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A FIXED-POINT PRINCIPLE FOR A PAIR OF NON-COMMUTATIVE OPERATORS

PENUMARTHY PARVATEESAM MURTHY[1,*], TANMOY SOM[2], ERDAL KARAPINAR[3]

[1]Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavidyalaya (A Central University), Koni, Bilaspur-495009, India

[2]Department of Mathematical Sciences, Indian Institute of Technology(BHU),Varanasi-221005, India

[3]Department of Mathematics, Atilim University 06836, Incek, Ankara, Turkey

Abstract. In this paper, a fixed point principle for a pair of operators $(f_i, X, d), i = 1, 2$, where (X, d) is a metric space and $f_1, f_2 : X \to X$, is established under the generalized uniform equivalence condition of different orbits generated by the maps f_1 and f_2 separately, which gives another generalization of the fixed point principle of Leader [1] and estimates approximations to the fixed points of both the operators simultaneously.

Keywords: operator; orbit, weakly commuting maps; fixed point.

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1. Introduction

Consider two operators $(f_i, X, d), i = 1, 2$, where (X, d) is a metric space and $f_1, f_2 : X \to X$. From Meir and Keeler [2], an operator is said to have a contractive fixed point if the limit of every orbit generated by the operator is fixed. This can be easily obtained by imposing graph

*Corresponding author

E-mail address: ppmurthy@gmail.com

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completeness condition on the operator. The fixed point principle given by Leader [1] needs the uniform equivalence condition of all orbits generated by the operator to have a contractive fixed point. In the present paper, the idea of equivalence condition of orbits by a single operator is further extended to generalized equivalence condition of two different orbits generated separately by two mappings f_1 and f_2 and a fixed point principle for them is derived.

Let $x, y \in X, \{x, f_1x, f_1^2x, ...\}$ and $\{y, f_2y, f_2^2y, ...\}$ be the orbits of x and y generated by the repeated application of f_1 and f_2 separately on x and y, respectively. We say that the above two orbits are generalized equivalent if $d(f_1^m x, f_2^m y) \to 0$ as $m, n \to \infty$.

2. Main Result

Now we are ready to prove the main Theorem in this section.

Theorem 2.1. *Let $(f_i, X, d), i = 1, 2$ be a pair of operators on a metric space (X, d). Given $c > 0$, define a sequence of positive real numbers $\{\varepsilon_n\}$ by*

$$\varepsilon_n = sup\{d(f_1^i x, f_2^i y) : i \geq n, d(x, y) \leq c\}. \tag{2.1}$$

If $(m+1)\varepsilon_n + 2\varepsilon_m \leq c$ and $d(x, y) \leq c$, $d(x, f_2y) \leq c$, $d(f_1x, y) \leq c$ then

$$d(f_1^i x, f_1^{i+j} x) \leq (m+1)\varepsilon_n + 2\varepsilon_m, \tag{2.2}$$

$$d(f_2^i y, f_2^{i+j} y) \leq (m+1)\varepsilon_n + 2\varepsilon_m, \tag{2.3}$$

for all $i \geq n$ and all $j \in N$. Further if

$$d(f_1^n x, f_2^n y) \to 0 \ \ as \ \ n \to \infty \tag{2.4}$$

uniformly for all $x, y \in X$ with $d(x, y) \leq c$, then the orbits

$$\{f_1^n x\} \text{ and } \{f_2^n y\} \text{ are uniformly Cauchy.} \tag{2.5}$$

If the graphs of both $(f_i, X, d), i = 1, 2$ are complete and (2.4) holds, then $d(x, y) \leq c$, $d(x, f_2y) \leq c$ and $d(f_1x, y) \leq c$ imply that the orbits $\{f_1^n x\}$ and $\{f_2^n y\}$ converge to the fixed points $p = f_1 p$

and $q = f_2 q$, respectively, where p and q are the limits of $\{f_1^n x\}$ and $\{f_2^n y\}$ respectively. So $p = q$. Further for ε_n as defined in (2.1), we have

$$d(f_1^n x, p) \leq (m+1)\varepsilon_n + 2\varepsilon_m \text{ if } (m+1)\varepsilon_n + 2\varepsilon_m \leq c, \tag{2.6}$$

$$d(f_2^n y, q) \leq (m+1)\varepsilon_n + 2\varepsilon_m \text{ if } (m+1)\varepsilon_n + 2\varepsilon_m \leq c. \tag{2.7}$$

Proof. Using induction on k, we prove (2.2) and (2.3) for $j \leq km$ for all $k \in N$ under the given condition that $(m+1)\varepsilon_n + 2\varepsilon_m \leq c$ for a given m, n and $d(x,y) \leq c$, $d(x, f_2 y) \leq c$ and $d(f_1 x, y) \leq c$ $x, y \in X$. Let $x_i = f_1^i x$ and $y_i = f_2^i y$. Then for $k = 1$, (2.1) implies for all $i \geq n$ and $j \leq m$, where m is even. It follows that

$$\begin{aligned} d(x_i, x_{i+j}) &\leq d(x_i, y_{i+1}) + d(y_{i+1}, x_{i+2}) + \ldots + d(y_{i+j-1}, x_{i+j}) \\ &\leq d((x)_i, (y_1)_i) + d((y)_{i+1}, (x_1)_{i+1}) + \ldots + d((y)_{i+j-1}, (x_1)_{i+j-1}) \\ &\leq j\varepsilon_n \leq m\varepsilon_n. \end{aligned}$$

If m is odd, we get

$$\begin{aligned} d(x_i, x_{i+j}) &\leq d(x_i, y_{i+1}) + d(y_{i+1}, x_{i+2}) + \ldots + d(x_{i+j-1}, y_{i+j}) + d(y_{i+j}, x_{i+j}) \\ &\leq d((x)_i, (y_1)_i) + d((y)_{i+1}, (x_1)_{i+1}) + \ldots + d((x)_{i+j-1}, (y_1)_{i+j-1}) + d((y)_{i+j}, (x)_{i+j}) \\ &\leq (j+1)\varepsilon_n \leq (m+1)\varepsilon_n. \end{aligned}$$

Thus, we have

$$d(x_i, x_{i+j}) \leq (m+1)\varepsilon_n \forall i \geq n \text{ and } j \leq m \tag{2.8}$$

independent of m even or odd, that is, (2.2) holds for all $j \leq m$. Similarly, we find that (2.3) holds for all $j \leq m$. Now, suppose for a given $k \in N$ that (2.2), (2.3) hold for all $j \leq km$, we prove it for $j \leq (k+1)m$. Taking $km < j \leq (k+1)m$, we find $0 < j - m \leq (k+1)m$ and so the induction process gives that

$$d(x_i, x_{i+j-m}) \leq (m+1)\varepsilon_n + 2\varepsilon_m \leq c$$

for all $i \geq n$. Then iterating x_i and x_{i+j-m} by f_1 and f_2 respectively m times, we get

$$d(x_{i+m}, x'_{p+m}) \leq \varepsilon_m \text{ for all } i \geq n \text{ where } x'_p = x_{i+j-m}.$$

Note that

$$\begin{aligned} d(x_{i+m},x_{i+j}) &\leq d(x_{i+m},x'_{p+m})+d(x'_{p+m},x_{i+j}) \\ &= d(x_{i+m},x'_{p+m})+d(f_2^m(x_{i+j-m}),f_1^m(x_{i+j-m})) \\ &\leq \varepsilon_m+\varepsilon_m = 2\varepsilon_m \text{ for all } i\geq n. \end{aligned} \tag{2.9}$$

Therefore, from (2.8) with $j=m$ and (2.9), we get

$$d(x_i,x_{i+j}) \leq (m+1)\varepsilon_n + 2\varepsilon_m \text{ for all } i\geq n.$$

Thus (2.2) holds for all $j\leq (k+1)m$ and hence for all $j\in N$. (2.3) can be proved in a similar way. So (2.2) and (2.3) holds for all $i\geq n$ and $j\in N$. Now (2.1) and (2.4) gives $\varepsilon_n \downarrow 0$. Then for a given $0<\varepsilon<c$, we take m so large that $2\varepsilon_m<\varepsilon$. Further choose n so large that $\varepsilon_n<(m+1)^{-1}(\varepsilon-2\varepsilon_m)$ giving $(m+1)\varepsilon_n+2\varepsilon_m<\varepsilon<c$ and therefore $d(f_1^i x, f_1^{i+j}x)<\varepsilon$ and $d(f_2^i y, f_2^{i+j}y)<\varepsilon$ for all $i\geq n$ and all $j\in N$. Hence (2.5) holds. Further considering graph completeness of both the maps it can be easily obtained that $f_i p_i = p_i, i=1,2$ and that $p_1=p_2$ by (2.4). Finally in (2.2) and (2.3) taking $i=n$ and letting $j\to\infty$ we obtain (2.6) and (2.7). The theorem is completed.

3. A Fixed-point principle

In this section, we extend Theorem 3 of Som and Mukherjee [3] to three and four mappings under some weaker condition than the condition of commutativity of the mappings, used in Theorem 3 of Som and Mukherjee [3]. Next, we give here the definition of a weakly commutative pair of mappings with an example [4].

Definition 3.1. Let S and T be a pair of self mappings of a metric space (X,d). Then $\{S,T\}$ is said to be a weakly commutative pair if

$$d(STx,TSx)\leq d(Tx,Sx), \quad \forall x\in X.$$

Clearly every commutative pair of mappings is weakly commutative but the converse is not true in general.

Example 3.2. Let $X = [0,1]$ with the usual metric. Let $T,S :\to X$ be defined by $Tx = \frac{3x}{5}$, $Sx = \frac{x}{x+3}$ for every $x \in X$. Then for all $x \in X$, we have

$$\begin{aligned} d(STx, TSx) &= \frac{3x}{3x+15} - \frac{3x}{5x+15} \\ &\leq \frac{3x^2+4x}{5x+15} \\ &= \frac{3x}{5} - \frac{x}{x+3} \\ &= d(Tx, Sx) \end{aligned}$$

So, S and T are commute weakly. However S and T are not a commuting pair for

$$STx = \frac{3x}{3x+15} > \frac{3x}{5x+15} = TSx, \forall x \ \ (x \neq 0) \in X.$$

Theorem 3.3. *Let (X,d) be a metric space and f,g and h be three self mappings of X with f continuous and $g(X) \subset f(X), h(X) \subset f(X)$. Let for some $x_0 \in X, \{y_n\}$ be a sequence defined by*

$$y_1 = f(x_1) = g(x_0), y_2 = f(x_2) = h(x_1),$$

and in general,

$$y_{2n+1} = f(x_{2n+1}) = g(x_{2n}), y_{2n+2} = f(x_{2n+2}) = h(x_{2n+1}), n = 0,1,...$$

Similarly, for some $u_0 \in X$, we have a sequence $\{z_n\}$, that is, for $n = 0,1,....$

$$z_{2n+1} = f(u_{2n+1}) = g(u_{2n}), z_{2n+2} = f(u_{2n+2}) = h(u_{2n+1}).$$

For some $c > 0$, define

$$\varepsilon_{n+1} = sup\{d(y_{p+i}, z_{q+i}) : i \geq n, d(y_p, z_q) \leq c \, for \; some \; p,q \in N\}. \tag{3.1}$$

If $m\varepsilon_n + \varepsilon_{m+1} \leq c$ and $d(f(x),g(x)) \leq c, d(f(x),h(x)) \leq c$, then for all $i \geq n$ and all $j \in N$,

$$(y_i, y_{i+j}) \leq m\varepsilon_n + \varepsilon_{m+1}. \tag{3.2}$$

Hence if $d(y_n, z_n) \to 0$ uniformly for all $x_0, u_0 \in X$ with $d(y_p, z_q) \leq c$ for some $p,q \in N$ then the sequence $\{y_n\}$ is uniformly Cauchy. Further if g,h satisfy

$$d(g(x),h(y)) \leq d(f(x),f(y)) \; for \; all \; x \neq y \in X \tag{3.3}$$

and either $\{f,g\}$ or $\{f,h\}$ is a weakly commutative pair then f,g and h have a coincidence point. Moreover if

$$d(fx,y) \leq d(x,y), x \neq y \in X, \tag{3.4}$$

then f,g and h have a common fixed point in X.

Proof. The proofs of (3.2) and that $\{y_n\}$ is Cauchy follows in the lines of Theorem 3 of Som and Mukherjee [4]. So we omit the proof here. Let $y_n \to t \in X$. Since f is continuous, we have $f(y_n) \to f(t)$. From (3.3), we have

$$d(g(y_n),h(t)) \leq d(f(y_n),f(t)),$$

which in the limiting case implies that $g(y_n) \to h(t)$. Similarly it can be shown that $h(y_n) \to g(t)$. Further putting $x = y_n, y = y_{n+1}$ in (3.3) and taking the limits we get $g(t) = h(t)$. Let $\{f,g\}$ be weakly commutative. then we have

$$d(fg(x_{2n}),gf(x_{2n})) \leq d(g(x_{2n}),f(x_{2n})),$$

which in the limiting case gives that $d(f(t),h(t)) \leq d(t,t)$ and therefore $g(t) = h(t) = f(t)$. Similarly, we have the same result if $\{f,g\}$ is weakly commutative. Thus we conclude that t is a coincidence point of f,g and h. Finally, putting $x = t, y = y_n$ in (3.4) and taking the limit, we obtain a common fixed point for f,g and h. This completes the proof of the theorem.

Remark 3.4. If $g = h$ in theorem 3.3, then our theorem improves theorem 3 of Som and Mukherjee [4]. Moreover from (3.4), we observe that f is not necessarily an identity mapping to have a common fixed point result.

Theorem 3.5. *Let (X,d) be a metric space and $g_k, f_k, k = 1,2$, be four self mappings of X with each f_k continuous for each $k = 1,2$ and $g_k(X) \subset f_k(X)$. Let for some $x_0 \in X$, $\{y_n\}$ be a sequence defined by*

$$y_1 = f_1(x_1) = g_1(x_0), y_2 = f_2(x_2) = g_2(x_1), \ldots$$

and in general

$$y_{2n+1} = f_1(x_{2n+1}) = g_1(x_{2n}) \text{ and } y_{2n+2} = f_2(x_{2n+2}) = g_2(x_{2n+1}), n = 0,1,\ldots$$

Similarly, for some $u_0 \in X$, *define a sequence* $\{z_n\}$, *that is, for* $n = 0, 1, ...,$ $z_{2n+1} = f_1(u_{2n+1}) = g_1(u_{2n})$ *and* $z_{2n+2} = f_2(u_{2n+2}) = g_2(u_{2n+1})$. *For some* $c > 0$, *we define*

$$\varepsilon_{n+1} = sup\{d(y_{p+i}, z_{q+i}) : i \geq n, d(y_p, z_q) \leq c \text{ for some } p, q \in N\}.$$

If $m\varepsilon_n + \varepsilon_{m+1} \leq c$ *and* $d(f_k(x), g_l(x)) \leq c, k \neq l \ \ (k, l = 1, 2)$, *then for all* $i \geq n$ *and all* $j \in N$,

$$d(y_i, y_{i+j}) \leq m\varepsilon_n + \varepsilon_{m+1}.$$

Hence, if $d(y_n, z_n) \to 0$ *uniformly for all* $x_0, u_0 \in X$ *with* $d(y_p, z_q) \leq c$ *for some* $p, q \in N$, *then the sequence* $\{y_n\}$ *is uniformly Cauchy. Further if* g_1, g_2 *satisfy*

$$d(g_1(x), g_2(y)) \leq d(x, y) \ \ \forall\, x, y \in X$$

and $\{f_1, g_2\}, \{f_2, g_1\}$ *are weakly commutative pairs, then* $f_k, g_k, k = 1, 2$ *have a coincidence point. Moreover if*

$$d(f_k x, y) \leq d(x, y) \ \ \forall\, x \neq y \in X \text{ for } k = 1, 2,$$

then f_k, g_k *have a common fixed point in* X.

Proof. From Theorem 4 of Som and Mukherjee [3], we find the desired conclusion immediately.

Conflict of Interests

The authors declare that there is no conflict of interests.

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