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COUPLED COMMON FIXED POINT THEOREMS FOR (ψ, ϕ)-CONTRACTIVE MIXED MONOTONE MAPPINGS IN PARTIALLY ORDERED G-METRIC SPACES

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Abstract. In this paper, we establish some coupled common fixed point results on a generalized complete metric spaces (X,G). These results extend and generalize well-known comparable results in the literature.

Keywords: fixed point, coincidence point, partially G-metric spaces, contraction.

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1. Introduction

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity; see, [1-20]. The notion of D-metric space is a generalization of usual metric spaces and it is introduced by Dhage [13] and [14]. Recently, Mustafa and Sims [15-17] have shown that most of the results concerning Dhage's D-metric spaces are invalid. In [16] and [17], they introduced a improved version of the generalized metric space structure which they called G-metric spaces. For more results on G-metric spaces, one can refer to the articles [18-20]. Subsequently, several authors proved fixed point results in these spaces. Some

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of them have been applied to solve matrix equations, ordinary differential equations and integral equations.

2. Preliminaries

Definition 2.1. [22] Let X be a non-empty set and let $G : X \times X \times X \to R_+$ be a function satisfying the following properties:

(G1) $G(x,y,z) = 0$ if $x = y = z$.

(G2) $0 < G(x,x,y)$ for all $x;y \in X$ with $x \neq y$.

(G3) $G(x,x,y) \leq G(x,y,z)$ forall $x,y,z \in X$ with $y \neq z$.

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x)$.

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$.

(G6) $G(x,y,y) \leq 2G(y,x,x)$ for all $x;y \in X$.

Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 2.2. [22]. Let (X,G) be a G-metric space, and let (x_n) be a sequence of points of X. We say that (x_n) is $G-$convergent to $x \in X$ if $\lim_{n,m\to\infty} G(x;x_n,x_m) = 0$, that is, for any ε > 0, there exists $N \in \mathbb{N}$ such that $G(x;x_n,x_m) < \varepsilon$, for all $n,m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n,m\to\infty} x_n = x$.

Proposition 2.3. [22]. Let (X,G) be a G-metric space. The following are equivalent:

(1) (x_n) is G-convergent to x.

(2) $G(x_n,x_n,x) \to 0$ as $n \to \infty$.

(3) $G(x_n,x,x) \to 0$ as $n \to \infty$.

(4) $G(x_n,x_m,x) \to 0$ as $n,m \to \infty$.

Definition 2.4. [22]. Let (X,G) be a G-metric space. A sequence (x_n) is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $m,n,l \geq N$, that is $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

Proposition 2.5. [22]. Let (X,G) be a G-metric space. Then the following are equivalent

(1) The sequence (x_n) is G-Cauchy

(2) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n; m \geq N$.

Proposition 2.6. [22]. Let (X, G) be a G-metric space. A mapping $f : X \to X$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever (x_n) is G-convergent to x, $f(x_n)$ is G-convergent to $f(x)$.

Proposition 2.7. [10]. Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous all three of its variables.

Definition 2.8. [10]. A G-metric space (X, G) is called $G-$ complete if every G-Cauchy sequence is G-convergent in (X, G).

Definition 2.9. [22]. Two mappings $f, g : X \to X$ are weakly compatible if they commute at their coincidence points, that is $ft = gt$ for some $t \in X$ implies that $fgt = gft$.

Definition 2.10. [22]. Suppose $(X, \preceq)$ is a partially ordered set and $f, g : X \to X$ are mappings. f is said to be g-Nondecreasing if for $x, y \in X$, $gx \preceq gy$ implies $fx \preceq fy$.

Definition 2.11. [21,22]. An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 2.12. [21,22]. An element $(x, y) \in X \times X$ is called:

(C1) A coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and (gx, gy) is called coupled point of coincidence.

(C2) A common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Definition 2.13. [21,22]. Let $(X, \leq)$ be a partially ordered set. A map $F : X \times X \to X$ is said to has the g-mixed monotone property where $g : X \to X$ if for $x_1, x_2, y_1, y_2 \in X$, $gx_1 \leq gx_2$ implies $F(x_1, y) \leq F(x_2, y)$ for all $y \in X$ and $gy_1 \leq gy_2$ implies $F(x, y_2) \leq F(x, y_1)$ for all $x \in X$.

Definition 2.14. [21,22]. Let X be a nonempty set. Mappings $F : X \times X \to X$ and $g : X \to X$ are said to be commutative if $g(F(x, y)) = F(gx, gy)$ for all $x, y \in X$.

Now, we are ready to state our results.

Let Φ denotes the class of the function $\varphi : [0,+\infty[\to [0,+\infty[$ which satisfies the following conditions:

(1) φ is nondecreasing and continuous.

(2) $\varphi(t+s) \leq \varphi(t)+\varphi(s)$, for all $t,s \in [0,+\infty[$.

(3) $\varphi(t) = 0 \iff t = 0$.

The elements of Φ are called altering distance functions.

Let Ψ denotes the class of the function $\psi : [0,+\infty[\to [0,+\infty[$, which satisfies the following conditions:

$$\lim_{t\to r}\psi(t) > 0, \quad \forall r > 0, \quad \lim_{t\to 0}\psi(t) = 0.$$

Let $(X,\leq)$ be a partially ordered set and endow the product space $X\times X$ with the following partial order: For (x,y), $(u,v) \in X\times X$, $(u,v) \leq (x,y) \iff x \geq u$ and $y \leq v$.

3. Main results

Theorem 3.1. *Let $(X,\leq)$ be a partially ordered set and suppose that there is a G-metric on X such that (X,G) is a complete G-metric space. Let $F : X\times X \to X$ and g: $X \to X$ be two mappings such that F has the mixed g-monotone property for which there exist $\psi \in \Psi$ and $\varphi \in \Phi$, such that*

$$\varphi\left(\alpha G(F(x,y),F(u,v),F(z,w))+\beta G(F(y,x),F(v,u),F(w,z))\right)$$
$$\leq \varphi\left(\alpha G(gx,gu,gz)+\beta G(gy,gv,gw)\right)-\psi\left(\alpha G(gx,gu,gz)+\beta G(gy,gv,gw)\right)$$

for all $x,y,z,u,v,w \in X$ with $gx \geq gu \geq gw$ and $gy \leq gv \leq gz$ with $\alpha,\beta \in \mathbb{R}_+^*$. We suppose $F(X\times X)$ is contained in a closed subspace $g(X)$ and g is $G-$continuous, injective and commutes with F and we suppose either

(a) *F is continuous*

(b) *for a nondecreasing sequence $\{x_n\}$ with $x_n \to x$, we have $x_n \leq x$ for all n;*

(c) *for a nonincreasing sequence $\{y_n\}$ with $y_n \to x$, we have $y \leq y_n$ for all n.*

Then F and g have a coupled common fixed point provided that there exist x_0, $y_0 \in X$ such that $gx_0 \leq F(x_0,y_0)$ and $F(y_0,x_0) \leq gy_0$ or $gx_0 \geq F(x_0,y_0)$ and $F(y_0,x_0) \geq gy_0$.

Proof. Consider the functional $G_{\alpha,\beta} : X^2 \times X^2 \times X^2 \to \mathbb{R}_+$ defined by

$$\begin{aligned} G_{\alpha,\beta}(X,Y,Z) &= \alpha G(x,u,z) + \beta G(y,v,w), \text{ for all } X = (x,y) \in X^2, \\ Y &= (u,v) \in X^2, \ Z = (z,w) \ \in X^2. \end{aligned}$$

It is easy to see that $G_{\alpha,\beta}$ is a G-metric space on X^2 and moreover, if (X,G) is a complete space, then $(X^2, G_{\alpha,\beta})$ is a complete metric space, too. Now consider the operator $T : X^2 \to X^2$ defined by

$$T(X) = (F(x,y), F(y,x)) \text{ for all } X = (x,y) \in X^2.$$

Clearly, for $X = (x,y)$,$Y = (u,v)$, $Z = (z,w)$. In view of the definition of $G_{\alpha,\beta}$, we have

$$G_{\alpha,\beta}(T(X),T(Y),T(Z)) = \alpha G(F(x,y),F(u,v),F(z,w)) + \beta G(F(y,x),F(v,u),F(w,z)).$$

Thus, by the contractive condition, we obtain that F satisfies the following (φ,ψ)-contractive condition:

$$\varphi\left(G_{\alpha,\beta}(T(X),T(Y),T(Z))\right) \leq \varphi\left(G_{\alpha,\beta}(gX,gY,gZ)\right) - \psi\left(G_{\alpha,\beta}(gX,gY,gZ)\right) \tag{3.1}$$

for all $gX \geq gY \geq gZ$ and $gX, gY, gZ \in X^2$. Assume that there exist $x_0,\ y_0 \in X$ such that $gx_0 \leq F(x_0,y_0)$ and $F(y_0,x_0) \leq gy_0$. Denote $gX_0 = (gx_0, gy_0) \in X^2$ and consider the Picard iteration associated to T with the initial value gX_0, that is the sequence $\{gX_n\} \subset X^2$, defined by

$$gX_{n+1} = TgX_n, \quad \forall n \geq 0. \tag{3.2}$$

with $gX_n = (gx_n, gy_n) \in X^2$, $n \geq 0$. Since F is mixed monotone, we have

$$gX_0 = (gx_0, gy_0) \leq (F(x_0,y_0), F(y_0,x_0)) = (gx_1, gy_1) = gX_1.$$

By induction, we have

$$gX_n = (gx_n, gy_n) \leq (F(x_n,y_n), F(y_n,x_n)) = (gx_{n+1}, gy_{n+1}) = gX_{n+1},$$

which shows that the mapping T is monotone and the sequence $\{X_n\}$ is nondecreasing. Take $X = X_n$ and $Y = Z = X_{n+1}$ in (3.1), we obtain

$$\begin{aligned} \varphi\left(G_{\alpha,\beta}(T(gX_n),T(gX_{n+1}),T(gX_{n+1}))\right) &\leq \varphi\left(G_{\alpha,\beta}(gX_n,gX_{n+1},gX_{n+1})\right) \\ &\quad - \psi\left(G_{\alpha,\beta}(gX_n,gX_{n+1},gX_{n+1})\right) \end{aligned} \tag{3.3}$$

with $X = gX_n \geq Y = Z = gX_{n+1}$. Since $\psi \geq 0$, (3.3) implies that

$$\varphi\left(G_{\alpha,\beta}(gX_{n+1}, gX_{n+2}, gX_{n+2})\right) \leq \varphi\left(G_{\alpha,\beta}\left(gX_n, gX_{n+1}, gX_{n+1}\right)\right), \quad \forall n \geq 0.$$

So, by the property of monotonocity of φ, we have

$$G_{\alpha,\beta}(X_{n+1}, X_{n+2}, X_{n+2}) \leq G_{\alpha,\beta}\left(X_n, X_{n+1}, X_{n+1}\right), \quad \forall n \geq 0. \tag{3.4}$$

This shows that the sequence $\left\{\delta^n_{\alpha,\beta} = G_{\alpha,\beta}\left(gX_n, gX_{n+1}, gX_{n+1}\right)\right\}$, $n \geq 0$, is nondecreasing. Therefore, there exists $\delta_{\alpha,\beta} \geq 0$ such that

$$\lim_{n\to\infty} \delta^n_{\alpha,\beta} = \alpha G(gx_n, gx_{n+1}, gx_{n+1}) + \beta G(gy_n, gy_{n+1}, gy_{n+1}) = \delta_{\alpha,\beta}. \tag{3.5}$$

We shall prove that $\delta_{\alpha,\beta} = 0$. Assume that we have the contrary, that is $\delta_{\alpha,\beta} > 0$. Then by letting $n \to \infty$ in (3.3), we have

$$\begin{aligned}\varphi\left(\delta_{\alpha,\beta}\right) &= \lim_{n\to\infty} \varphi\left(\delta^n_{\alpha,\beta}\right) \leq \lim_{n\to\infty} \varphi\left(\delta^n_{\alpha,\beta}\right) - \lim_{n\to\infty} \psi\left(\delta^n_{\alpha,\beta}\right) \\ &= \varphi\left(\delta_{\alpha,\beta}\right) - \lim_{\delta^n_{\alpha,\beta}\to\delta^+_{\alpha,\beta}} \psi\left(\delta^n_{\alpha,\beta}\right) < \varphi\left(\delta_{\alpha,\beta}\right),\end{aligned}$$

which is a contradiction. Thus $\delta_{\alpha,\beta} = 0$ and hence

$$\lim_{n\to\infty} \delta^n_{\alpha,\beta} = \alpha G(gx_n, gx_{n+1}, gx_{n+1}) + \beta G(gy_n, gy_{n+1}, gy_{n+1}) = 0. \tag{3.6}$$

Now we prove that $\{gX_n\}$ is a Cauchy sequence in $\left(X^2, G_{\alpha,\beta}\right)$ that is $\{gx_n\}, \{gy_n\}$ are Cauchy sequence in (X, G). Suppose that the contrary, that is at least one of the sequences $\{gx_n\}, \{gy_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we find subsequences $\{gx_{n_k}\}, \{gx_{m_k}\}$ of $\{gx_n\}$ and $\{gy_{n_k}\}, \{gy_{m_k}\}$ of $\{gy_n\}$ with $n_k \geq m_k \geq k$ such that

$$\alpha G(gx_{m_k}, gx_{n_k}, gx_{n_k}) + \beta G(gy_{m_k}, gy_{n_k}, gy_{n_k}) \geq \varepsilon. \tag{3.7}$$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ which satisfy (3.6). Then

$$\alpha G(gx_{m_k}, gx_{n_{k-1}}, gx_{n_{k-1}}) + \beta G(gy_{m_k}, g_{y_{n_{k-1}}}, gy_{n_{k-1}}) < \varepsilon. \tag{3.8}$$

By using the rectangle inequality of generalized metric and (3.8) we have

$$\begin{aligned}
\varepsilon &\leq \alpha G(gx_{m_k}, gx_{n_k}, gx_{n_k}) + \beta G(gy_{m_k}, gy_{n_k}, gy_{n_k}) \\
&\leq \alpha G(gx_{m_k}, gx_{n_k-1}, gx_{n_k-1}) + \beta G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}) \\
&\quad + \alpha G(gy_{m_k}, gy_{n_{k-1}}, gy_{n_k-1}) + \beta G(gy_{n_k-1}, gy_{n_k}, gy_{n_k)} \\
&\leq \varepsilon + \alpha G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}) + \beta G(gy_{n_k-1}, gy_{n_k}, gy_{n_k)}.
\end{aligned}$$

Letting $k \to \infty$ and using (3.6), we have

$$\lim_{k\to\infty} r_k^{\alpha,\beta} := \alpha G(gx_{m_k}, gx_{n_k}, gx_{n_k}) + \beta G(gy_{m_k}, gy_{n_k}, gy_{n_k} = \varepsilon. \tag{3.9}$$

By using the rectangular inequality and the property (G_6), we have

$$\begin{aligned}
G(gx_{m_k}, gx_{n_k}, gx_{n_k}) &\leq G(gx_{m_k}, gx_{n_k}, gx_{n_{k+1}}) + G(gx_{m_k}, gx_{n_k+1}, gx_{n_k+1}) \\
&\leq 2G(gx_{n_k+1}, gx_{n_k+1}, gx_{n_k}) + G(gx_{n_k+1}, gx_{n_k+1}, gx_{m_k+1}) \\
&\quad + G(gx_{m_k+1}, gx_{m_k+1}, gx_{m_k})
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
G(gy_{m_k}, gy_{n_k}, gy_{n_k}) &\leq G(gy_{m_k}, gy_{n_k}, gy_{n_{k+1}}) + G(gy_{m_k}, gy_{n_k+1}, gy_{n_k+1}) \\
&\leq 2G(gy_{n_k+1}, gy_{n_k+1}, gy_{n_k}) + G(gy_{n_k+1}, gy_{n_k+1}, gy_{m_k+1}) \\
&\quad + G(gy_{m_k+1}, gy_{m_k+1}, gy_{m_k}).
\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we have

$$\begin{aligned}
&\alpha G(gx_{m_k}, gx_{n_k}, gx_{n_k}) + \beta G(gy_{m_k}, gy_{n_k}, gy_{n_k}) \\
&\leq \left(2\delta_{\alpha,\beta}^{n_k} + \delta_{\alpha,\beta}^{m_k} + \alpha G(gx_{n_k+1}, gx_{n_k+1}, gx_{m_k+1}) + \beta G(gy_{n_k+1}, gy_{n_k+1}, gy_{m_k+1})\right).
\end{aligned} \tag{3.12}$$

Since $n_k \geq m_k$, $gx_{n_k} \geq gx_{m_k}$ and $gy_{n_k} \leq gy_{m_k}$ and hence, we find from $x = x_{n_k}, y = y_{n_k}, u = x_{m_k}, v = y_{m_k}, z = y_{n_{k+1}}, w = y_{m_{k+1}}$ that

$$\begin{aligned}
&\varphi\left(\alpha G(gx_{n_{k+1}}, gx_{n_{k+1}}, gx_{m_{k+1}}) + \beta G(gy_{n_{k+1}}, gy_{n_{k+1}}, gy_{m_{k+1}})\right) \\
= \;& \varphi\begin{pmatrix} \alpha G(F(x_{n_k}y_{n_k}), F(x_{n_k}y_{n_k}), F(x_{m_k}, y_{m_k})) \\ +\beta G(F(y_{n_k}, x_{n_k}), F(y_{n_k}, x_{n_k}), F(y_{m_k}, x_{m_k})) \end{pmatrix} \\
\leq \;& \varphi\left(\alpha G(gx_{n_k}, gx_{n_k}, gx_{m_k}) + \beta G(gy_{n_k}, gy_{n_k}, gy_{m_k})\right) \\
& -\psi\left(\alpha G(gx_{n_k}, gx_{n_k}, gx_{m_k}) + \beta G(gy_{n_k}, gy_{n_k}, gy_{m_k})\right) \\
= \;& \varphi\left(r_k^{\alpha,\beta}\right) - \psi\left(r_k^{\alpha,\beta}\right).
\end{aligned}$$

Therefore, we have

$$\varphi\left(\alpha G(gx_{n_{k+1}}, gx_{n_{k+1}}, gx_{m_{k+1}}) + \beta G(gy_{n_{k+1}}, gy_{n_{k+1}}, gy_{m_{k+1}})\right) \leq \varphi\left(r_k^{\alpha,\beta}\right) - \psi\left(r_k^{\alpha,\beta}\right). \quad (3.13)$$

On the other hand, by (3.12) and using the property of φ, we get

$$\varphi\left(r_k^{\alpha,\beta}\right) \leq \varphi\left(2\delta_{\alpha,\beta}^{n_k} + \delta_{\alpha,\beta}^{m_k}\right) + \varphi\left(r_k^{\alpha,\beta}\right) - \psi\left(r_k^{\alpha,\beta}\right). \quad (3.14)$$

Let $k \to \infty$ in (3.14). Using (3.6), (3.9) and the properties of φ, we get

$$\varphi(\varepsilon) \leq \varphi(0) + \varphi(\varepsilon) - \lim_{k\to\infty} \psi\left(r_k^{\alpha,\beta}\right) = \varphi(\varepsilon) - \lim_{r_k \to \varepsilon^+} \psi\left(r_k^{\alpha,\beta}\right) < \varphi(\varepsilon),$$

which is a contradiction. This shows that $\{gx_n\}$ and $\{gy_n\}$ are G-Cauchy sequences in complete subspace $g(X)$. And this implies that there exist a, b in X such that

$$ga = \lim_{n\to\infty} gx_{n+1} \text{ and } gb = \lim_{n\to\infty} gy_{n+1}.$$

Now, let us suppose that F is continuous. Then

$$ga = \lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(gx_n, gy_n) = F(ga, gb)$$

and

$$gb = \lim_{n\to\infty} gy_{n+1} = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(gy_n, gx_n) = F(gb, ga).$$

Suppose now the assumption (ba) holds. Since $\{gx_n\}_{n\geq 0}$ is a nondecreasing sequence that converges to ga, we have $gx_n \leq ga$ for all $n \geq 0$. Similarly, $gy_n \geq gb$ for all $n \geq 0$. Then

$$\begin{aligned} G(ga,ga,F(ga,gb)) &\leq G(ga,ga,gx_{n+1})) + G(gx_{n+1},ga,F(ga,gb)) \\ &= G(ga,ga,gx_{n+1})) + G(F(gx_n,gy_n),F(ga,gb),F(ga,gb)). \end{aligned}$$

So, $G(ga,ga,F(ga,gb)) - G(ga,ga,gx_{n+1})) \leq G(F(gx_n,gy_n),F(ga,gb),F(ga,gb))$ and

$$G(gb,gb,F(gb,ga)) - G(gb,gb,gy_{n+1})) \leq G(F(gy_n,gx_n),F(gb,ga),F(gb,ga)).$$

Hence, we have

$$\begin{aligned} &\alpha G(ga,ga,F(ga,gb)) - \alpha G(ga,ga,gx_{n+1})) \\ &+\beta G(gb,gb,F(gb,ga)) - \beta G(gb,gb,gy_{n+1})) \\ \leq\ &\alpha G(F(gx_n,gy_n),F(ga,gb),F(ga,gb)) + \beta G(F(gy_n,gx_n),F(gb,ga),F(gb,ga)), \end{aligned}$$

which implies by monotonoticity of φ

$$\begin{aligned} &\varphi\begin{pmatrix} \alpha G(ga,ga,F(ga,gb)) - \alpha G(ga,ga,gx_{n+1})) \\ +\beta G(gb,gb,F(gb,ga)) - \beta G(gb,gb,gy_{n+1})) \end{pmatrix} \\ \leq\ &\varphi\left(\alpha G(F(gx_n,gy_n),F(ga,gb),F(ga,gb)) + \beta G(F(gy_n,gx_n),F(gb,ga),F(gb,ga))\right) \\ \leq\ &\varphi\left(\alpha G(gx_n,ga,ga) + \beta G(gy_n,gb,gb)\right) - \psi\left(\alpha G(gx_n,ga,ga) + \beta G(gy_n,gb,gb)\right). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned} &\varphi\begin{pmatrix} \alpha G(F(gx_n,gy_n),F(ga,gb),F(ga,gb)) \\ +\beta G(F(gy_n,gx_n),F(gb,ga),F(gb,ga)) \end{pmatrix} \\ \leq\ &\varphi(0) - 0 = 0, \end{aligned}$$

which implies by the properties of φ that $G(ga,ga,F(ga,gb)) = 0$ and $G(gb,gb,F(gb,ga))$. Hence $ga = F(ga,gb)$ and $gb = F(gb,ga)$.

Corollary 3.2. *Let $(X,\leq)$ be a partially ordered set and suppose that there is a G-metric on X such that (X,G) is a complete G-metric space. Let $F: X\times X \to X$ and g: $X \to X$ be two* *mappings such that F has the mixed g-monotone property for which there exist $\psi \in \Psi$ and $\varphi \in \Phi$, such that*

$$\varphi\left(\frac{G(F(x,y),F(u,v),F(z,w))+G(F(y,x),F(v,u),F(w,z))}{2}\right)$$
$$\leq \varphi\left(\frac{G(gx,gu,gz)+G(gy,gv,gw)}{2}\right)-\psi\left(\frac{G(gx,gu,gz)+G(gy,gv,gw)}{2}\right)$$

for all $x,y,z,u,v,w \in X$ with $gx \geq gu \geq gw$ and $gy \leq gv \leq gz$. We suppose $F(X\times X)$ is contained in a closed subspace $g(X)$ and g is $G-$continuous, injective map which commutes with F and we suppose either

(a) *F is continuous*

(b) *for a nondecreasing sequence $\{x_n\}$ with $x_n \to x$, we have $x_n \leq x$ for all n;*

(c) *for a nonincreasing sequence $\{y_n\}$ with $y_n \to x$, we have $y \leq y_n$ for all n.*

Then F and g have a coupled common fixed point provided that there exist x_0, $y_0 \in X$ such that $gx_0 \leq F(x_0,y_0)$ and $F(y_0,x_0) \leq gy_0$ or $gx_0 \geq F(x_0,y_0)$ and $F(y_0,x_0) \geq gy_0$.

Proof. Putting $\alpha=\beta=\frac{1}{2}$ in Theorem 3.1, we can conclude the desired conclusion immediately.

Corollary 3.3. *Let $(X,\leq)$ be a partially ordered set and suppose that there is a G-metric on X such that (X,G) is a complete $G-$metric space. Let $F: X\times X \to X$ and g: $X \to X$ be two mappings such that F has the mixed g-monotone property for which there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\begin{aligned} & G(F(x,y),F(u,v),F(z,w))+G(F(y,x),F(v,u),F(w,z)) \\ \leq\ & G(gx,gu,gz)+G(gy,gv,gw)-\psi(G(gx,gu,gz)+G(gy,gv,gw)) \end{aligned}$$

for all $x,y,z,u,v,w \in X$ with $gx \geq gu \geq gw$ and $gy \leq gv \leq gz$ with $\alpha,\beta \in \mathbb{R}_+^*$. We suppose $F(X\times X)$ is contained in a closed subspace $g(X)$. and g is a $G-$continuous, injective maps which commutes with $F$and we suppose either

(a) *F is continuous*

(b) *for a nondecreasing sequence $\{x_n\}$ with $x_n \to x$, we have $x_n \leq x$ for all n;*

(c) *for a nonincreasing sequence $\{y_n\}$ with $y_n \to x$, we have $y \leq y_n$ for all n.*

Then F and g have a coupled common fixed point provided that there exist x_0, $y_0 \in X$ such that $gx_0 \leq F(x_0,y_0)$ and $F(y_0,x_0) \leq gy_0$ or $gx_0 \geq F(x_0,y_0)$ and $F(y_0,x_0) \geq gy_0$.

Proof. Putting $\varphi = id_X$ and $\alpha = \beta = 1$ in Theorem 3.1, we can conclude the desired conclusion immediately.

Example 3.4. Let $X = \mathbb{R}$, $G(x,y,z) = |x-y| + |y-z| + |x-z|$, $F(x,y) = \frac{x-2y}{4}$, $\varphi(t) = \frac{1}{2}t$, $\psi(t) = \frac{5}{16}t$, $g(x) = 2x$
$\alpha = \beta = \frac{1}{2}$, g commutes with F. F has the g-mixed monotone property

$$\begin{aligned}
G(F(x,y),F(u,v),F(z,w)) &= G(\frac{x-2y}{4},\frac{u-2v}{4},\frac{z-2w}{4}) \\
&= \left|\frac{x-2y}{4} - \frac{u-2v}{4}\right| + \left|\frac{u-2z}{4} - \frac{v-2w}{4}\right| \\
&\quad + \left|\frac{x-2z}{4} - \frac{y-2w}{4}\right| \\
&\leq \frac{1}{4}|x-u| + \frac{1}{2}|y-v| + \frac{1}{4}|u-z| + \frac{1}{2}|v-w| \\
&\quad \frac{1}{4}|z-x| + \frac{1}{2}|w-y| \ + \frac{1}{4}|x-u| + \frac{1}{2}|y-v| \\
&\quad + \frac{1}{4}|u-z| + \frac{1}{2}|v-w| \\
&\leq \frac{1}{8}G(gx,gu,gz) + \frac{1}{4}G(gy,gv,gw), \quad \forall gx \geq gu \geq gz
\end{aligned}$$

and

$$G(F(y,x),F(v,u),F(w,z)) \leq \frac{1}{4}G(gx,gu,gz) + \frac{1}{8}G(gy,gv,gw), \quad \forall gy \leq gv \leq gw.$$

Hence, we have

$$\begin{aligned}
&\varphi\left(\frac{G(F(x,y),F(u,v),F(z,w)) + G(F(y,x),F(v,u),F(w,z))}{2}\right) \\
\leq\ & \frac{3}{16}\left(\frac{G(gx,gu,gz) + G(gy,gv,gw}{2})\right) \\
& \frac{1}{2}\left(\frac{G(gx,gu,gz) + G(gy,gv,gw)}{2}\right) - \frac{5}{16}\left(\frac{G(gx,gu,gz) + G(gy,gv,gw)}{2}\right) \\
& \varphi\left(\frac{G(gx,gu,gz) + G(gy,gv,gw)}{2}\right) - \psi\left(\frac{G(gx,gu,gz) + G(gy,gv,gw)}{2}\right).
\end{aligned}$$

We choose $x_0 = -2 \leq F(-2,3)$ and $3 \geq F(3,-2)$. So by Corollary 3.2, we obtain that F and g have $(0,0)$ as coincidence point.

From the previous obtained results, we deduce some coincidence point results for mappings satisfying a contraction of an integral type. For this purpose, let

$$Y = \left\{ \begin{array}{c} \chi, \chi : \mathbb{R}^+ \to \mathbb{R}^+, \text{ satisfies that } \chi \text{ is Lesbesgue integrable,} \\ \text{summable on each compact of subset of } \mathbb{R}^+\text{, subadditive} \\ \text{and } \int_0^{\varepsilon} \chi(t)\,dt > 0 \text{ for each } \varepsilon > 0 \end{array} \right\}.$$

Theorem 3.4. *Let $(X,\leq)$ be a partially ordered set and suppose that there is a G-metric on X such that (X,G) is a complete G-metric space. Let $F : X \times X \to X$ and g: $X \to X$ be two mappings such that F has the mixed g-monotone property for which there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\int_0^{\varphi(\alpha G(F(x,y),F(u,v),F(z,w))+\beta G(F(y,x),F(v,u),F(w,z)))} \chi(t)\,dt$$
$$\leq \int_0^{\varphi(\alpha G(gx,gu,gz)+\beta G(gy,gv,gw))} \chi(t)\,dt - \int_0^{\psi(\alpha G(gx,gu,gz)+\beta G(gy,gv,gw))} \chi(t)\,dt, \quad \forall \chi \in Y \tag{3.15}$$

for all $x,y,z,u,v,w \in X$ with $gx \geq gu \geq gw$ and $gy \leq gv \leq gz$ with $\alpha, \beta \in \mathbb{R}_+^*$. We suppose $F(X \times X)$ is contained in a closed subspace $g(X)$ and g is a G-continuous, injective map which commutes with F and we suppose either

(a) *F is continuous*

(b) *for a nondecreasing sequence $\{x_n\}$ with $x_n \to x$, we have $x_n \leq x$ for all n;*

(c) *for a nonincreasing sequence $\{y_n\}$ with $y_n \to x$, we have $y \leq y_n$ for all n.*

Then F and g have a coupled common fixed point provided that there exist x_0, $y_0 \in X$ such that $gx_0 \leq F(x_0,y_0)$ and $F(y_0,x_0) \leq gy_0$ or $gx_0 \geq F(x_0,y_0)$ and $F(y_0,x_0) \geq gy_0$.

Proof. For $\chi \in Y$, consider the function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\Lambda(x) = \int_0^x \chi(t)\,dt$. We note that $\Lambda \in \Psi$. Thus the inequality (31.5) becomes

$$\Lambda(\varphi(\alpha G(F(x,y),F(u,v),F(z,w)) + \beta G(F(y,x),F(v,u),F(w,z)))$$
$$\leq \Lambda(\varphi(\alpha G(gx,gu,gz) + \beta G(gy,gv,gw))) - \Lambda(\psi(\alpha G(gx,gu,gz) + \beta G(gy,gv,gw))). \tag{3.16}$$

Setting $\Lambda\circ\psi=\psi_1$, $\psi_1\in\Psi$ and $\Lambda\circ\varphi=\varphi_1$, $\varphi_1\in\Phi$, we obtain

$$\begin{aligned}&\varphi_1\left(\alpha G(F(x,y),F(u,v),F(z,w))+\beta G(F(y,x),F(v,u),F(w,z))\right)\\ \leq\;&\varphi_1\left(\alpha G(gx,gu,gz)+\beta G(gy,gv,gw)\right)-\psi_1\left(\alpha G(gx,gu,gz)+\beta G(gy,gv,gw)\right).\end{aligned}$$

Using Theorem 3.1, we see that F and g have a coupled common fixed point.

Corollary 3.5. *Let $(X,\leq)$ be a partially ordered set and suppose that there is a $G-$metric on X such that (X,G) is a complete G-metric space. Let $F:X\times X\to X$ and g: $X\to X$ be two mappings such that F has the mixed g-monotone property for which there exist $\psi\in\Psi$ and $\varphi\in\Phi$ such that*

$$\int_0^{\varphi\left(\frac{G(F(x,y),F(u,v),F(z,w))+G(F(y,x),F(v,u),F(w,z))}{2}\right)}\chi(t)\,dt\leq\int_0^{\varphi\left(\frac{G(gx,gu,gz)+G(gy,gv,gw)}{2}\right)}\chi(t)\,dt-\int_0^{\psi\left(\frac{G(gx,gu,gz)+G(gy,gv,gw)}{2}\right)}\chi(t)\,dt.$$

for all $\chi\in Y$ and $x,y,z,u,v,w\in X$ with $gx\geq gu\geq gw$ and $gy\leq gv\leq gz$ with $\alpha,\beta\in\mathbb{R}_+^*$. We suppose $F(X\times X)$ is contained in a closed subspace $g(X)$and g is a $G-$continuous, injective map which commutes with $F$and we suppose either

(a) *F is continuous*

(b) *for a nondecreasing sequence $\{x_n\}$ with $x_n\to x$, we have $x_n\leq x$ for all n;*

(c) *for a nonincreasing sequence $\{y_n\}$ with $y_n\to x$, we have $y\leq y_n$ for all n.*

Then F and g have a coupled common fixed point provided that there exist x_0, $y_0\in X$ such that $gx_0\leq F(x_0,y_0)$ and $F(y_0,x_0)\leq gy_0$ or $gx_0\geq F(x_0,y_0)$ and $F(y_0,x_0)\geq gy_0$.

Remark 3.6. Let us note that, since the contractivity condition in Theorem 3.1 is valid only for comparable elements of X^2, Theorem 3.1 cannot guarantee in general the uniqueness of the coupled fixed point.

Let us add hypothesis of Theorem 3.1, the following condition. Every pair of elements in X^2 has either a lower bound or an upper bound, which is known to be equivalent to the following condition: For all $Y=(x,y),A=(a,b)\in X^2,\exists Z=(z_1,z_2)\in X^2$, that is comparable to Y and A.

Theorem 3.7 *In addition to the hypotheses of Theorem 15, suppose that the above condition holds. Then F has a unique coincidence point.*

Proof. From Theorem 3.1, we have the set of coupled fixed point of F is nonempty. Assume that $A_1 = (a_1, a_2) \in X^2$ and $B = (b_1, b_2) \in X^2$ are two coupled coincidence points of F. We shall prove that $A = B$. By the above assumption, there exists $(u, v) \in X^2$ that is comparable to (a_1, a_2) and (b_1, b_2). We define the sequence $\{u_n\}$, $\{v_n\}$ as follows:

$$u_0 = u,\, v_0 = v,\, gu_{n+1} = F(u_n, v_n),\, gv_{n+1} = F(v_n, u_n),\;\, n \geq 0.$$

Since (u, v) is comparable to (b_1, b_2), we may assume $(b_1, b_2) \geq (u, v) = (u_0, v_0)$. Using Theorem 3.1, we obtain inductively

$$(b_1, b_2) \geq (gu_n, gv_n), \quad \forall n \geq 0. \tag{3.17}$$

It follows that

$$\begin{aligned} &\varphi\left(\alpha G(gb_1, gu_{n+1},\, gu_{n+1}) + \beta G(gb_2,\, gv_{n+1},\, gv_{n+1})\right) \\ &= \varphi\begin{pmatrix} \alpha G(F(b_1, b_2),\, F(u_n, v_n),\, F(u_n, v_n)) \\ +\beta G(F(b_2, b_1),\, F(v_n, u_n),\, F(v_n, u_n) \end{pmatrix} \\ &\leq \varphi\left(\alpha G(gb_1,\, gu_n, gu_n) + \beta G(gb_2,\, gv_n,\, gv_n)\right) \\ &\quad - \psi\left(\alpha G(gb_1, gu_n,\, gu_n) + \beta G(gb_2,\, gv_n,\, gv_n)\right), \end{aligned} \tag{3.18}$$

which implies from $\psi \geq 0$ that

$$\begin{aligned} &\varphi\left(\alpha G(gb_1, gu_{n+1},\, gu_{n+1}) + \beta G(gb_2,\, gv_{n+1},\, gv_{n+1})\right) \\ \leq\;\; &\varphi\left(\alpha G(gb_1,\, gu_n, gu_n) + \beta G(gb_2,\, gv_n,\, gv_n)\right). \end{aligned}$$

Thus, by the monotonicity of φ, we obtain that the sequence $\{\eta^n_{\alpha,\beta}\}$ defined by

$$\eta^n_{\alpha,\beta} = \alpha G(gb_1,\, gu_n, gu_n) + \beta G(gb_2,\, gv_n,\, gv_n),\;\, n \geq 0$$

is nonincreasing. Hence there exists $\eta_{\alpha,\beta} \geq 0$ such that $\lim\limits_{n\to\infty} \eta^n_{\alpha,\beta} = \eta_{\alpha,\beta}$. We shall prove that $\eta_{\alpha,\beta} = 0$. Suppose, to the contrary, that is $\eta_{\alpha,\beta} > 0$. Letting $n \to \infty$ in (3.18), we get

$$\varphi\left(\eta_{\alpha,\beta}\right) \leq \varphi\left(\eta_{\alpha,\beta}\right) - \lim_{n\to\infty} \psi\left(\eta^n_{\alpha,\beta}\right) = \varphi\left(\eta_{\alpha,\beta}\right) - \lim_{\eta^n_{\alpha,\beta} \to \eta^+_{\alpha,\beta}} \psi\left(\eta^n_{\alpha,\beta}\right) < \varphi\left(\eta_{\alpha,\beta}\right),$$

which is a contradiction. Thus $\eta_{\alpha,\beta} = 0$. That is

$$\lim_{n\to\infty} \alpha G(gb_1,\, gu_n, gu_n) = \lim_{n\to\infty} \beta G(gb_2,\, gv_n,\, gv_n) = 0.$$

Similarly, we obtain

$$\lim_{n\to\infty} \alpha G(a_1,\ u_n,\ u_n) = \lim_{n\to\infty} \beta G(a_2,\ v_n,\ v_n) = 0.$$

By the uniqueness of the limit, we have $ga_1 = gb_1$ and $ga_2 = gb_2$.

Theorem 3.8 *In addition to the hypotheses of Theorem 3.1, suppose that gx_0, gy_0 are comparable. Then F has a unique fixed point, that is, there exists γ such that $g\gamma = F(\gamma,\gamma)$.*

Proof. Assume $gx_0 \le F(x_0,y_0)$ and $F(y_0,x_0) \le gy_0$. Since gx_0, gy_0 are comparable, we have $gx_0 \ge gy_0$ or $gx_0 \le gy_0$. Suppose we are in the second case. Then, by the mixed monotone property of F, we have

$$gx_1 = F(x_0,y_0) \ge F(y_0,x_0) = gy_1$$

and hence, by induction one obtains

$$gx_{n+1} = F(x_n,y_n) \ge F(y_n,x_n) = gy_{n+1}, \quad n \ge 0.$$

Since

$$ga_1 = \lim_{n\to\infty} F(x_n,y_n) \text{ and } gb_1 = \lim_{n\to\infty} F(y_n,x_n),$$

we find from the continuity of the distance G that

$$\begin{aligned} G(ga_1,gb_1,gb_1) &= G(\lim_{n\to\infty} F(x_n,y_n), \lim_{n\to\infty} F(y_n,x_n), \lim_{n\to\infty} F(y_n,x_n)) \\ &= \lim_{n\to\infty} G(F(x_n,y_n),F(y_n,x_n),F(y_n,x_n)) \\ &= \lim_{n\to\infty} G(gx_{n+1},gy_{n+1},gy_{n+1}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \varphi\left((\alpha+\beta)\, G(F(x_n,y_n),F(y_n,x_n),F(y_n,x_n))\right) \\ \le\ & \varphi\left((\alpha+\beta)\, G(gx_n,gy_n,gy_n)\right) - \psi\left((\alpha+\beta)\, G(gx_n,gy_n,gy_n)\right),\ n \ge 0, \end{aligned}$$

which means

$$\begin{aligned} & \varphi((\alpha+\beta)\, G(gx_{n+1},gy_{n+1},gy_{n+1})) \\ \le\ & \varphi\left((\alpha+\beta)\, G(gx_n,gy_n,gy_n)\right) - \psi\left((\alpha+\beta)\, G(gx_n,gy_n,gy_n)\right),\ n \ge 0. \end{aligned}$$

Suppose that $G(ga_1, gb_1, gb_1) > 0$. Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} & \varphi((\alpha+\beta)\,G(ga_1, gb_1, gb_1)) \\ \leq \quad & \varphi\left((\alpha+\beta)\,G(ga_1, gb_1, gb_1)\right) - \lim_{n\to\infty}\psi\left((\alpha+\beta)\,G(gx_n, gy_n, gy_n)\right), n \geq 0, \end{aligned}$$

which leads to $\lim_{n\to\infty}\psi\left((\alpha+\beta)\,G(gx_n, gy_n, gy_n)\right) \leq 0$, which contradicts the hypothese of ψ. So $G(ga_1, gb_1, gb_1) = 0$, hence $ga_1 = gb_1$.

Conflict of Interests

The author declares that there is no conflict of interests.

REFERENCES

[1] Harjani, B. Lopez, K. Sadrangani, Fixed point theorems for mixed monotone operators and applications to integral equations, Nonlinear Anal. 74 (2011), 1749-1760.

[2] Z. Kadelburg, S. Radenovic, Coupled fixed point results under tvs-cone metric and w-cone-distance, Adv. Fixed Point Theory 2 (2012), 29-46.

[3] N. V. Luong, N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal. 74 (2011), 983-992.

[4] B. S. Choudhury, A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Anal. 73 (2010), 2524-2531.

[5] V. Lakshmikantham, L. Ciric, Coupled fixed point theorems for non-linear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009), 4341-4349.

[6] L. Ciric, M. O. Olatinwo, D. Gopal, G. Akinbo, Coupled fixed point theorems for mappings satisfying a contractive condition of rational type on a partially ordered metric space, Adv. Fixed Point Theory 2 (2012), 1-8.

[7] B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal. 72 (2010), 4508-4517.

[8] M. Jain, K. Tas, S. Kumar, N. Gupta, Coupled common fixed points involving a (ψ, ϕ)-contractive condition for mixed g-monotone operators in partially ordered metric spaces, J. Inequal. Appl. 2012 (2012), Article ID 285.

[9] W. Sintunavarat, Y.J. Cho, P. Kumam, Coupled fixed point theorems for weak contraction mapping under F-invariant set, Abst. Appl. Anal. 2012 (2012), Article ID 324874.

[10] W. Sintunavarat, P. Kumam, Coupled best proximity point theorem in metric spaces, Fixed Point Theory Appl. 2012 (2012), Article ID 93.

[11] Y. Qing, J.K. Kim, X. Qin, Fixed point theorems and stability of iterations in cone metric spaces, Adv. Fixed Point Theory 2 (2012), 58-63.

[12] W. Sintunavarat, Y. J. Cho, P. Kumam, Coupled fixed point theorems for contraction mapping induced by cone ball-metric in partially ordered spaces, Fixed Point Theory Appl. 2012 (2012), Article ID 128.

[13] B.C. Dhage, Generalized metric space and mapping with fixed point, Bull. Calcutta Math. Soc. 84 (1992), 329-336.

[14] B.C. Dhage, Generalized metric spaces and topological structure I, Annalele Stintifice ale Universitatii Al.I. Cuza, 46 (2000), 3-24.

[15] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289-297.

[16] Z. Mustafa, B. Sims, Some remarks concerning *D*-metric spaces, in: Proc. Int. Conf. on Fixed Point Theory and Applications, Valencia, Spain, July, 2003.

[17] Z. Mustafa, B. Sims, Fixed point theorems for contractive mappings in complete *G*-metric spaces, Fixed Point Theory Appl. 2009 (2009), Article ID 917175.

[18] Z. Mustafa, H. Obiedat, F. Awawdeh, Some fixed point theorem for mapping on complete *G*-metric spaces, Fixed Point Theory Appl. 2008 (2008), Article ID 189870.

[19] Z. Mustafa, W. Shatanawi, M. Bataineh, Existence of fixed point results in *G*-metric spaces, Int. J. Math. Math. Sci. 2009 (2009), Article ID 283028.

[20] M. Bousselsal, Z. Mostefaoui, $(\psi,\alpha,\beta)-$weak contraction in partially ordered *G*-metric spaces, Thai J. Math. 12 (2014), 71-80.

[21] A. Alotaibi, S.M. Alsulami, Coupled coincidence points for monotone operators in partially ordered metric spaces. Fixed Point Theory Appl. 2011 (2011), Article ID 44.

[22] T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006), 1379-1393.