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COMMON FIXED POINT THEOREMS FOR A FAMILY OF MAPPINGS IN CONE METRIC SPACES

YIZHUO MA, YUBO LIU*, MEIMEI SONG

College of Science, Tianjin University of Technology, Tianjin 300384, China

Abstract. Huang and Zhang obtained some fixed point theorems in cone metric spaces. Abbas and Rhoades generalized those theorems and obtained common fixed points for mappings with normal cone conditions satisfying certain contractive conditions. In this paper, we prove some theorems by removing the normal cone conditions and extend these theorems from a pair of single valued maps to a family of mappings in cone metric spaces.

Keywords: cone metric space; non-normal cones; fixed point; a family of mappings.

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1. Introduction

Recently, Huang and Zhang [1] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and proved some fixed point theorems for mappings satisfying different contractive conditions. They also discussed several properties of convergence of sequences in cone metric spaces. Subsequently, many authors extended different kinds of contractive mappings in cone metric spaces, see [2]-[10] and the references therein. Abbas

*Corresponding author
E-mail address: liuyubo2003@126.com (Y. Liu)
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and Jungck [2] presented the existence of coincidence points and proposed a pair of weakly compatible maps. Abbas and Rhoades [3] obtained some fixed point theorems for mappings without commutativity conditions, but still rely on the assumption of normality in cone metric spaces. The purpose of this paper is to extend and improve the results in [3] with non-normal cone conditions and obtain common fixed point theorems for a family of mappings.

2. Preliminaries

Let E always be a real Banach space and let P be a subset of E. Recall that P is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{\theta\}$;

(b) $\forall a,b \in R, a,b \geq 0, \forall x,y \in P$ imply that $ax+by \in P$;

(c) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$. While $x \prec\prec y$ stands for $y - x \in intP$ (interior of P).

A cone P is called normal if there is a number $K > 0$ such that for all $x,y \in E$,

$$\theta \preceq x \preceq y \text{ implies } \| x \| \leq K \| y \|.$$

The least positive number satisfying the above inequality is called the normal constant of P.

In the following we always suppose E be a Banach space, P is a cone in E with $intP \neq \emptyset$ and $\preceq$ is a partial ordering with respect to P .

Definition 1.1. [1] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $d(x,y) \succeq \theta$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(d3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space. The concept of a cone metric space is more general than that of a metric space.

Definition 1.2. [1] Let (X,d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \prec\prec c$, there is an N such that for all $n > N, d(x_n,x) \prec\prec c$, then $\{x_n\}$ is said to be a convergent sequence.

Definition 1.3. [1] Let (X,d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \prec\prec c$, there is an N such that for all $n,m > N, d(x_n,x_m) \prec\prec c$, then $\{x_n\}$ is called a Cauthy sequence in X.

Definition 1.4. [1] Let (X,d) be a cone metric space, if every Cauchy sequence is convergent in X , then X is called a complete cone metric space.

3. Main results

Theorem 3.1. *Let (X,d) be a complete cone metric space. Suppose that the mapping f and g are two self-maps of X satisfying*

$$d(fx,gy) \preceq \alpha d(x,y) + \beta[d(x,fx)+d(y,gy)] + \gamma[d(x,gy)+d(y,fx)] \tag{3.1}$$

for all $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha+2\beta+2\gamma < 1$. Then f and g have a unique common fixed point in X. Moreover, any fixed point of f is the fixed point of g, and conversely.

Proof. Suppose x_0 is an arbitrary point of X, and define $\{x_n\}$ by

$$x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}, n = 0,1,2,\cdots.$$

It follows that

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &= d(fx_{2n},gx_{2n+1}) \\ &\preceq \alpha d(x_{2n},x_{2n+1}) + \beta[d(x_{2n},fx_{2n}) + d(x_{2n+1},gx_{2n+1})] \\ &\quad +\gamma[d(x_{2n},gx_{2n+1}) + d(x_{2n+1},fx_{2n})] \\ &\preceq \alpha d(x_{2n},x_{2n+1}) + \beta[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})] \\ &\quad +\gamma[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})] \\ &\preceq (\alpha+\beta+\gamma)d(x_{2n},x_{2n+1}) + (\beta+\gamma)d(x_{2n+1},x_{2n+2}), \end{aligned}$$

which implies that

$$d(x_{2n+1},x_{2n+2}) \preceq td(x_{2n},x_{2n+1}),$$

where $t = \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} < 1$. Similarly, we find that

$$d(x_{2n+2},x_{2n+3}) \preceq td(x_{2n+1},x_{2n+2}).$$

Therefore, for all n, we see that

$$\begin{aligned} d(x_{n+1},x_{n+2}) &\preceq td(x_n,x_{n+1}) \\ &\preceq \cdots \preceq t^{n+1}d(x_0,x_1). \end{aligned}$$

Now, for any $n > m$, we see that

$$\begin{aligned} d(x_n,x_m) &\preceq d(x_n,x_{n-1}) + d(x_{n-1},x_{n-2}) + \cdots + d(x_{m+1},x_m) \\ &\preceq (t^{n-1}+t^{n-2}+\cdots+t^m)d(x_1,x_0) \\ &\preceq \frac{t^m}{1-t}d(x_1,x_0). \end{aligned}$$

Let $\theta \prec\prec c$ be given. Choose $\delta > 0$ such that $c + N_\delta(\theta) \subseteq P$, where $N_\delta(\theta) = \{y \in E : \| y \| < \delta\}$. Also, choose a natural number N_1, such that

$$\tfrac{t^m}{1-t}d(x_1,x_0) \in N_\delta(\theta), \text{ for all } m \geq N_1.$$

Then, $\frac{t^m}{1-t}d(x_1,x_0) \prec\prec c$, for all $m \geq N_1$. Thus, $d(x_n,x_m) \preceq \frac{t^m}{1-t}d(x_1,x_0) \prec\prec c$, for all $n > m$. Therefore, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X,d). Since (X,d) is a complete cone metric space, there exists $x^* \in X$ such that $x_n \to x^*$. Choose a natural number N_2 such that $d(x_n,x^*) \prec\prec \frac{\sigma}{6}c$, where $\sigma = 1-\beta-\gamma$, for all $n \geq N_2$. Hence, we have

$$\begin{aligned} d(x^*,gx^*) &\preceq d(x^*,x_{2n+1}) + d(x_{2n+1},gx^*) \\ &= d(x^*,x_{2n+1}) + d(fx_{2n},gx^*) \\ &\preceq d(x^*,x_{2n+1}) + \alpha d(x_{2n},x^*) + \beta[d(x_{2n},x_{2n+1}) + d(x^*,gx^*)] \\ &\quad +\gamma[d(x_{2n},gx^*) + d(x^*,x_{2n+1})] \\ &\preceq d(x^*,x_{2n+1}) + \alpha d(x_{2n},x^*) + \beta[d(x_{2n},x_{2n+1}) + d(x^*,gx^*)] \\ &\quad +\gamma[d(x_{2n},x^*) + d(x^*,gx^*) + d(x^*,x_{2n+1})], \end{aligned}$$

which further implies that

$$\begin{aligned}
d(x^*,gx^*) &\preceq \frac{1}{1-\beta-\gamma}[d(x^*,x_{2n+1})+\alpha d(x_{2n},x^*)+\beta d(x_{2n},x_{2n+1})+\gamma[d(x_{2n},x^*) \\
&\quad +d(x^*,x_{2n+1})] \\
&\preceq \frac{1}{\sigma}[d(x^*,x_{2n+1})+d(x_{2n},x^*)+d(x_{2n},x_{2n+1})+d(x_{2n},x^*)+d(x^*,x_{2n+1})] \\
&\preceq \frac{1}{\sigma}[d(x^*,x_{2n+1})+d(x_{2n},x^*)+d(x_{2n},x^*)+d(x^*,x_{2n+1})+d(x_{2n},x^*) \\
&\quad +d(x^*,x_{2n+1})] \\
&\prec\prec \frac{1}{\sigma}[\frac{3\sigma}{6}c+\frac{3\sigma}{6}c] \\
&= c
\end{aligned}$$

for all $n \geq N_2$. Thus, $d(x^*,gx^*) \prec\prec \frac{c}{m}$, for all $m \geq 1$. So $\frac{c}{m}-d(x^*,gx^*) \in intP$, for all $m \geq 1$. Since $\frac{c}{m} \to \theta$ (as $m \to \infty$) and P is closed, $-d(x^*,gx^*) \in P$. But $d(x^*,gx^*) \in P$. Therefore, $d(x^*,gx^*) = \theta$, and $gx^* = x^*$. Now, we will show that x^* is also a fixed point of f :

$$\begin{aligned}
d(fx^*,x^*) &= d(fx^*,gx^*) \\
&\preceq \alpha d(x^*,x^*)+\beta[d(x^*,fx^*)+d(x^*,gx^*)]+\gamma[d(x^*,gx^*)+d(x^*,fx^*)] \\
&= (\beta+\gamma)d(x^*,fx^*),
\end{aligned}$$

which, using the definition of partial ordering on E and properties of cone P, gives that $d(fx^*,x^*) = \theta$, and $fx^* = x^*$. Conversely, any fixed point of f is also that of g. To prove uniqueness, we suppose that if y^* is another common fixed point of g and f, then

$$\begin{aligned}
d(x^*,y^*) &= d(fx^*,gy^*) \\
&\preceq \alpha d(x^*,y^*)+\beta[d(x^*,fx^*)+d(y^*,gy^*)]+\gamma[d(x^*,gy^*)+d(y^*,fx^*)] \\
&\preceq (\alpha+2\gamma)d(x^*,y^*),
\end{aligned}$$

which gives $d(x^*,y^*) = \theta$, and so $x^* = y^*$. This completes the proof.

Corollary 3.2. *Let (X,d) be a complete cone metric space. Suppose that mapping $f : X \to X$ satisfies*

$$d(f^px,f^qy) \preceq \alpha d(x,y)+\beta[d(x,f^px)+d(y,f^qy)]+\gamma[d(x,f^qy)+d(y,f^px)] \qquad (3.2)$$

for all $x, y \in X$, *where* $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + 2\beta + 2\gamma < 1$, *and* p *and* q *are fixed positive integers. Then* f *has a unique fixed point in* X.

Proof. Set $f = f^p$ and $g = f^q$ in (3.1), we find that f^p and f^q have a unique common fixed point x^*. It follows that $f^p x^* = f^q x^* = x^*$. Since any fixed point of f^p is the fixed point of f^q, we only need to take account of the fixed point of f^p. Note that $f^p(fx^*) = f(f^p x^*) = fx^*$. So fx^* is also a fixed point of f^p. In view of the uniqueness, $fx^* = x^*$, we find that x^* is a fixed point of f. Since the fixed point of f is also that of f^p, the fixed point of f is unique. This completes the proof.

Corollary 3.3. *Let* (X, d) *be a complete cone metric space. Suppose that mapping* $f : X \to X$ *satisfies*

$$d(fx, fy) \preceq \alpha d(x,y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)] \tag{3.3}$$

for all $x, y \in X$, *where* $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + 2\beta + 2\gamma < 1$. *Then* f *has a unique fixed point in* X.

Proof. Set $p = q = 1$ in (3.2) and $\beta = \gamma = 0$. Then the above corollary is the classical Banach contraction principle in cone metric space.

Corollary 3.4. *Let* (X, d) *be a complete cone metric space. Suppose that mapping* $f : X \to X$ *satisfies*

$$d(fx, fy) \preceq a_1 d(x,y) + a_2 d(x, fx) + a_3 d(y, fy) + a_4 d(x, fy) + a_5 d(y, fx) \tag{3.4}$$

for all $x, y \in X$, *where* $a_i \geq 0$ *for each* $i \in \{1, 2, \cdots, 5\}$, *and* $a_1 + 2max\{a_2, a_3\} + 2max\{a_4, a_5\} < 1$. *Then* f *has a unique fixed point in* X.

Proof. From (3.4), we find that

$$\begin{aligned} d(fx, fy) &\preceq a_1 d(x,y) + a_2 d(x, fx) + a_3 d(y, fy) + a_4 d(x, fy) + a_5 d(y, fx) \\ &\preceq a_1 d(x,y) + max\{a_2, a_3\}[d(x, fx) + d(y, fy)] + max\{a_4, a_5\}[d(x, fy) + d(y, fx)]. \end{aligned}$$

Set $max\{a_2, a_3\} = \beta, max\{a_4, a_5\} = \gamma$ in (3.3), we can easily conclude the conclusion.

Theorem 3.5. *Let* (X, d) *be a complete cone metric space. Suppose that a family of self-maps* $\{f_\lambda\}, \lambda \in \wedge$ *(* $\wedge$ *is a subscript set) satisfies*

$$d(f_\lambda x, f_\mu y) \preceq \alpha d(x,y) + \beta[d(x, f_\lambda x) + d(y, f_\mu y)] + \gamma[d(x, f_\mu y) + d(y, f_\lambda x)] \tag{3.5}$$

for all $x,y \in X$*, all* $\lambda,\mu \in \wedge$*, where* $\alpha,\beta,\gamma \geq 0$ *and* $\alpha+2\beta+2\gamma < 1$*. Then a family of maps* $\{f_\lambda\}_{\lambda\in\wedge}$ *have a unique common fixed point in* X*.*

Proof. Suppose that $f = f_\lambda, g = f_\mu$. Based on Theorem 3.1, we can get that there exists $x^* \in X$, such that $f_\lambda(x^*) = f_\mu(x^*) = x^*$. Now choose an arbitrary $\nu \in \wedge$, and $\nu \neq \lambda,\mu$. It follows that f_λ, f_ν satisfied the conditions in (3.5). Therefore, we can obtain that there exists $x^{*'} \in X$ such that $f_\lambda(x^{*'}) = f_\nu(x^{*'}) = x^{*'}$. This implies that f_λ and f_ν have a unique common fixed point in X. Note that

$$\begin{aligned}
\theta \preceq d(x^*,x^{*'}) &= d(f_\mu x^*, f_\nu x^{*'}) \\
&\preceq d(f_\mu x^*, f_\lambda x^*) + d(f_\lambda x^*, f_\nu x^{*'}) \\
&\preceq \alpha d(x^*,x^*) + \beta[d(x^*, f_\mu x^*) + d(x^*, f_\lambda x^*)] + \gamma[d(x^*, f_\lambda x^*) + d(x^*, f_\mu x^*)] \\
&+ \alpha d(x^*,x^{*'}) + \beta[d(x^*, f_\lambda x^*) + d(x^{*'}, f_\nu x^{*'})] + \gamma[d(x^*, f_\nu x^{*'}) + d(x^{*'}, f_\lambda x^*)] \\
&= \alpha d(x^*,x^{*'}) + 2\gamma d(x^*,x^{*'}) \\
&= (\alpha+2\gamma) d(x^*,x^{*'}),
\end{aligned}$$

which implies that

$$\theta \preceq (1-\alpha-2\gamma)d(x^*,x^{*'}) \preceq \theta.$$

Since $1-\alpha-2\gamma > 0$, we find from the definition of partial ordering on E and the properties of cone P that $d(x^*,x^{*'}) = \theta$, and $x^* = x^{*'}$. It shows that f_λ, f_μ, f_ν have a unique common fixed point x^*. By the arbitrary of ν, gives that $\{f_\lambda\}_{\lambda\in\wedge}$ have a unique common fixed point in X. This completes the proof.

Theorem 3.6. *Let* (X,d) *be a complete cone metric space, and there exists non-negative sequences* $\{\alpha_k\},\{\beta_k\},\{\gamma_k\}(k=1,2,\cdots)$*. Suppose that a family of self-maps* $\{f_k\}$ $(k=1,2,\cdots)$ *satisfies*

$$d(f_k x, f_{k+1} y) \preceq \alpha_k d(x,y) + \beta_k[d(x, f_k x) + d(y, f_{k+1} y)] + \gamma_k[d(x, f_{k+1} y) + d(y, f_k x)] \tag{3.6}$$

for all $x,y \in X$*, where* $\alpha_k + 2\beta_k + 2\gamma_k < 1(k=1,2,\cdots)$*. Then* $\{f_k\}$ *have a unique common fixed point in* X*. In other words, there exists* $x^* \in X$*, such that* $f_k(x^*) = x^*(k=1,2,\cdots)$*.*

Proof. When $k=1$, we can get f_1, f_2 have a unique fixed point $x^* \in X$ by setting $f=f_1, g=f_2$ in (3.1). It follows that $f_1(x^*)=f_2(x^*)=x^*$. When $k=2$, we can get f_2, f_3 have a unique fixed point $x^{*'} \in X$ by setting $f=f_2, g=f_3$ in (3.1). According to Theorem 3.5, we obtain that $x^*=x^{*'}$. Therefore f_1, f_2, f_3 have a unique fixed point in X. Suppose that $k=n$, we can get the conclusion. Suppose that $k=n+1$, for all $x, y \in X$,

$$d(f_n x, f_{n+1} y) \preceq \alpha_n d(x,y) + \beta_n [d(x, f_n x) + d(y, f_{n+1} y)] + \gamma_n [d(x, f_{n+1} y) + d(y, f_n x)].$$

Based on Theorem 3.1 we can obtain that f_n and f_{n+1} have a unique common fixed point $x^{*''} \in X$. According to inductive assumption, we can get $f_1, f_2, \cdots, f_n$ have a common fixed point x^*. Using Theorem 3.5, we find that $x^* = x^{*''}$. It follows that $f_1, f_2, \cdots, f_{n+1}$ have a unique common fixed point in X. Based on the inductive method, we can obtain that $\{f_k\}(k=1,2,\cdots)$ have a unique common fixed point in X. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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