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SOME NOTES ON FIXED POINT SETS IN $CAT(0)$ SPACES

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Abstract. In this paper we verify some of important relations between $\triangle$-convergent sequences, $\triangle$-closed sets, $\triangle$-closed fixed point sets of mappings on subset of a $CAT(0)$ space X. In the sequel, we obtain a topology on $\triangle$-closed fixed point sets in $CAT(0)$ space.

Keywords: CAT(0) space; $\triangle$-closed set; fixed point; topology.

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1. Introduction

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subseteq R$ to X such that $c(0)=x, c(l)=y$, and $d(c(t),c(t_0)) = |t-t_0|$ for all $t,t_0 \in [0,l]$. In particular, c is an isometry and $d(x,y)=l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A

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comparison triangle for a geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i,\bar{y}_j) = d(x_i,y_j)$ for $i,j \in \{1,2,3\}$.

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom: "Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x,y \in \triangle$ and all comparison points $\bar{x},\bar{y} \in \overline{\triangle}$, $d(x,y) \leq d_{\mathbb{E}^2}(\bar{x},\bar{y})$." Here we recall some useful lemmas which play an important role in this paper.

Lemma 1.1. [1] *Let (X,d) be a CAT(0) space. For $x,y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that*

$$d(x,z) = td(x,y), \quad d(y,z) = (1-t)d(x,y).$$

We use the notation $(1-t)x \oplus ty$ for the unique point z of the above lemma.

Lemma 1.2. [1] *Let (X,d) be a CAT(0) space. Then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z),$$

for $x,y,z \in X$ and $t \in [0,1]$.

Lemma 1.3. [1] *Let (X,d) be a CAT(0) space. Then*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2,$$

for all $x,y,z \in X$ and $t \in [0,1]$.

In particular by Lemma 1.3 we have

$$d(z, \frac{1}{2}x \oplus \frac{1}{2}y)^2 \leq \frac{1}{2}d(z,x)^2 + \frac{1}{2}d(z,y)^2 - \frac{1}{4}d(x,y)^2,$$

for all $x,y,z \in X$, which is called (CN) inequality of Bruhat-Tits, as it was shown in [2]. In fact (cf. [3], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality. Let $\{x_n\}$ be a bounded sequence in X and K be a nonempty bounded subset of X. We associate this sequence with the number

$$r = r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\},$$

where

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x),$$

and the set

$$A = A(K,\{x_n\}) = \{x \in K : r(x,\{x_n\}) = r\}.$$

The number r is known as the *asymptotic radius* of $\{x_n\}$ relative to K. Similarly, set A is called the *asymptotic center* of $\{x_n\}$ relative to K.

In the CAT(0) space, the asymptotic center $A = A(K,\{x_n\})$ of $\{x_n\}$ consists of exactly one point whenever K is closed and convex. A sequence $\{x_n\}$ in a CAT(0) space X said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. Notice that given $\{x_n\} \subset X$ such that $\{x_n\}$ is $\triangle$-convergent to x and given $y \in X$ with $x \neq y$,

$$\limsup_{n\to\infty} d(x_n,x) < \limsup_{n\to\infty} d(x_n,y).$$

So every CAT(0) space X satisfies the Opial property.

Lemma 1.4. [4] *Every bounded sequence in a complete CAT(0) space has a $\triangle$-convergent subsequence.*

Lemma 1.5. [5] *If K is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of is in K.*

Theorem 1.6. [6] *Let A be a nonempty subset of a $CAT(0)$ space (X,d). Then there exists a continuous map $T : X \to X$ such that $F(T) = \overline{A}$.*

2. Main results

Lemma 2.1. *Let A be a nonempty subset of a $CAT(0)$ space (X,d). Then there exists a continuous map $T : X \to X$ such that $F(T) = \overline{\overline{A}^{\triangle}}$.*

Proof. Replacing A by $\overline{A}^{\triangle}$ in the Theorem 1.6, we obtain the desired conclusion immediately.

Lemma 2.2. *If for each $\{x_n\} \subseteq A \subseteq X$ and $x \in A$ such that $x_n \to x$. Then $x_n \xrightarrow{\triangle} x$.*

Proof. Let $x_n \to x$, so sequence $\{x_n\}$ is bounded in a $CAT(0)$ space (X,d). Therefore $A(\{x_n\}) = \{x\}$, namely $r(x,\{x_n\}) = r(\{x_n\})$ and or

$$0 = \lim_{n\to\infty} d(x_n,x) = \limsup_{n\to\infty} d(x_n,x) = \inf_{y\in X} r(y,\{x_n\}),$$

so for some $y \in X$ we have $r(y,\{x_n\}) = 0$ thus $\limsup_{n\to\infty} d(x_n,y) = 0$.

Now let $\{u_n\}$ be a arbitrary subsequence of bounded sequence $\{x_n\}$ we show that $A(\{u_n\}) = \{x\}$. But

$$\begin{aligned}\limsup_{n\to\infty} d(u_n,x) &\leq \limsup_{n\to\infty}\Big(d(u_n,x_n)+d(x_n,x)\Big)\\ &\leq \limsup_{n\to\infty} d(u_n,x_n)+\limsup_{n\to\infty} d(x_n,x) = 0.\end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty} d(u_n,x) = 0 = \inf_{y\in X}\limsup_{n\to\infty} d(u_n,y)$$

or

$$r(x,\{u_n\}) = r(\{u_n\})$$

for every subsequence $\{u_n\}$ of $\{x_n\}$. So $x_n \xrightarrow{\triangle} x$. This completes the proof.

Lemma 2.3. *If $A = \overline{A}^{\triangle}$. Then $A = \overline{A}$.*

Proof. Suppose $\{x_n\}$ is a sequence in A and convergent to some $x \in X$. We show $x \in A$. In order to prove this, by Lemma 2.2, we have $x_n \xrightarrow{\triangle} x$, where $x_n \in A$ and $A = \overline{A}^{\triangle}$ so $x \in A$.

Example 2.4. *Let $\{e_n\}$ be a orthonormal base in Hilbert space H. For every $x \in H$, we have $\|x\|^2 = \sum|\langle e_n,x\rangle|^2$, so*

$$\forall x\in H \quad \langle e_n,x\rangle \to 0.$$

Since $H^* = H$ namely for every functional f on H, we have $f(e_n) \to 0$ as $n \to \infty$ so $e_n \xrightarrow{w.} 0$ and according to

$$e_n \xrightarrow{w.} 0 \iff e_n \xrightarrow{\triangle} 0,$$

we obtain $e_n \xrightarrow{\triangle} 0$, but $e_n \overset{\|\cdot\|}{\not\longrightarrow} 0$.

Corollary 2.5. *If A be a nonempty and convex subset of a $CAT(0)$ space (X,d). Then $\overline{A}^{\triangle} = \overline{A}$.*

Corollary 2.6. *If A be a nonempty, $\triangle$-closed and convex subset of a $CAT(0)$ space (X,d). Then $\overline{A} = A$, i.e. A is closed set.*

Proof. Using Corollary 2.5 and Lemma 2.3 we have $\overline{A}^{\triangle} = \overline{A}$ and $\overline{A}^{\triangle} = A$ respectively, so $A = \overline{A}$.

Lemma 2.7. *Let A be a nonempty and $\triangle$-closed subset of a $CAT(0)$ space (X,d). Then there exists a continuous map $T : X \to X$ such that $F(T) = A$.*

Proof. By Theorem 1.6 and Lemma 2.3, there exists a continuous map $T : X \to X$ such that $F(T) = \overline{A} = A$.

Lemma 2.8. *Let A be a nonempty and convex subset of a $CAT(0)$ space (X,d). Then there exists a continuous map $T : X \to X$ such that $F(T) = \overline{A}^{\triangle}$.*

Proof. By Theorem 1.6 and Corollary 2.5, there exists a continuous map $T : X \to X$ such that $F(T) = \overline{A} = \overline{A}^{\triangle}$.

Theorem 2.9. *Let A be a nonempty subset of a $CAT(0)$ space (X,d). Then there exists a quasi-nonexpansive map $T : X \to X$ such that if $x_0 \in A$ and $y \in F(T)$ we have $d(x_0,y) \leq d(x,y)$ for all $x \in X$, further $F(T) = \overline{\overline{A}^{\triangle}}$.*

Proof. For each $x \in X$, let $k_x = \frac{d(x,\overline{A}^{\triangle})}{1+d(x,\overline{A}^{\triangle})} \in [0,1]$. First, note that for each $x, \in X$,

$$|k_x - k_y| \leq |d(x,\overline{A}^{\triangle}) - d(y,\overline{A}^{\triangle})| \leq d(x,y).$$

Now, fix $x_0 \in A$ and define $T : X \to X$ by $T(x) = (1-k_x)x \oplus k_x x_0$ for all $x \in X$. To see that T is quasi-nonexpansive, we let $x \in X$. Then, for each $y \in F(T)$, we have

$$\begin{aligned} d(Tx,Ty) &= d(Tx,y) \\ &= d((1-k_x)x \oplus k_x x_0, y) \\ &\leq (1-k_x)d(x,y) + k_x d(x_0,y) \\ &\leq \max\{d(x,y), d(x_0,y)\} \\ &\leq d(x,y). \end{aligned}$$

Finally, it is easy to see that

$$T(x) = x \iff (1-k_x)x \oplus k_x x_0 = x \iff k_x = 0$$

$$\iff d(x,\overline{A}^{\triangle}) = 0 \iff x \in \overline{\overline{A}^{\triangle}} \iff F(T) = \overline{\overline{A}^{\triangle}}$$

as desired.

3. A Topology on $\triangle$-Closed Fixed Point Sets

Lemma 3.1. *Let (X,d) be a CAT(0) space, $F_\alpha := Fix(T_\alpha)$ and*

$$\Im := \{Fix(T) | T : X \to X,\ \emptyset \neq Fix(T) = \overline{Fix(T)}^{\triangle}\} \cup \{\emptyset, X\}.$$

Then

(1) If $F_\alpha \in \Im$ for every $\alpha \in I$, then $\bigcap_{\alpha \in I} F_\alpha \in \Im$.
(2) If $F_i \in \Im$ for $1 \leq i \leq n$, then $\bigcup_{i=1}^{n} F_i \in \Im$.

Proof. If $\bigcap_\alpha F_\alpha = \emptyset$, then $\bigcap_\alpha F_\alpha \in \Im$. Otherwise $\bigcap_{\alpha \in I} F_\alpha$ is nonempty and $\triangle$-closed so by Theorem 1.6 there exists continuous map $T : X \to X$ such that $Fix(T) = \overline{\overline{\bigcap_\alpha F_\alpha}^{\triangle}} = \overline{\bigcap_\alpha F_\alpha}^{\triangle} = \bigcap_\alpha F_\alpha$ by Lemma 2.3, so $\bigcap_\alpha F_\alpha \in \Im$. This completes (1).

Lemma 3.2. *With assumptions of Lemma 3.1, if $\Im_0 := \{F | F^c \in \Im\}$, Then $\Im_0$ is a topology on X.*

Proof. By Lemma 3.1, $\Im$ is a topology on X.

Corollary 3.3. [7] *Let (X,d) be a CAT(0) space, $F_\alpha := Fix(T_\alpha)$ and*

$$\Im := \{Fix(T) | T : X \to X,\ \emptyset \neq Fix(T) = \overline{Fix(T)}\} \cup \{\emptyset, X\}.$$

Then

(1) If $F_\alpha \in \Im$ for every $\alpha \in I$, then $\bigcap_{\alpha \in I} F_\alpha \in \Im$.
(2) If $F_i \in \Im$ for $1 \leq i \leq n$, then $\bigcup_{i=1}^{n} F_i \in \Im$.
(3) If $\Im_0 := \{F | F^c \in \Im\}$, Then $\Im_0$ is a topology on X.

Proof. By Lemma 2.3, Corollary is clear.

Conflict of Interests

The author declares that there is no conflict of interests.

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