}) = 0,$$

then $S(fx_{2k},fx_{2k},gx_{2k+1}) = 0$, which implies $x_{2k} = fx_{2k} = x_{2k+1} = gx_{2k+1} = x_{2k+2}$. Therefore, $x_{2k+1} = fx_{2k} = x_{2k}$, hence there exist n_1, m_1 such that $n_1 = fm_1 = m_1$. Similarly, there exist n_2, m_2 such that $n_2 = gm_2 = m_2$. Note that

$$S(m_1,m_1,gm_2) + S(m_2,m_2,fm_1) + S(m_1,m_1,m_2) = 0.$$

We deduce that $S(fm_1, fm_1, gm_2) = 0$, which implies that $n_1 = fm_1 = gm_2 = n_2$. Therefore, $n_1 = fm_1 = fn_1$. Similarly, we get $n_2 = gm_2 = gn_2$. Since $n_1 = n_2$, we deduce that $fn_1 = gn_1 = n_1$. Thus, n_1 is a common fixed point of f and g. To show uniqueness, we assume there exist u, v common fixed points of $f$and g. Note that $S(u,u,gv) + S(v,v,fu) + S(u,u,v) = 0$. Thus, $S(u,u,v) = S(fu, fu, gv) = 0$, which implies that $u = v$ as required. This completes the proof.

Next, we present a trivial and useful corollary of Theorem 2.1, which is the case when $f = g$.

Corollary 2.2. *Let (X,S) be a complete complex valued S-metric space and f be a self mapping on X satisfying the following contraction condition:*

$$S(fx, fx, fy) \precsim \alpha S(x,x,y) + \frac{\beta S(x,x,fx)S(y,y,fy)}{2S(x,x,fy) + S(y,y,fx) + S(x,x,y)}$$

for all $x, y \in X$ such that $x \neq y$, $S(x,x,fy) + S(y,y,fx) + S(x,x,y) \neq 0$ where α, β are two nonnegative real numbers with $\alpha + \beta < 1$ or $S(fx, fx, fy) = 0$ if $S(x,x,fy) + S(y,y,fx) + S(x,x,y) = 0$. Then f have a unique common fixed point.

Now, as an application of Theorem 2.1, we prove the following for two finite families of self mappings on a complex valued S-metric space (X,S).

Theorem 2.3. *If $\{f_i\}_1^m$ and $\{g_i\}_1^n$ are two positive commuting families of self mappings defined on a complete complex valued metric space (X,S) such that the mappings $f = f_1 f_2 \cdots f_m$ and $g = g_1 g_2 \cdots g_m$ satisfies the contraction condition $(\star)$ in Theorem 2.1, then the component maps of the two families$\{f_i\}_1^m$ and $\{g_i\}_1^n$ have a unique common fixed point.*

Proof. Note that the maps f and g satisfy all the hypothesis of Theorem 2.1. Thus, f and g has a unique common fixed point, that is there exists $u \in X$ such that $fu = gu = u$. Since $\{f_i\}_1^m$ and $\{g_i\}_1^n$ are two positive commuting families, we have

$$f_k u = f_k g u = g f_k u \quad \text{and} \quad f_k u = f_k f u = f f_k u,$$

which implies that for all k, $f_k u$ is also a common fixed point of f and g. By the uniqueness of the common fixed point we deduce that for all k, $f_k u = u$ and hence u is a common fixed point of the family $\{f_i\}_1^m$. Similarly, u is a common fixed point of the family $\{g_i\}_1^n$ as required.

The following result is a corollary of Theorem 2.3.

Corollary 2.4. *Let (X,S) be a complete complex valued S-metric space and F,G be two self mappings on X satisfying the following contraction condition:*

$$S(F^m x, F^m x, G^n y) \precsim \alpha S(x,x,y) + \frac{\beta S(x,x,F^m x) S(y,y,G^n y)}{2S(x,x,G^n y) + S(y,y,F^m x) + S(x,x,y)}$$

for all $x,y \in X$ and α, β are two nonnegative real numbers with $\alpha + \beta < 1$ or $S(F^m x, F^m x, G^n y) = 0$ if $S(x,x,G^n y) + S(y,y,F^m x) + S(x,x,y) = 0$. Then F,G have a unique common fixed point.

Proof. Note that this corollary is just a special case of Theorem 2.3, just take $F = f_1 = f_2 = \cdots = f_m$ and $G = g_1 = g_2 = \cdots = g_n$ and the result follows as desired.

Notice that if we assume that $\beta = 0$, $f = g$ and $n = m$ in Corollary 2.4, we obtain the following nice contraction principle result in complex valued S-metric space.

Corollary 2.5. *If f is a self mapping on a complete complex valued S-metric space (X,S) that satisfies:*

$$S(f^n x. f^n x. f^n y) \precsim \alpha S(x,x,y)$$

for all $x,y \in X$ and α a nonnegative real number such that $\alpha < 1$, then f has a unique fixed point in X.

Next we prove the existence and the uniqueness of a common fixed point for a two self mappings on a complex valued S-metric space under a contraction principle that is different from $(\star)$.

Theorem 2.6. *Let (X,S) be a complete complex valued S-metric space and f,g be two self mappings on X that satisfy:*

$$S(fx,fx,gy) \precsim \alpha S(x,x,y) + \frac{\beta [S^2(x,x,gy) + S^2(y,y,fx)]}{S(x,x,gy) + S(y,y,fx)} + \gamma [S(x,x,fx) + S(y,y,gy)] \quad (\star\star)$$

for all $x,y \in X$ such that $x \neq y$, where α, β, γ are nonnegative real numbers with the property $\alpha + 4\beta + 2\gamma < 1$ or $S(fx,fx,gy) = 0$ if $S(x,x,gy) + S(y,y,fx) = 0$. Then f,g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ and let $x_{2k+1} = fx_{2k}$, $x_{2k+2} = gx_{2k+1}$, $k \in \{0,1,2,\cdots\}$. Thus,

$$\begin{aligned}
S(x_{2k+1},x_{2k+1},x_{2k+2}) &= S(fx_{2k},fx_{2k},gx_{2k+1})\\
&\precsim \alpha S(x_{2k},x_{2k},x_{2k+1})\\
&+\frac{\beta[S^2(x_{2k},x_{2k+1},fx_{2k})+S^2(x_{2k+1},x_{2k+1},gx_{2k})]}{S(x_{2k},x_{2k},gx_{2k+1})+S(x_{2k+1},x_{2k+1},fx_{2k})}\\
&+\gamma[S(x_{2k},x_{2k},fx_{2k})+S(x_{2k+1},x_{2k+1},gx_{2k+1})\\
&= \alpha S(x_{2k},x_{2k},x_{2k+1})\\
&+\frac{\beta[S^2(x_{2k},x_{2k},x_{2k+2})+S^2(x_{2k+1},x_{2k+1},x_{2k+1})]}{S(x_{2k},x_{2k},x_{2k+2})+S(x_{2k+1},x_{2k+1},x_{2k+1})}\\
&+\gamma[S(x_{2k},x_{2k},x_{2k+1})+S(x_{2k+1},x_{2k+1},x_{2k+2})].
\end{aligned}$$

Using the fact that $S(x,x,x)=0$ for all $x \in X$, we get

$$\begin{aligned}
|S(x_{2k+1},x_{2k+1},x_{2k+2})| &\leq \alpha|S(x_{2k},x_{2k},x_{2k+1})|\\
&+\beta|S(x_{2k},x_{2k},x_{2k+2})|\\
&+\gamma[|S(x_{2k},x_{2k},x_{2k+1})|+|S(x_{2k+1},x_{2k+1},x_{2k+2})|].
\end{aligned}$$

By condition (iii) in Definition 1.1, we obtain

$$|S(x_{2k},x_{2k},x_{2k+2})| \leq 2|S(x_{2k},x_{2k},x_{2k+1})|+|S(x_{2k+1},x_{2k+1},x_{2k+2})|.$$

Hence,

$$\begin{aligned}
|S(x_{2k+1},x_{2k+1},x_{2k+2})| &\leq \alpha|S(x_{2k},x_{2k},x_{2k+1})|\\
&+\beta[2|S(x_{2k},x_{2k},x_{2k+1})|+|S(x_{2k+1},x_{2k+1},x_{2k+2})|]\\
&+\gamma[|S(x_{2k},x_{2k},x_{2k+1})|+|S(x_{2k+1},x_{2k+1},x_{2k+2})|]\\
&\leq \alpha|S(x_{2k},x_{2k},x_{2k+1})|\\
&+2\beta[|S(x_{2k},x_{2k},x_{2k+1})|+|S(x_{2k+1},x_{2k+1},x_{2k+2})|]\\
&+\gamma[|S(x_{2k},x_{2k},x_{2k+1})|+|S(x_{2k+1},x_{2k+1},x_{2k+2})|].
\end{aligned}$$

Thus,

$$|S(x_{2k+1},x_{2k+1},x_{2k+2})| \leq (\frac{\alpha+2\beta+\gamma}{1-2\beta-\gamma})|S(x_{2k},x_{2k},x_{2k+1})|.$$

Using the argument we obtain

$$|S(x_{2k+2},x_{2k+2},x_{2k+3})| \leq (\frac{\alpha+2\beta+\gamma}{1-2\beta-\gamma})|S(x_{2k+1},x_{2k+1},x_{2k+2})|.$$

Now, let $\eta = (\frac{\alpha+2\beta+\gamma}{1-2\beta-\gamma})$. Note that $\eta < 1$. Therefore,

$$|S(x_{2n+1},x_{2n+1},x_{2n+2})| \leq \eta |S(x_{2n},x_{2n},x_{2n+1})| \leq \cdots \leq \eta^{n+1}|S(x_0,x_0,x_1)|.$$

So, for any two natural numbers $0 < n < m$ and by using Lemma 1.3, and the condition (iii) of Definition 1.1, we obtain

$$\begin{aligned}
|S(x_n,x_n,x_m)| &\leq 2|S(x_n,x_n,x_{n+1})| + |S(x_{n+1},x_{n+1},x_m)| \\
&\leq 2|S(x_n,x_n,x_{n+1})| + 2|S(x_{n+1},x_{n+1},x_{n+2})| + |S(x_{n+2},x_{n+2},x_m)| \\
&\leq \cdots \\
&\leq 2|S(x_n,x_n,x_{n+1})| + 2|S(x_{n+1},x_{n+1},x_{n+2})| + \cdots + 2|S(x_{m-1},x_{m-1},x_m)| \\
&\leq 2[\eta^n + \eta^{n+1} + \cdots + \eta^{m-1}]|S(x_0,x_0,x_1)| \\
&\leq 2(\frac{\eta^n}{1-\eta})|S(x_0,x_0,x_1)| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. We claim that u is a fixed point of f, so if $z = S(u,u,fu)$, then

$$\begin{aligned}
z &\precsim 2S(u,u,x_{2k+2}) + S(x_{2k+2},x_{2k+2},fu) = 2S(u,u,x_{2k+2}) + S(gx_{2k+1},gx_{2k+1},fu) \\
&\precsim 2S(u,u,x_{2k+2}) + \alpha S(u,u,x_{2k+1}) \\
&+ \frac{\beta[S^2(u,u,gx_{2k+1}) + S^2(x_{2k+1},x_{2k+1},fu)]}{S(u,u,gx_{2k+1}) + S(x_{2k+1},x_{2k+1},fu)} \\
&+ \gamma[S(u,u,fu) + S(x_{2k+1},x_{2k+1},gx_{2k+1})] \\
&\precsim 2S(u,u,x_{2k+2}) + \alpha S(u,u,x_{2k+1}) \\
&+ \frac{\beta[S^2(u,u,gx_{2k+1}) + S^2(x_{2k+1},x_{2k+1},fu)]}{S(u,u,gx_{2k+1}) + S(x_{2k+1},x_{2k+1},fu)} \\
&+ \gamma[z + S(x_{2k+1},x_{2k+1},gx_{2k+1})].
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}|z| &\precsim 2|S(u,u,x_{2k+2})| + \alpha|S(u,u,x_{2k+1})| \\ &+ \frac{\beta[|S^2(u,u,gx_{2k+1})| + |S^2(x_{2k+1},x_{2k+1},fu)|]}{|S(u,u,gx_{2k+1}) + S(x_{2k+1},x_{2k+1},fu)|} \\ &+ \gamma[|z| + |S(x_{2k+1},x_{2k+1},gx_{2k+1})|].\end{aligned}$$

Note that as $n \to \infty$ we have $|z| = |S(u,u,fu)| \to 0$. Thus, $fu = u$ as required. Similarly, we obtain $gu = u$. Therefore, f and g has a fixed point. Now, to show uniqueness assume there exist two common fixed point of f and g say v and w. Hence,

$$S(v,v,w) = S(fv,fv,gw).$$

So, condition $(\star\star)$ implies that $|S(v,v,w)| = \alpha|S(v,v,w)|$, but given the fact that $\alpha < 1$ we deduce that $S(v,v,w) = 0$ and thus $v = w$ as desired. Now, we assume that for all natural numbers k if we have:

$$S(x_{2k},x_{2k},gx_{2k+1}) + S(x_{2k+1},x_{2k+1},fx_{2k}) = 0,$$

then $S(fx_{2k},fx_{2k},gx_{2k+1}) = 0$, which implies $x_{2k} = fx_{2k} = x_{2k+1} = gx_{2k+1} = x_{2k+2}$. Therefore, $x_{2k+1} = fx_{2k} = x_{2k}$, hence there exist n_1, m_1 such that $n_1 = fm_1 = m_1$. Similarly, there exist n_2, m_2 such that $n_2 = fm_2 = m_2$. Note that

$$S(m_1,m_1,gm_2) + S(m_2,m_2,fm_1) = 0.$$

We deduce that $S(fm_1,fm_1,gm_2) = 0$, which implies that $n_1 = fm_1 = gm_2 = n_2$. Therefore, $n_1 = fm_1 = fn_1$, similarly we get $n_2 = gm_2 = gn_2$. Since $n_1 = n_2$ we deduce that $fn_1 = gn_1 = n_1$. Thus, n_1 is a common fixed point of f and g. To show uniqueness assume there exist u, v common fixed points of $f$and g. Note that

$$S(u,u,gv) + S(v,v,fu) = 0.$$

Thus, $S(u,u,v) = S(fu,fu,gv) = 0$ which implies that $u = v$ as required.

As a consequence of Theorem 2.6, we obtain the following useful corollary.

Corollary 2.7. *Let (X,S) be a complete complex valued S-metric space and f be two self mappings on X that satisfy:*

$$S(fx,fx,fy) \precsim \alpha S(x,x,y) + \frac{\beta[S^2(x,x,fy)+S^2(y,y,fx)]}{S(x,x,fy)+S(y,y,fx)} + \gamma[S(x,x,fx)+S(y,y,fy)]$$

for all $x,y \in X$ such that $x \neq y$, where α,β,γ are nonnegative real numbers with the property $\alpha+4\beta+2\gamma<1$ or $S(fx,fx,fy)=0$ if $S(x,x,fy)+S(y,y,fx)=0$. Then f have a unique fixed point in X.

Proof. Putting $f=g$ in Theorem 2.6, we find the desired conclusion immediately.

Next, we prove the following result.

Theorem 2.8. *If $\{f_i\}_1^m$ and $\{g_i\}_1^n$ are two positive commuting families of self mappings defined on a complete complex valued metric space (X,S) such that the mappings $f=f_1f_2\cdots f_m$ and $g=g_1g_2\cdots g_m$ satisfies the contraction condition $(\star\star)$ in Theorem 2.6, then the component maps of the two families$\{f_i\}_1^m$ and $\{g_i\}_1^n$ have a unique common fixed point.*

Proof. Note that the maps f and g satisfy all the hypothesis of Theorem 2.6. Thus, f and g has a unique common fixed point, that is there exists $u \in X$ such that $fu=gu=u$. Since $\{f_i\}_1^m$ and $\{g_i\}_1^n$ are two positive commuting families, we have

$$f_ku = f_kgu = gf_ku \quad \text{and} \quad f_ku = f_kfu = ff_ku.$$

Which implies that for all k, f_ku is also a common fixed point of f and g. By the uniqueness of the common fixed point we deduce that for all k, $f_ku=u$ and hence u is a common fixed point of the family $\{f_i\}_1^m$. Similarly, u is a common fixed point of the family $\{g_i\}_1^n$ as required.

The following result is a corollary of Theorem 2.8.

Corollary 2.9. *Let (X,S) be a complete complex valued S-metric space and F,G be two self mappings on X satisfying the following contraction condition $(\star\star)$ in Theorem 2.6. Then F,G have a unique common fixed point.*

Proof. Note that this corollary is just a special case of Theorem 2.8, just take $F=f_1=f_2=\cdots=f_m$ and $G=g_1=g_2=\cdots=g_n$ and the result follows as desired.

Notice that if we assume that $\beta = \gamma = 0$, $f = g$ and $n = m$ in Corollary 2.9, we obtain the following nice contraction principle result in complex valued S-metric space.

Corollary 2.10. *If f is a self mapping on a complete complex valued S-metric space (X,S) that satisfies:*

$$S(f^n x.f^n x.f^n y) \precsim \alpha S(x,x,y)$$

for all $x,y \in X$ and α a nonnegative real number such that $\alpha < 1$, then f has a unique fixed point in X.

In closing, we give the following example which is an application of Theorem 2.1.

Example 1.1. Consider

$$X_1 = \{z \in \mathbb{C} : Re(z) \geq 0, Im(z) = 0\} \text{ and } X_2 = \{z \in \mathbb{C} : Im(z) \geq 0, Re(z) = 0\}.$$

Now, let $X = X_1 \cup X_2$ and define $S : X^3 \to \mathbb{C}$ by:

$$S(z_1,z_2,z_2) = \begin{cases} max\{x_1,x_2,x_3\} + i\ \ max\{x_1,x_2,x_3\} & \text{if } \ z_1,z_2,z_3 \in X_1, \\ max\{max\{y_1,y_2\},y_3\} + i\ \ max\{y_1,y_2,y_3\} & \text{if } \ z_1,z_2,z_3 \in X_2, \\ (max\{x_1,x_2\} + y_3) + i\ \ (max\{x_1,x_2\} + y_3) & \text{if } \ z_1,z_2 \in X_1, z_3 \in X_2, \\ (x_3 + max\{y_1,y_2\}) + i\ \ (x_3 + max\{y_1,y_2\}) & \text{if } \ z_1,z_2 \in X_2, z_3 \in X_1, \end{cases}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$. It is not difficult to see that (X,S) is a complete complex valued S-metric space. Now, to apply Theorem 2.1, we set $f = g$ and define f by:

$$fz = \begin{cases} \frac{Re(z)}{2} & \text{if } \ z \in X_1, \\ i\frac{Im(z)}{2} & \text{if } \ z \in X_2. \end{cases}$$

Note that,

$$0 \precsim S(z_1,z_1,z_2),\ 0 \precsim S(fz_1,fz_1,fz_2),\ 0 \precsim \frac{S(z_1,z_1,fz_1)S(z_2,z_2,fz_2)}{S(z_1,z_1,fz_2)+S(z_2,z_2,fz_1)+S(z_1,z_1,z_2)}.$$

Now, let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Hence, we have four cases:

<u>**Case 1:**</u>

If $z_1, z_2 \in X_1$, then

$$\begin{aligned} S(fz_1, fz_1, fz_2) &= S(\frac{x_1}{2}, \frac{x_1}{2}, \frac{x_2}{2}) \\ &= max\{\frac{x_1}{2}, \frac{x_2}{2}\} + i \ \ max\{\frac{x_1}{2}, \frac{x_2}{2}\} \\ &= max\{\frac{x_1}{2}, \frac{x_2}{2}\}(1+i) \\ &\precsim \frac{1}{2}S(z_1, z_1, z_2). \end{aligned}$$

Case 2:

If $z_1, z_2 \in X_2$, then

$$\begin{aligned} S(fz_1, fz_1, fz_2) &= S(i\frac{y_1}{2}, i\frac{y_1}{2}, i\frac{y_2}{2}) \\ &= max\{\frac{y_1}{2}, \frac{y_2}{2}\} + i \ \ max\{\frac{y_1}{2}, \frac{y_2}{2}\} \\ &= max\{\frac{y_1}{2}, \frac{y_2}{2}\}(1+i) \\ &\precsim \frac{1}{2}S(z_1, z_1, z_2). \end{aligned}$$

Case 3:

If $z_1 \in X_1, z_2 \in X_2$, then

$$\begin{aligned} S(fz_1, fz_1, fz_2) &= S(\frac{x_1}{2}, \frac{x_1}{2}, i\frac{y_2}{2}) \\ &= (\frac{x_1}{2} + \frac{y_2}{2}) + i \ \ (\frac{x_1}{2} + \frac{y_2}{2}) \\ &= (\frac{x_1}{2} + \frac{y_2}{2})(1+i) \\ &\precsim \frac{1}{2}S(z_1, z_1, z_2). \end{aligned}$$

Case 4:

If $z_2 \in X_1, z_1 \in X_2$, then

$$\begin{aligned} S(fz_1, fz_1, fz_2) &= S(i\frac{y_1}{2}, i\frac{y_1}{2}, \frac{x_2}{2}) \\ &= (\frac{y_1}{2} + \frac{x_2}{2}) + i \ \ (\frac{y_1}{2} + \frac{x_2}{2}) \\ &= (\frac{y_1}{2} + \frac{x_2}{2})(1+i) \\ &\precsim \frac{1}{2}S(z_1, z_1, z_2). \end{aligned}$$

Thus, the self mapping $f = g$ satisfies all the conditions of $(\star)$, with $\alpha = \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$. Also, notice that all the condition of Theorem 2.1, are satisfied and $0 \in X$ is the unique fixed point.

Conflict of Interests

The author declares that there is no conflict of interests.

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