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AN ITERATIVE METHOD FOR NONEXPANSIVE SEMIGROUPS, VARIATIONAL INCLUSIONS AND GENERALIZED EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we introduce a viscosity iterative scheme for finding a common element of the set of common fixed points of a one-parameter nonexpansive semigroup, the set of solutions to variational inclusions and the set of solutions to generalized equilibrium problems in a real Hilbert space. Strong convergence theorems for the common element are obtained.

Keywords: nonexpansive semigroup; variational inclusion; inverse strongly monotone mapping; generalized equilibrium problem.

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1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Recall the following definitions.

A mapping $T : C \to H$ is said to be

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(1) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(2) α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

(3) α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \quad \forall x, y \in C.$$

(4) k-Lipschitz continuous if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

If T is α-inverse strongly monotone, then T is $\frac{1}{\alpha}$-Lipschitz continuous. In the case that $k = 1$, $T : C \to H$ is said to be nonexpansive.

A (one parameter) nonexpansive semigroup is a family $\Gamma = \{S(t) : t \geq 0\}$ of self-mapping of C if the following conditions are satisfied:

(a) $S(0)x = x$ for all $x \in C$;

(b) $S(s+t) = S(s)S(t)$ for all $s, t \geq 0$;

(c) for each $t > 0$, $\|S(t)x - S(t)y\| \leq \|x - y\|, x, y \in C$;

(d) for each $x \in C$, the mapping $S(\cdot)x$ is continuous.

We use $F(\Gamma)$ to denote the common fixed point set of the semigroup Γ, that is,

$$F(\Gamma) = \{x \in C : S(t)x = x, t \geq 0\}.$$

Let $A : H \to H$ be a single-valued nonlinear mapping and let $M : H \to 2^H$ be a set-valued mapping. The variational inclusion is to find $x \in H$ such that

$$\theta \in A(x) + M(x), \tag{1.1}$$

where θ is a zero vector in H. The set of solutions to variational inclusion (1.1) is denoted by $I(A, M)$. When $A = 0$, then (1.1) becomes the inclusion problem introduced by Rockafellar [1].

Let $\varphi : C \to H$ be a nonlinear mapping. The variational inequality problem is to find $x \in C$ such that

$$\langle \varphi x, y - x\rangle \geq 0, \forall y \in C. \tag{1.2}$$

The set of solutions to variational inequality problem (1.2) is denoted by $VI(C,\varphi)$. Finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions to a variational inequality problem has been studied extensively in the literature; see, for example, [2] and the references therein.

A set-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if the graph $G(M)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y,g) \in G(M)$ implies $f \in Mx$. The resolvent operator $J_{M,\lambda}$ associated with M and λ is the mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I+\lambda M)^{-1}(u), \quad u \in H, \lambda > 0. \tag{1.3}$$

It is known that the resolvent operator $J_{M,\lambda}$ is single-valued, nonexpansive and 1-inverse-strongly monotone and that a solution of (1.1) is a fixed point of $J_{M,\lambda}(I-\lambda A), \forall\lambda > 0$, see [3]. If $0 < \lambda < 2\alpha$, it is easy to see that $J_{M,\lambda}(I-\lambda A)$ is nonexpansive and $I(A,M)$ is closed and convex.

Let G be a bi-function of $C \times C$ into R, the set of reals and $\varphi : C \to H$ be a nonlinear mapping. The generalized equilibrium problem is to find $x \in C$ such that

$$G(x,y) + \langle \varphi x, y-x\rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions to this generalized equilibrium problem (1.4) is denoted by EP. Thus

$$EP := \{x \in C : G(x,y) + \langle \varphi x, y-x\rangle \geq 0, \forall y \in C\}.$$

In the case of $\varphi \equiv 0$, the problem (1.4) reduces to an equilibrium problem, which is to find $x \in C$ such that

$$G(x,y) \geq 0, \forall y \in C. \tag{1.5}$$

and EP is then denoted by $EP(G)$. In the case of $G \equiv 0$, the problem (1.4) reduces to the variational inequality problem (1.2) and EP is denoted by $VI(C,\varphi)$. Numerous problems in physics, optimization and economics can be reduced to the generalized equilibrium problem

(1.4). Some methods have been proposed to solve the generalized equilibrium problems and equilibrium problems.

For solving the equilibrium problem for a bifunction $G : C \times C \to R$, let us assume that F satisfies the following conditions:

(A1) $G(x,x) = 0$ for all $x \in C$;

(A2) G is monotone, i.e., $G(x,y) + G(y,x) \leq 0$ for all $x,y \in C$;

(A3) For each $x,y,z \in C$, $\limsup_{t \to 0} G(tz + (1-t)x, y) \leq G(x,y)$;

(A4) For each $x \in C$, $y \mapsto G(x,y)$ is convex and lower semicontinuous.

For finding an element of $F(T)$, where T is a nonexpansive mapping. Moudafi [4] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H, starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfies certain conditions, the sequence $\{x_n\}$ converges strongly to the unique solution q in $F(T)$.

Tian [5] consider the following general iterative method

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \quad n \geq 0.$$

where F is a k-Lipschitzian and η-strongly monotone operator with $k > 0, \eta > 0, 0 < \mu < \frac{2\eta}{k^2}$. If the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ converges strongly to the unique solution $q \in F(T)$ of the variational inequality

$$\langle (\gamma f - \mu F)q, p - q \rangle \leq 0, \quad \forall p \in F(T).$$

For finding a common element in $F(S) \cap EP$, Takahashi and Takahashi [6] introduced the following iterative scheme:

$$\begin{cases} G(u_n, y) + \langle \varphi x_n, y - u_n \rangle + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n) u_n]. \end{cases} \tag{1.6}$$

Under the suitable conditions, some strong theorems are proved.

For finding a common element in $F(S)\cap EP\cap I(A,M)$, Shehu [7] introduced the following iterative scheme:

$$\begin{cases} G(u_n,y)+\langle \varphi x_n, y-u_n\rangle+\dfrac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, \quad \forall y\in C, \\ x_{n+1}=\beta_n x_n+(1-\beta_n)S[\alpha_n f(x_n)+(1-\alpha_n)J_{M,\lambda}(u_n-\lambda A u_n)]. \end{cases} \tag{1.7}$$

Under the suitable conditions, some strong theorems are proved which extend the results of Takahashi and Takahashi [6].

For finding a common element in $F(\Gamma)\cap EP\cap I(A,M)$, Shehu [8] introduced the following iterative scheme:

$$\begin{cases} G(u_n,y)+\langle \varphi x_n, y-u_n\rangle+\dfrac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, \quad \forall y\in C, \\ x_{n+1}=\beta_n x_n+(1-\beta_n)(\dfrac{1}{t_n}\displaystyle\int_0^{t_n} S(u)[\alpha_n f(x_n)+(1-\alpha_n)J_{M,\lambda}(u_n-\lambda A u_n)]du). \end{cases} \tag{1.8}$$

Under the suitable conditions, some strong theorems are proved which extend the results of Shehu [7].

A nonexpansive semigroup is said to be uniformly asymptotically regular if for any $t\geq 0$ and for any bounded subset D of C,

$$\lim_{s\to\infty}\sup_{x\in D}\|S(t+s)x-S(s)x\|=0.$$

Let C be a nonempty closed convex sunset of a real Hilbert space H, and $\Gamma=\{S(t):t>0\}$ a nonexpansive semigroup on C such that $F(\Gamma)$ is nonempty. Let $\sigma(t)x=\frac{1}{t}\int_0^t S(u)xdu$ is an uniformly asymptotically regular nonexpansive semigroup; see [9].

In this paper, motivated and inspired by the above results, we introduce an iterative scheme for finding a common element of the set of common fixed points of nonexpansive semigroups, the set of solutions to variational inclusions and the set of solutions to generalized equilibrium problems. Our results improved and extend many recent results in the literature.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. In a real Hilbert space, the following inequality holds:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H. \tag{2.1}$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C. P_C is characterized by the following properties:

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

In the context of the variational inequality problem, this implies

$$p \in VI(C, \varphi) \Leftrightarrow p = P_C(p - \lambda \varphi p), \quad \forall \lambda > 0. \tag{2.3}$$

It is well known that H satisfies the Opial's condition [3], *i.e.*, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \forall y \in H, y \neq x.$$

In order to prove our main results, w shall make use of the following lemmas.

Lemma 2.1. [10] *Let $\{x_n\}$, $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1-\beta_n) y_n, \forall\, n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.2. [11] *Let $\{s_n\}$ be a sequence of nonnegative real numbers such that:*

$$s_{n+1} \leq (1 - \lambda_n) s_n + \beta_n, \quad n \geq 0,$$

where $\{\lambda_n\}, \{\beta_n\}$ satisfy the conditions:

(i) $\{\lambda_n\} \subset (0,1)$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$,

(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\lambda_n} \leq 0$ or $\sum_{n=1}^{\infty} |\beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. [2] *Let C be a nonempty closed subset of H and let G be a bifunction of $C\times C$ into R satisfying (A1)-(A4). Let $r>0$ and $x\in H$. Then, there exists $z\in C$ such that*

$$G(z,y)+\frac{1}{r}\langle y-z,z-x\rangle \geq 0, \quad \forall y\in C.$$

Lemma 2.4. [12] *Assume that $G: C\times C\to R$ satisfies (A1)-(A4). For $r>0$ and $x\in H$, define a mapping $T_r: H\to C$ as follows:*

$$T_r x=\{z\in C: G(z,y)+\frac{1}{r}\langle y-z,z-x\rangle \geq 0, \forall\, y\in C\}$$

for all $x\in H$. Then the following hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, i.e., for any $x,y\in H$,

$$\|T_r x-T_r y\|^2 \leq \langle T_r x-T_r y, x-y\rangle;$$

(3) $F(T_r)=EP(G)$;

(4) $EP(G)$ is closed and convex.

Lemma 2.5. [13] *Let $M: H\to 2^H$ be a maximal monotone mapping and let $A: H\to H$ be a Lipschitz continuous mapping. Then the mapping $M+A: H\to 2^H$ is a maximal monotone mapping.*

Lemma 2.6. [14] *Let H be a real Hilbert space and let $F: H\to H$ be a k-Lipschitz and η-strongly monotone operator with $k>0$, $\eta>0$. Let $0<\mu<2\eta/k^2$, $B=I-t\mu F$ and $\mu(\eta-\frac{\mu k^2}{2})=\tau$. Then for $t\in(0,\min\{1,\frac{1}{\tau}\})$, B is a contraction with a constant $1-t\tau$.*

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G: C\times C\to R$ be a bifunction satisfying conditions (A1)-(A4). Let $\varphi: H\to H$ be an θ-inverse-strongly monotone mapping, $A: C\to H$ be an α-inverse-strongly monotone mapping and $M: H\to 2^H$ be a maximal monotone mapping. Let $\Gamma=\{S(t): t\geq 0\}$ be uniform asymptotically regular nonexpansive semigroup on C such that $\Omega=F(\Gamma)\cap EP\cap I(A,M)\neq\emptyset$. Let $f: H\to H$*

is Lipschitz mapping with the coefficient L, and $F : H \to H$ be a k-Lipschitz and η-strongly monotone operator. Suppose that the sequences $\{x_n\}$, $\{u_n\}$, $\{z_n\}$ are generated by $x_1 \in H$

$$\begin{cases} G(u_n,y)+\langle \varphi x_n, y-u_n\rangle + \dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = \beta_n u_n + (1-\beta_n)S(t_n)u_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + [(1-\delta_n)I - \alpha_n \mu F]J_{M,\lambda}(z_n - \lambda A z_n), \end{cases}$$

where the sequences $\{\alpha_n\},\{\beta_n\},\{\delta_n\} \subseteq (0,1)$ and $\{r_n\} \subseteq (0,\infty)$, $\{t_n\} \subseteq [0,\infty)$ satisfying the following restrictions:

(C1) $0 < a \leq r_n \leq b < 2\theta$;

(C2) $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$;

(C3) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(C4) $\lambda \in (0, 2\alpha]$;

(C5) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;

(C6) $\lim_{n\to\infty} \beta_n = 0$;

(C7) $\{t_n\} \subseteq [0,\infty)$ *be a real increasing sequence such that* $\lim_{n\to\infty} t_n = \infty$;

(C8) $0 < \mu < \frac{2\eta}{k^2}$, $0 < \gamma < \tau/L$, $\mu(\eta - \frac{\mu k^2}{2}) = \tau$.

Then the sequence $\{x_n\}$ converges strongly to $q \in \Omega$, which is the unique solution in the Ω to the following variational inequality

$$\langle \gamma f(q) - \mu F q, p - q\rangle \leq 0, \quad \forall\, p \in \Omega. \tag{3.1}$$

Equivalently, we have $q = P_{\Omega}(I + \gamma f - \mu F)q$.

Proof. We divide the proof into five steps.

Step 1. Show that $\{x_n\}$ is bounded.

For all $x, y \in C$ and $\lambda > 0$, we obtain

$$\begin{aligned} &\|(I-\lambda A)x - (I-\lambda A)y\|^2 \\ &= \|(x-y) - \lambda(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 + \lambda(\lambda - 2\alpha)\|Ax-Ay\|^2. \end{aligned}$$

So, $I-\lambda A$ is a nonexpansive mapping. Using (C1), we see that $I-r_n\varphi$ is also nonexpansive. Let $v_n = J_{M,\lambda}(z_n-\lambda Az_n)$ and let $p\in\Omega$. It follows that

$$\begin{aligned}\|v_n-p\|^2 &= \|J_{M,\lambda}(z_n-\lambda Az_n)-J_{M,\lambda}(p-\lambda Ap)\|^2\\ &\le \|(z_n-\lambda Az_n)-(p-\lambda Ap)\|^2\\ &\le \|z_n-p\|^2+\lambda(\lambda-2\alpha)\|Az_n-Ap\|^2\\ &\le \|z_n-p\|^2\end{aligned}$$

and $\|z_n-p\|\le(1-\beta_n)\|S(t_n)u_n-p\|+\beta_n\|u_n-p\|\le\|u_n-p\|$. From Lemma 2.4, $u_n=T_{r_n}(x_n-r_n\varphi x_n)$ and T_{r_n} is nonexpansive. Hence, we have

$$\begin{aligned}\|u_n-p\|^2 &= \|T_{r_n}(x_n-r_n\varphi x_n)-T_{r_n}(p-r_n\varphi p)\|^2\\ &\le \|(x_n-r_n\varphi x_n)-(p-r_n\varphi p)\|^2\\ &\le \|x_n-p\|^2+r_n(r_n-2\theta)\|\varphi x_n-\varphi p\|^2\\ &\le \|x_n-p\|^2.\end{aligned}\tag{3.2}$$

From (C3), (C8) and Lemma 2.6, we have $\|(1-\delta_n)I-\alpha_n\mu F\|\le 1-\delta_n-\alpha_n\tau$. Further

$$\begin{aligned}&\|x_{n+1}-p\|\\ &=\|\alpha_n(\gamma f(x_n)-\mu Fp)+\delta_n(x_n-p)+[(1-\delta_n)I-\alpha_n\mu F](v_n-p)\|\\ &\le \alpha_n\|\gamma f(x_n)-\mu Fp\|+\delta_n\|x_n-p\|+(1-\delta_n-\alpha_n\tau)\|v_n-p\|\\ &\le \alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-\mu Fp\|+\delta_n\|x_n-p\|+(1-\delta_n-\alpha_n\tau)\|x_n-p\|\\ &\le \alpha_n\gamma L\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu Fp\|+\delta_n\|x_n-p\|+(1-\delta_n-\alpha_n\tau)\|x_n-p\|\\ &=[1-\alpha_n(\tau-\gamma L)]\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu Fp\|.\end{aligned}$$

By induction, we have

$$\|x_n-p\|\le\max\{\|x_1-p\|,\frac{1}{\tau-\gamma L}\|\gamma f(p)-\mu Fp\|\}.$$

Therefore $\{x_n\}$ is bounded, we have $\{v_n\}$,$\{u_n\}$, $\{S(t_n)u_n\}$, $\{Ax_n\}$, $\{Fv_n\}$, $\{f(x_n)\}$ are also bounded.

Step 2. Show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Put $l_n = \frac{x_{n+1}-\delta_n x_n}{1-\delta_n}$, this is, $x_{n+1} = (1-\delta_n)l_n + \delta_n x_n$. Observing that

$$\begin{aligned}\|l_{n+1}-l_n\| \leq & \frac{\alpha_{n+1}}{1-\delta_{n+1}}(\|\gamma f(x_{n+1})\| + \|\mu F v_{n+1}\|) + \frac{\alpha_n}{1-\delta_n}(\|\gamma f(x_n)\| + \|\mu F v_n\|) \\ & + \|v_{n+1}-v_n\|.\end{aligned} \tag{3.3}$$

Since $I-\lambda A$ is a nonexpansive, we have

$$\begin{aligned}\|v_{n+1}-v_n\| &= \|J_{M,\lambda}(z_{n+1}-\lambda A z_{n+1}) - J_{M,\lambda}(z_n - \lambda A z_n)\| \\ &\leq \|(z_{n+1}-\lambda A z_{n+1}) - (z_n-\lambda A z_n)\| \\ &\leq \|z_{n+1}-z_n\| \\ &\leq (1-\beta_{n+1})\|S(t_{n+1})u_{n+1} - S(t_n)u_n\| + \beta_{n+1}\|u_{n+1}-u_n\| \\ &+ |\beta_{n+1}-\beta_n|\|S(t_n)u_n - u_n\| \\ &\leq \|S(t_{n+1})u_{n+1} - S(t_{n+1})u_n\| + \|S[(t_{n+1}-t_n)+t_n]u_n - S(t_n)u_n\| \\ &+ \beta_{n+1}\|u_{n+1}-u_n\| + |\beta_{n+1}-\beta_n|\|S(t_n)u_n - u_n\| \\ &\leq \|u_{n+1}-u_n\| + \sup_{x\in\{u_n\}, t\geq 0}\|S(t+t_n)x - S(t_n)x\| \\ &+ \beta_{n+1}\|u_{n+1}-u_n\| + |\beta_{n+1}-\beta_n|\|S(t_n)u_n - u_n\|.\end{aligned} \tag{3.4}$$

On the other hand, from $u_n = T_{r_n}(x_n - r_n\varphi x_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1}\varphi x_{n+1})$, we obtain

$$G(u_n, y) + \langle \varphi x_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C \tag{3.5}$$

and

$$G(u_{n+1}, y) + \langle \varphi x_{n+1}, y-u_{n+1}\rangle + \frac{1}{r_{n+1}}\langle y-u_{n+1}, u_{n+1}-x_{n+1}\rangle \geq 0, \quad \forall y \in C. \tag{3.6}$$

Substituting $y = u_{n+1}$ in (3.5) and $y = u_n$ in (3.6), we have

$$G(u_n, u_{n+1}) + \langle \varphi x_n, u_{n+1}-u_n\rangle + \frac{1}{r_n}\langle u_{n+1}-u_n, u_n-x_n\rangle \geq 0$$

and

$$G(u_{n+1}, u_n) + \langle \varphi x_{n+1}, u_n-u_{n+1}\rangle + \frac{1}{r_{n+1}}\langle u_n-u_{n+1}, u_{n+1}-x_{n+1}\rangle \geq 0.$$

So, from (A2), we have

$$0 \leq \langle \varphi x_{n+1} - \varphi x_n, u_n - u_{n+1}\rangle + \langle u_{n+1}-u_n, \frac{u_n-x_n}{r_n} - \frac{u_{n+1}-x_{n+1}}{r_{n+1}}\rangle$$

and hence,

$$0 \leq \langle u_{n+1}-u_n, r_n(\varphi x_n-\varphi x_{n+1})+(u_n-x_n)-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\rangle$$
$$= \langle u_{n+1}-u_n, u_n-u_{n+1}+x_{n+1}-r_n\varphi x_{n+1}-(x_n-r_n\varphi x_n)+(1-\frac{r_n}{r_{n+1}})(u_{n+1}-x_{n+1})\rangle.$$

It follows that

$$\|u_{n+1}-u_n\|^2 \leq \|u_{n+1}-u_n\|\{\|x_{n+1}-x_n\|+(1-\frac{r_n}{r_{n+1}})\|u_{n+1}-x_{n+1}\|\}.$$

From (C1), we have

$$\begin{aligned}\|u_{n+1}-u_n\| &\leq \|x_{n+1}-x_n\|+|1-\frac{r_n}{r_{n+1}}|\|u_{n+1}-x_{n+1}\| \\ &\leq \|x_n-x_{n+1}\|+\frac{1}{a}|r_n-r_{n+1}|\|u_{n+1}-x_{n+1}\|.\end{aligned} \tag{3.7}$$

Substituting (3.4) and (3.7) into (3.3), we have

$$\begin{aligned}&\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\| \\ &\leq \frac{\alpha_{n+1}}{1-\delta_{n+1}}(\|\gamma f(x_{n+1})\|+\|\mu F v_{n+1}\|)+\frac{\alpha_n}{1-\delta_n}(\|\gamma f(x_n)\|+\|\mu F v_n\|) \\ &+\beta_{n+1}\|u_{n+1}-u_n\|+\sup_{x\in\{u_n\},t\geq 0}\|S(t+t_n)x-S(t_n)x\| \\ &+|\beta_{n+1}-\beta_n|\|S(t_n)u_n-u_n\|+\frac{1}{a}|r_n-r_{n+1}|\|u_{n+1}-x_{n+1}\|.\end{aligned}$$

Since (C2), (C3), (C6), (C7) and the uniform asymptotic regularity of nonexpansive semigroup, we have

$$\limsup_{n\to\infty}(\|l_{n+1}-l_n\|-\|x_n-x_{n+1}\|) \leq 0.$$

By Lemma 2.1, we have $\lim_{n\to\infty}\|l_n-x_n\|=0$. Consequently, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|l_n-x_n\| = 0. \tag{3.8}$$

Step 3. Show that $\lim_{n\to\infty}\|x_n-S(t)x_n\|=0, \forall t\geq 0$.

Observing that

$$\begin{aligned}\|x_n-v_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-v_n\| \\ &\leq \|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-\mu F v_n\|+\delta_n\|x_n-v_n\|.\end{aligned}$$

It follows that

$$(1-\delta_n)\|x_n-v_n\| \leq \|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-\mu F v_n\|.$$

From (C3), (C5) and (3.8), we have

$$\lim_{n\to\infty}\|v_n - x_n\| = 0. \tag{3.9}$$

Let $M > 0$ be a constant such that $M > \sup_{n\geq 1}\max\{\|\gamma f(x_n) - \mu F v_n\|, \|x_n - p\|\}$. From (2.1), (3.2) and the convexity of $\|.\|^2$, we obtain

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \|\delta_n(x_n - p) + (1-\delta_n)(v_n - p)\|^2 + 2\alpha_n\langle \gamma f(x_n) - \mu F v_n, x_{n+1} - p\rangle \\
&\leq (1-\delta_n)\|v_n - p\|^2 + \delta_n\|x_n - p\|^2 + 2\alpha_n M^2 \\
&\leq (1-\delta_n)\|u_n - p\|^2 + \delta_n\|x_n - p\|^2 + 2\alpha_n M^2 \\
&\leq (1-\delta_n)[\|x_n - p\|^2 + r_n(r_n - 2\theta)\|\varphi x_n - \varphi p\|^2] + \delta_n\|x_n - p\|^2 + 2\alpha_n M^2 \\
&\leq \|x_n - p\|^2 + a(b-2\theta)\|\varphi x_n - \varphi p\|^2 + 2\alpha_n M^2.
\end{aligned}$$

Hence,

$$a(2\theta - b)\|\varphi x_n - \varphi p\|^2 \leq 2\alpha_n M^2 + (\|x_{n+1} - p\| + \|x_n - p\|)\|x_n - x_{n+1}\|.$$

Using (C3) and (3.8), we have $\|\varphi x_n - \varphi p\| \to 0, n \to \infty$. Similarly, we also have $\|Az_n - Ap\| \to 0$, as $n \to \infty$. Since T_{r_n} is 1-inverse-strongly-monotone, we have

$$\begin{aligned}
&\|u_n - p\|^2 = \|T_{r_n}(x_n - r_n\varphi x_n) - T_{r_n}(p - r_n\varphi p)\|^2 \\
&\leq \langle x_n - r_n\varphi x_n - (p - r_n\varphi p), u_n - p\rangle \\
&= \frac{1}{2}[\|x_n - r_n\varphi x_n - (p - r_n\varphi p)\|^2 + \|u_n - p\|^2 - \|x_n - r_n\varphi x_n - (p - r_n\varphi p) - (u_n - p)\|^2] \\
&\leq \frac{1}{2}[\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - r_n(\varphi x_n - \varphi p)\|^2] \\
&= \frac{1}{2}[\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, \varphi x_n - \varphi p\rangle - r_n^2\|\varphi x_n - \varphi p\|^2].
\end{aligned}$$

This implies that

$$\begin{aligned}
\|u_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, \varphi x_n - \varphi p\rangle - r_n^2\|\varphi x_n - \varphi p\|^2 \\
&\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|\varphi x_n - \varphi p\|.
\end{aligned} \tag{3.10}$$

From (2.1) and (3.10), we obtain

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|\delta_n(x_n-p)+(1-\delta_n)(v_n-p)+\alpha_n(\gamma f(x_n)-\mu F v_n)\|^2\\
&\le \|\delta_n(x_n-p)+[(1-\delta_n)(v_n-p)]\|^2+2\alpha_n\langle \gamma f(x_n)-\mu F v_n, x_{n+1}-p\rangle\\
&\le (1-\delta_n)\|v_n-p\|^2+\delta_n\|x_n-p\|^2+2\alpha_n M^2\\
&\le (1-\delta_n)\|u_n-p\|^2+\delta_n\|x_n-p\|^2+2\alpha_n M^2\\
&\le (1-\delta_n)[\|x_n-p\|^2-\|x_n-u_n\|^2+2r_n\|x_n-u_n\|\|\varphi x_n-\varphi p\|]+\delta_n\|x_n-p\|^2+2\alpha_n M^2\\
&\le \|x_n-p\|^2-(1-\delta_n)\|x_n-u_n\|^2+2r_n\|x_n-u_n\|\|\varphi x_n-\varphi p\|+2\alpha_n M^2.
\end{aligned}
$$

This implies that

$$(1-\delta_n)\|x_n-u_n\|^2\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2r_n\|x_n-u_n\|\|\varphi x_n-\varphi p\|+2\alpha_n M^2.$$

Since (C3), (C5), (3.8) and $\lim_{n\to\infty}\|\varphi x_n-\varphi p\|=0$, we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.11}$$

Since $J_{M,\lambda}$ is also 1-inverse-strongly-monotone, By the similar argument above, we also have

$$\lim_{n\to\infty}\|v_n-z_n\|=0. \tag{3.12}$$

From $z_n=\beta_n x_n+(1-\beta_n)S(t_n)u_n$, we have from (C6)

$$\lim_{n\to\infty}\|z_n-S(t_n)u_n\|=\lim_{n\to\infty}\beta_n\|u_n-S(t_n)u_n\|=0. \tag{3.13}$$

From (3.9), (3.11), (3.12) and (3.13), we have

$$
\begin{aligned}
\|x_n-S(t_n)x_n\|&\le \|x_n-v_n\|+\|v_n-z_n\|+\|z_n-S(t_n)u_n\|+\|S(t_n)u_n-S(t_n)x_n\|\\
&\le \|x_n-v_n\|+\|v_n-z_n\|+\|z_n-S(t_n)u_n\|+\|u_n-x_n\|\to 0.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
\|x_n-S(t)x_n\|&\le \|x_n-S(t_n)x_n\|+\|S(t_n)x_n-S(t)S(t_n)x_n\|+\|S(t)S(t_n)x_n-S(t)x_n\|\\
&\le 2\|x_n-S(t_n)x_n\|+\|S(t_n)x_n-S(t)S(t_n)x_n\|\\
&\le 2\|x_n-S(t_n)x_n\|+\sup_{x\in\{x_n\},t\ge 0}\|S(t+t_n)x-S(t_n)x\|.
\end{aligned}
$$

From (C7) and the uniform asymptotic regularity of the nonexpansive semigroup, we get

$$\lim_{n\to\infty} \|x_n - S(t)x_n\| = 0, \quad \forall t \geq 0. \tag{3.14}$$

Step 4. We show that $\limsup_{n\to\infty}\langle \gamma f(q) - \mu F q, x_n - q\rangle \leq 0$.

From (C8), we obtain $\mu F - \gamma f$ is strongly monotone. Then q is the uniqueness of a solution of (3.1). Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(q) - \mu F q, x_n - q\rangle = \lim_{i\to\infty}\langle \gamma f(q) - \mu F q, x_{n_i} - q\rangle.$$

As $\{x_{n_i}\}$ is bounded, Without loss of generality that $x_{n_i} \rightharpoonup z$, We first show that $z \in I(A,M)$. Since A is $\frac{1}{\alpha}$ Lipschitz monotone and $D(A) = H$, we obtain from Lemma 2.5, $M + A$ is maximal monotone. Let $(v,g) \in G(M+A)$, that is, $g - Av \in M(v)$. Since $v_{n_i} = J_{M,\lambda}(z_{n_i} - \lambda A z_{n_i})$, we get $(I - \lambda A)z_{n_i} \in (I + \lambda M)v_{n_i}$, that is,

$$\frac{z_{n_i} - \lambda A z_{n_i} - v_{n_i}}{\lambda} \in M(v_{n_i}).$$

Using the maximal monotonicity of $M + A$, we obtain

$$\langle v - v_{n_i}, g - Av - \frac{z_{n_i} - \lambda A z_{n_i} - v_{n_i}}{\lambda}\rangle \geq 0. \tag{3.15}$$

By the monotonicity of A and (3.15), we have

$$\begin{aligned}
\langle v - v_{n_i}, g\rangle &\geq \langle v - v_{n_i}, Av + \frac{z_{n_i} - \lambda A z_{n_i} - v_{n_i}}{\lambda}\rangle \\
&= \langle v - v_{n_i}, Av - Av_{n_i} + Av_{n_i} - Az_{n_i} + \frac{z_{n_i} - v_{n_i}}{\lambda}\rangle \\
&\geq \langle v - v_{n_i}, Av_{n_i} - Az_{n_i}\rangle + \langle v - v_{n_i}, \frac{z_{n_i} - v_{n_i}}{\lambda}\rangle.
\end{aligned}$$

It follow from (3.12), $\lim_{i\to\infty}\|Av_{n_i} - Az_{n_i}\| = 0$. From (3.9), $v_{n_i} \rightharpoonup z$, we have

$$\lim_{n\to\infty}\langle v - v_{n_i}, g\rangle = \langle v - z, g\rangle \geq 0.$$

Using the maximal monotonicity of $M + A$, we obtain $\theta \in (M+A)(v)$, this implies $z \in I(A,M)$. Since $u_n = T_{r_n}(x_n - r_n \varphi x_n)$, for any $y \in C$

$$G(u_n, y) + \langle \varphi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0.$$

Replace n by n_i and using (A2), we have

$$\langle \varphi x_{n_i}, y-u_{n_i}\rangle + \langle y-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle \geq G(y,u_{n_i}). \tag{3.16}$$

Let $y_t = ty+(1-t)z$ for all $0<t\leq 1$ and $y\in C$. Since $y\in C$ and $z\in C$, we have $y_t\in C$. From (3.16), we have

$$\begin{aligned}\langle y_t-u_{n_i}, \varphi y_t\rangle &\geq \langle y_t-u_{n_i}, \varphi y_t\rangle - \langle y_t-u_{n_i}, \varphi x_{n_i}\rangle - \langle y_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle + G(y_t,u_{n_i})\\ &= \langle y_t-u_{n_i}, \varphi y_t-\varphi u_{n_i}\rangle + \langle y_t-u_{n_i}, \varphi u_{n_i}-\varphi x_{n_i}\rangle\\ &- \langle y_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle + G(y_t,u_{n_i}).\end{aligned}$$

From the monotonicity of φ, we have $\langle y_t-u_{n_i}, \varphi y_t-\varphi u_{n_i}\rangle\geq 0$. From (3.11), we have $u_{n_i}\rightharpoonup z$ and $\|\varphi x_{n_i}-\varphi u_{n_i}\|\to 0$. From (A4), we have

$$\langle y_t-z, \varphi y_t\rangle \geq G(y_t,z). \tag{3.17}$$

From (A1), (A4) and (3.17), we have

$$\begin{aligned}0 = G(y_t,y_t) &\leq tG(y_t,y)+(1-t)G(y_t,z)\\ &\leq tG(y_t,y)+(1-t)\langle y_t-z,\varphi y_t\rangle\\ &= tG(y_t,y)+(1-t)t\langle y-z,\varphi y_t\rangle\end{aligned}$$

and hence $G(y_t,y)+(1-t)\langle y-z,\varphi y_t\rangle\geq 0$. Letting $t\to 0$ and (A3), we have $G(z,y)+\langle y-z,\varphi z\rangle\geq 0$ for all $y\in C$ and hence $z\in EP$.

Finally, we show that $z\in F(\Gamma)$. Assume the contrary that $z\neq S(t)z$ for some $t\in[0,+\infty)$. Then by the Opial's condition, we obtain from (3.14) that

$$\begin{aligned}\liminf_{i\to\infty}\|x_{n_i}-z\| &< \liminf_{i\to\infty}\|x_{n_i}-S(t)z\|\\ &\leq \liminf_{i\to\infty}(\|x_{n_i}-S(t)x_{n_i}\|+\|S(t)x_{n_i}-S(t)z\|)\\ &\leq \liminf_{i\to\infty}\|x_{n_i}-z\|.\end{aligned}$$

This is a contradiction. Hence $z\in F(\Gamma)$. Thus $z\in\Omega$. This follows that

$$\limsup_{n\to\infty}\langle \gamma f(q)-\mu Fq, x_n-q\rangle = \langle \gamma f(q)-\mu Fq, z-q\rangle \leq 0. \tag{3.18}$$

Step 5. Show that $x_n\to q$.

We compute that

$$
\begin{aligned}
&\|x_{n+1}-q\|^2\\
&=\alpha_n\langle\gamma f(x_n)-\mu Fq,x_{n+1}-q\rangle+\delta_n\langle x_n-q,x_{n+1}-q\rangle\\
&+\langle[(1-\delta_n)I-\alpha_n\mu F](v_n-q),x_{n+1}-q\rangle\\
&\le\alpha_n\gamma\langle f(x_n)-f(q),x_{n+1}-q\rangle+\alpha_n\langle\gamma f(q)-\mu Fq,x_{n+1}-q\rangle\\
&+\delta_n\|x_n-q\|\|x_{n+1}-q\|+(1-\delta_n-\alpha_n\tau)\|v_n-q\|\|x_{n+1}-q\|\\
&\le\alpha_n\gamma L\|x_n-q\|\|x_{n+1}-q\|+\alpha_n\langle\gamma f(q)-\mu Fq,x_{n+1}-q\rangle\\
&+\delta_n\|x_n-q\|\|x_{n+1}-q\|+(1-\delta_n-\alpha_n\tau)\|x_n-q\|\|x_{n+1}-q\|\\
&=(1-\alpha_n(\tau-\gamma L))\|x_n-q\|\|x_{n+1}-q\|+\alpha_n\langle\gamma f(q)-\mu Fq,x_{n+1}-q\rangle\\
&\le\frac{1-\alpha_n(\tau-\gamma L)}{2}\|x_n-q\|^2+\frac{1}{2}\|x_{n+1}-q\|^2+\alpha_n\langle\gamma f(q)-\mu Fq,x_{n+1}-q\rangle,
\end{aligned}
$$

which implies that

$$
\|x_{n+1}-q\|^2\le[1-\alpha_n(\tau-\gamma L)]\|x_n-q\|^2+2\alpha_n\langle\gamma f(q)-\mu Fq,x_{n+1}-q\rangle.
$$

By (3.18), (C3) and Lemma 2.2, we obtain $\lim_{n\to\infty}\|x_n-q\|=0$. This completes the proof.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G:C\times C\to R$ be a bifunction satisfying conditions (A1)-(A4), $\varphi:H\to H$ be an θ-inverse-strongly monotone mapping and let $A:C\to H$ be an α-inverse-strongly monotone mapping. Let $\Gamma=\{S(t):t\ge 0\}$ be uniform asymptotically regular nonexpansive semigroup on C such that $\Omega:=F(\Gamma)\cap EP\cap VI(C,A)\neq\emptyset$. Let $f:H\to H$ is Lipschitz mapping with the coefficient L and let $F:H\to H$ be a k-Lipschitz and η-strongly monotone operator. Suppose the sequences $\{x_n\}$, $\{u_n\}$, $\{z_n\}$ are generated by $x_1\in H$*

$$
\begin{cases}
G(u_n,y)+\langle\varphi x_n,y-u_n\rangle+\dfrac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\forall y\in C,\\
z_n=\beta_n u_n+(1-\beta_n)S(t_n)u_n,\\
x_{n+1}=\alpha_n\gamma f(x_n)+\delta_n x_n+[(1-\delta_n)I-\alpha_n\mu F]P_C(z_n-\lambda Az_n).
\end{cases}
$$

where the sequences $\{\alpha_n\},\{\beta_n\},\{\delta_n\}\subseteq(0,1)$ and $\{r_n\}\subseteq(0,\infty)$, $\{t_n\}\subseteq[0,\infty)$ satisfying the following restrictions:

(C1) $0 < a \le r_n \le b < 2\theta$;

(C2) $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$;

(C3) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(C4) $\lambda \in (0, 2\alpha]$;

(C5) $0 < \liminf_{n\to\infty} \delta_n \le \limsup_{n\to\infty} \delta_n < 1$;

(C6) $\lim_{n\to\infty} \beta_n = 0$;

(C7) $\{t_n\} \subseteq [0,\infty)$ *be a real increasing sequence such that* $\lim_{n\to\infty} t_n = \infty$;

(C8) $0 < \mu < \frac{2\eta}{k^2}$, $0 < \gamma < \tau/L$, $\mu(\eta - \frac{\mu k^2}{2}) = \tau$.

Then the sequence $\{x_n\}$ *converges strongly to* $q \in \Omega$*, which is the unique solution in the* Ω *to the variational inequality (3.1). Equivalently, we have* $q = P_{\Omega}(I + \gamma f - \mu F)q$.

Proof. Take $M = \partial\delta_C : H \to 2^H$, where $\delta_C : H \to [0,\infty)$ is the indicator function of C, the subdifferential $\partial\delta_C$ of δ_C is a maximal monotone operator. Then $J_{M,\lambda} = P_C$ and $I(A,M) = VI(C,A)$. From the Theorem 3.1, we have the desired conclusion immediately.

Recall that mapping $T : C \to C$ is called α-strictly pseudocontractive if there exists $\alpha \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + \alpha\|(T-I)x - (T-I)y\|^2, \quad \forall x,y \in C.$$

If $\alpha = 0$, then T is nonexpansive. Put $A = I - T$, Then, we have

$$\|(I-A)x - (I-A)y\|^2 \le \|x-y\|^2 + \alpha\|Ax - Ay\|^2, \quad \forall x,y \in C.$$

Hence we have

$$\langle x-y, Ax-Ay\rangle \ge \frac{1-\alpha}{2}\|Ax-Ay\|^2, \quad \forall x,y \in C.$$

Then A is $\frac{1-\alpha}{2}$-inverse strongly monotone.

Corollary 3.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $G : C \times C \to R$ *be a bifunction satisfying conditions (A1)-(A4),* $\varphi : H \to H$ *be an* θ*-inverse-strongly monotone mapping and let* $T : C \to C$ *be an* α*-strictly pseudocontractive maping. Let* $\Gamma = \{S(t) : t \ge 0\}$ *be uniform asymptotically regular nonexpansive semigroup on* C *such that* $\Omega := F(\Gamma) \cap EP \cap F(T) \ne \emptyset$*. Let* $f : H \to H$ *is Lipschitz mapping with the coefficient L, and let* $F : H \to H$ *be a k-Lipschitz and* η*-strongly monotone operator. Suppose the sequences* $\{x_n\}$,

$\{u_n\}$, $\{z_n\}$ are generated by $x_1 \in H$

$$\begin{cases} G(u_n,y)+\langle \varphi x_n, y-u_n\rangle + \dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \forall y\in C, \\ z_n=\beta_n u_n+(1-\beta_n)S(t_n)u_n, \\ x_{n+1}=\alpha_n\gamma f(x_n)+\delta_n x_n+[(1-\delta_n)I-\alpha_n\mu F][(1-\lambda)z_n+\lambda T z_n]. \end{cases}$$

where the sequences $\{\alpha_n\},\{\beta_n\},\{\delta_n\}\subseteq(0,1)$ and $\{r_n\}\subseteq(0,\infty)$, $\{t_n\}\subseteq[0,\infty)$ satisfying the following restrictions:

(C1) $0<a\leq r_n\leq b<2\theta$;
(C2) $\lim_{n\to\infty}|r_n-r_{n+1}|=0$;
(C3) $\lim_{n\to\infty}\alpha_n=0, \sum_{n=1}^{\infty}\alpha_n=\infty$;
(C4) $\lambda\in(0,2\alpha]$;
(C5) $0<\liminf_{n\to\infty}\delta_n\leq\limsup_{n\to\infty}\delta_n<1$;
(C6) $\lim_{n\to\infty}\beta_n=0$;
(C7) $\{t_n\}\subseteq[0,\infty)$ *be a real increasing sequence such that* $\lim_{n\to\infty}t_n=\infty$;
(C8) $0<\mu<\frac{2\eta}{k^2}$, $0<\gamma<\tau/L$, $\mu(\eta-\frac{\mu k^2}{2})=\tau$.

Then the sequence $\{x_n\}$ converges strongly to $q\in\Omega$, which is the unique solution in the Ω to the variational inequality (3.1). Equivalently, we have $q=P_\Omega(I+\gamma f-\mu F)q$.

Proof. Putting $A=I-T$, we have A is $\frac{1-\alpha}{2}$-inverse strongly monotone. We have $F(T)=VI(C,A)$ and $P_C(z_n-\lambda A z_n)=(1-\lambda)z_n+\lambda T z_n$. From Corollary 3.2, we have the desired conclusion immediately.

Conflict of Interests

The authors declare that there is no conflict of interests.

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