Available online at http://scik.org
Adv. Fixed Point Theory, 4 (2014), No. 2, 229-244
ISSN: 1927-6303

STRONG CONVERGENCE THEOREMS FOR COMMON SOLUTIONS OF VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS

CHANGQUN WU

School of Business and Administration, Henan University, Kaifeng 475000, China

Abstract. In this paper, we introduce a viscosity approximation method for solving common solutions of variational inequality and fixed point problems. Strong convergence theorems are established in the framework of Hilbert spaces.

Keywords: inverse-strongly monotone mapping; nonexpansive mapping; fixed point; Hilbert space.

2010 AMS Subject Classification: 47H09, 47H10.

1. Introduction

Fixed point theory has emerged as an effective and powerful tool for studying a wide class of real world problems which arise in economics, image reconstruction, transportation, network, elasticity and optimization; see [1-5] and the references therein.

The computation of fixed points is important in the study of many real world problems, including inverse problems; for instance, it is not hard to show that the split feasibility problem and the convex feasibility problem in signal processing and image reconstruction can both be formulated as a problem of finding fixed points of certain operators, respectively; see [6-8] for more details and the references therein.

E-mail address: hhdwucq@yeah.net

Received June 3, 2013

The aim of this paper is to investigate a viscosity approximation method for solving common solutions of variational inequality and fixed point problems. Strong convergence theorems are established in the framework of Hilbert spaces. The organization of this article is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a viscosity iterative method is discussed. Strong convergence theorems of common solutions are established in Hilbert spaces

2. Preliminaries

From now on, we always assume that H is a real Hilbert space, whose the inner product and the norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a closed convex subset of H and let $A : C \to H$ be a mapping. We denote by P_C be the projection of H onto the closed convex subset C. The classical variational inequality problem is to find $u \in C$ such that

$$\langle Au, v-u \rangle \geq 0, \quad \forall v \in C. \tag{2.1}$$

We denoted by $VI(C,A)$ the set of solutions of the variational inequality. One can see that the variational inequality problem (2.1) is equivalent to a fixed point problem, that is, an element $u \in C$ is a solution of the variational inequality (2.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant and I is the identity mapping.

Recall that a mapping $A : C \to H$ is said to be inverse-strongly monotone if there exists a positive real number μ such that

$$\langle Ax - Ay, x - y \rangle \geq \mu \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In this paper, we denote by $F(T)$ the set of fixed points of T. Recall that a mapping $f : C \to C$ is said to be contractive if there exists $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

Recall that a linear bounded operator $B : C \to C$ is said to be strongly positive if there exists a constant $\bar{\gamma} > 0$ such that $\langle Bx, x\rangle \geq \bar{\gamma}\|x\|^2$, $\forall x \in C$. Recall that monotone mapping $T : H \to 2^H$ is said to be maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. Recall that set-valued mapping $T : H \to 2^H$ is said to be monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. It is known that a monotone mapping T is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$.

Let A be a monotone mapping of C into H and $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0,\ \forall u \in C\}$ and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [9] and the references therein.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see [10-15] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in \Omega} \frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle, \tag{2.2}$$

where B is a linear bounded operator on H, Ω is the fixed point set of a nonexpansive mapping S and b is a given point in H.

In [14], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n b, \quad \forall n \geq 0,$$

converges strongly to the unique solution of the minimization problem (2.2) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

Recently, Marino and Xu [11] introduced a general iterative scheme by the viscosity approximation method:

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n \gamma f(x_n), \quad \forall n \geq 0,$$

where S is a nonexpansive mapping on H, f is a contraction on H with the coefficient α, B is a bounded linear strongly positive operator on H with the coefficient $\bar{\gamma}$ and γ is a constant such that $0 < \gamma < \bar{\gamma}/\alpha$. They proved that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the variational inequality: $\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0$, $\forall x \in F(S)$, which is the optimality condition for the minimization problem $\min_{x \in F(S)} \frac{1}{2} \langle Bx, x \rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$.)

Recently, variational inequalities and fixed point problems have been considered by many authors; see [16-20] and the references therein. For finding a common element of the sets of fixed points of nonexpansive mappings and solutions of variational inequalities for μ-inverse-strongly monotone mapping, Iiduka and Takahashi [21] proposed the following iterative scheme:

$$x_1 = x \in C, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\mu)$. They proved that the sequence $\{x_n\}$ converges strongly to some $z \in F(S) \cap VI(C, A)$.

In this paper, we consider a mapping W_n defined by

$$\begin{aligned}
&U_{n,n+1} = I, \\
&U_{n,n} = \gamma_n T_n U_{n,n+1} + (1 - \gamma_n) I, \\
&U_{n,n-1} = \gamma_{n-1} T_{n-1} U_{n,n} + (1 - \gamma_{n-1}) I, \\
&\cdots \\
&U_{n,k} = \gamma_k T_k U_{n,k+1} + (1 - \gamma_k) I, \\
&U_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1 - \gamma_{k-1}) I, \\
&\cdots \\
&U_{n,2} = \gamma_2 T_2 U_{n,3} + (1 - \gamma_2) I, \\
&W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1 - \gamma_1) I,
\end{aligned} \tag{2.3}$$

where $\gamma_1, \gamma_2, \ldots$ are real numbers such that $0 \le \gamma_n \le 1$ and $T_1, T_2, \cdots$ be an infinite family of mappings of C into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n.

Concerning W_n, we have the following lemmas which are important to prove our main results.

Lemma 2.1. [8] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \cdots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\gamma_1, \gamma_2, \cdots$ be real numbers such that $0 < \gamma_n \le b < 1$ for any $n \ge 1$. Then, for all $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.1, one can define the mapping W of C into itself as follows.

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C. \tag{2.4}$$

Such a mapping W is called the W-mapping generated by $T_1, T_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$.

Throughout this paper, we shall always assume that $0 < \gamma_i \le b < 1$ for all $i \ge 1$.

Lemma 2.2. [8] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \cdots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\gamma_1, \gamma_2, \cdots$ be real numbers such that $0 < \gamma_n \le b < 1$ for any $n \ge 1$. Then $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.3. [6] *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_1, T_2, \cdots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\gamma_1, \gamma_2, \cdots$ be a real sequence such that $0 < \gamma_n \le b < 1$ for all $n \ge 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

In order to prove our main results, we also need the following lemmas.

Lemma 2.4. [11] *Assume that B is a strong positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \|B\|^{-1}$. Then $\|I - \rho B\| \le 1 - \rho\bar{\gamma}$.*

Lemma 2.5 [11] *Let H be a Hilbert space, B be a strongly positive linear bounded self-adjoint operator on H with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $T : H \to H$ be a nonexpansive mapping with a fixed point x_t of the contraction $x \mapsto t\gamma f(x) + (I - tB)Tx$. Then*

$\{x_t\}$ *converges strongly as* $t \to 0$ *to a fixed point* $\bar{x}$ *of* T*, which solves the variational inequality:*

$$\langle (B - \gamma f)\bar{x}, \bar{x} - z\rangle \leq 0, \quad \forall z \in F(T).$$

Equivalently, $\bar{x} = P_F(\gamma f + I - B)\bar{x}$.

Lemma 2.6. [22] *Let* H *be a Hilbert space,* C *a closed convex subset of* H*,* $f : C \to C$ *a contraction with the coefficient* $\alpha \in (0,1)$ *and* B *a strongly positive linear bounded operator with the coefficient* $\bar{\gamma} > 0$*. Then, for any* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$*,*

$$\langle x - y, (B - \gamma f)x - (B - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad \forall x, y \in C.$$

That is, $B - \gamma f$ *is strongly monotone with the coefficient* $\bar{\gamma} - \alpha\gamma$*.*

Lemma 2.7. [23] *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main Results

Theorem 3.1. *Let* H *be a real Hilbert space and let* C *be be a nonempty closed convex subset of* H *such that* $C \pm C \subset C$*. Let* $A : C \to H$ *be a* μ*-inverse-strongly monotone mapping. Let* $f : C \to C$ *be a contraction with the coefficient* α *and let* $T_1, T_2, \cdots$ *be a sequence of nonexpansive self-mappings on* C*. Let* B *be a strongly positive linear bounded self-adjoint operator of* C *into itself with the coefficient* $\bar{\gamma} > 0$*. Assume that* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$*. Let the sequence* $\{x_n\}$ *be generated by*

$$x_1 \in C, \quad x_{n+1} = \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B)W_n P_C(I - \lambda_n A)x_n, \quad \forall n \geq 1,$$

where the mapping W_n *is defined by (2.3),* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\lambda_n\}$ *is a sequence in* $(0, 2\mu)$*. If* $F = \cap_{i=1}^{\infty} F(T_i) \cap VI(C,A) \neq \emptyset$ *and* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are chosen such that*

(a) $\lim_{n\to\infty} \alpha_n = 0$*;*

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(d) $\{\lambda_n\} \subset [u, v]$ *for some* u, v *with* $0 < u < v < 2\mu$,

then the sequence $\{x_n\}$ *converges strongly to some* $x^* \in F$*, which uniquely solves the following variation inequality:*

$$\langle Bx^* - \gamma f(x^*), x^* - p\rangle \leq 0, \quad \forall p \in F.$$

Equivalently, we have $x^* = P_F(\gamma f + I - B)x^*$.

Proof. First, we show that $\{x_n\}$ is bounded. Note that the mapping $I - \lambda_n A$ is nonexpansive for each $n \geq 1$. Indeed, from the condition (d), for $\forall x, y \in C$, we have

$$\begin{aligned}
&\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 \\
&\leq \|x - y\|^2 - 2\lambda_n\langle Ax - Ay, x - y\rangle + \lambda_n^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\lambda_n\mu\|Ax - Ay\|^2 + \lambda_n^2\|Ax - Ay\|^2 \\
&= \|x - y\|^2 + \lambda_n(\lambda_n - 2\mu)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

This shows that $I - \lambda_n A$ is a nonexpansive mapping. Noticing that condition (a), we may assume, with no loss of generality, that $\alpha_n \leq \|B\|^{-1}$ for all $n \geq 1$. Using Lemma 2.4, we know that, if $0 < \alpha_n \leq \|B\|^{-1}$ for all $n \geq 1$, then $\|I - \alpha_n B\| \leq 1 - \alpha_n\bar{\gamma}$. Fixing $p \in F$, we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\alpha_n(\gamma f(W_n x_n) - Bp) + (I - \alpha_n B)(W_n P_C(I - \lambda_n A)x_n - p)\| \\
&\leq \alpha_n\|\gamma f(W_n x_n) - Bp\| + (1 - \alpha_n\bar{\gamma})\|W_n P_C(I - \lambda_n A)x_n - p\| \\
&\leq \alpha_n\gamma\|f(W_n x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Bp\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\
&\leq [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)]\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\|.
\end{aligned}$$

By simple inductions, we obtain

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\|Bp - \gamma f(p)\|}{\bar{\gamma} - \gamma\alpha}\} \quad \forall n \geq 1,$$

which yields that the sequence $\{x_n\}$ is bounded. Putting $\rho_n = P_C(I - \lambda_n A)x_n$, we have

$$\begin{aligned}
&\|\rho_{n+1} - \rho_n\| \\
&= \|P_C(I - \lambda_{n+1}A)x_{n+1} - P_C(I - \lambda_n A)x_n\| \\
&\leq \|(I - \lambda_{n+1}A)x_{n+1} - (I - \lambda_n A)x_n\| \\
&= \|(I - \lambda_{n+1}A)x_{n+1} - (I - \lambda_{n+1}A)x_n + (I - \lambda_{n+1}A)x_n - (I - \lambda_n A)x_n\| \\
&= \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\|.
\end{aligned} \tag{3.1}$$

It follow that

$$\begin{aligned}
&\|x_{n+2} - x_{n+1}\| \\
&= \|(I - \alpha_{n+1}B)(W_{n+1}\rho_{n+1} - W_n\rho_n) - (\alpha_{n+1} - \alpha_n)BW_n\rho_n \\
&\quad + \gamma[\alpha_{n+1}(f(W_{n+1}x_{n+1}) - f(W_n x_n)) + f(x_n)(\alpha_{n+1} - \alpha_n)]\| \\
&\leq (1 - \alpha_{n+1}\bar{\gamma})(\|\rho_{n+1} - \rho_n\| + \|W_{n+1}\rho_n - W_n\rho_n\|) + |\alpha_{n+1} - \alpha_n|\|BW_n\rho_n\| \\
&\quad + \gamma[\alpha_{n+1}\alpha\|x_{n+1} - x_n\| + \|W_{n+1}x_n - W_n x_n\| + \|f(x_n)\||\alpha_{n+1} - \alpha_n|].
\end{aligned} \tag{3.2}$$

Since T_i and $U_{n,i}$ are nonexpansive, we find that

$$\begin{aligned}
\|W_{n+1}\rho_n - W_n\rho_n\| &= \|\gamma_1 T_1 U_{n+1,2}\rho_n - \gamma_1 T_1 U_{n,2}\rho_n\| \\
&\leq \gamma_1\|U_{n+1,2}\rho_n - U_{n,2}\rho_n\| \\
&= \gamma_1\|\gamma_2 T_2 U_{u+1,3}\rho_n - \gamma_2 T_2 U_{n,3}\rho_n\| \\
&\leq \gamma_1\gamma_2\|U_{u+1,3}\rho_n - U_{n,3}\rho_n\| \\
&\leq \cdots \\
&\leq \gamma_1\gamma_2\cdots\gamma_n\|U_{n+1,n+1}\rho_n - U_{n,n+1}\rho_n\| \\
&\leq M_1\prod_{i=1}^{n}\gamma_i,
\end{aligned}$$

where $M_1 \geq 0$ is an appropriate constant such that $\|U_{n+1,n+1}\rho_n - U_{n,n+1}\rho_n\| \leq M_1$ for all $n \geq 1$. Using (3.1) and (3.2), we arrive at

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\leq [1 - \alpha_{n+1}(\bar{\gamma} - \alpha\gamma)]\|x_{n+1} - x_n\| \\
&\quad + M_2(2\prod_{i=1}^{n}\gamma_i + 2|\alpha_{n+1} - \alpha_n| + |\lambda_{n+1} - \lambda_n|),
\end{aligned} \tag{3.3}$$

where M_2 is an appropriate constant such that

$$M_2 = \max\{M_1, \sup_{n\geq 1}\{\|Ax_n\|\}, \gamma \sup_{n\geq 1}\{\|f(x_n)\|\}, \sup_{n\geq 1}\{\|BW_n\rho_n\|\}\}.$$

Using the restrictions (a), (b) and (c), we find that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.4}$$

Note that

$$\begin{aligned}\|\rho_n - p\|^2 &\leq \|(x_n-p)-\lambda_n(Ax_n-Ap)\|^2\\ &= \|x_n-p\|^2 - 2\lambda_n\langle x_n-p, Ax_n-Ap\rangle + \lambda_n^2\|Ax_n-Ap\|^2\\ &\leq \|x_n-p\|^2 - 2\lambda_n\mu\|Ax_n-Ap\|^2 + \lambda_n^2\|Ax_n-Ap\|^2\\ &= \|x_n-p\|^2 + \lambda_n(\lambda_n-2\mu)\|Ax_n-Ap\|^2.\end{aligned} \tag{3.5}$$

On the other hand, we have

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &= \|\alpha_n\gamma f(x_n) + (I-\alpha_n B)W_n\rho_n - p\|^2\\ &\leq (\alpha_n\|\gamma f(W_nx_n) - Bp\| + (1-\alpha_n\bar{\gamma})\|W_n\rho_n - p\|)^2\\ &\leq (\alpha_n\|\gamma f(W_nx_n) - Bp\| + (1-\alpha_n\bar{\gamma})\|\rho_n - p\|)^2\\ &\leq \alpha_n\|\gamma f(W_nx_n) - Bp\|^2 + \|\rho_n-p\|^2 + 2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|.\end{aligned} \tag{3.6}$$

Substituting (3.5) into (3.6), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 \leq{}& \alpha_n\|\gamma f(W_nx_n)-Bp\|^2 + \|x_n-p\|^2 + \lambda_n(\lambda_n-2\mu)\|Ax_n-Ap\|^2\\ &+ 2\alpha_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|.\end{aligned}$$

It follows that

$$\begin{aligned}&u(2\mu - v)\|Ax_n-Ap\|^2\\ &\leq \alpha_n\|\gamma f(x_n)-Bp\|^2 + \|x_n-p\|^2 - \|x_{n+1}-p\|^2\\ &\quad + 2\alpha_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|\\ &\leq \alpha_n\|\gamma f(x_n)-Bp\|^2 + (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\ &\quad + 2\alpha_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|.\end{aligned}$$

Using (2.4), we see that

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{3.7}$$

Since P_C is firmly nonexpansive, we find that

$$\begin{aligned}
\|\rho_n - p\|^2 \leq& \langle (I-\lambda_n A)x_n - (I-\lambda_n A)p, \rho_n - p\rangle \\
=& \frac{1}{2}\{\|(I-\lambda_n A)x_n - (I-\lambda_n A)p\|^2 + \|\rho_n - p\|^2 \\
&- \|(I-\lambda_n A)x_n - (I-\lambda_n A)p - (\rho_n - p)\|^2\} \\
\leq& \frac{1}{2}\{\|x_n - p\|^2 + \|\rho_n - p\|^2 - \|(x_n - \rho_n) - \lambda_n(Ax_n - Ap)\|^2\} \\
=& \frac{1}{2}\{\|x_n - p\|^2 + \|\rho_n - p\|^2 - \|x_n - \rho_n\|^2 - \lambda_n^2\|Ax_n - Ap\|^2 \\
&+ 2\lambda_n\langle x_n - \rho_n, Ax_n - Ap\rangle\},
\end{aligned}$$

which yields that

$$\|\rho_n - p\|^2 \leq \|x_n - p\|^2 + 2\lambda_n\|x_n - \rho_n\|\|Ax_n - Ap\| - \|x_n - \rho_n\|^2. \tag{3.8}$$

This in turn implies that

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq \alpha_n\|\gamma f(W_n x_n) - Bp\|^2 + \|x_n - p\|^2 + 2\lambda_n\|x_n - \rho_n\|\|Ax_n - Ap\| \\
+ 2\alpha_n\|\gamma f(W_n x_n) - Bp\|\|\rho_n - p\| - \|x_n - \rho_n\|^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_n - \rho_n\|^2 \leq& \alpha_n\|\gamma f(W_n x_n) - Bp\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&+ 2\lambda_n\|x_n - \rho_n\|\|Ax_n - Ap\| + 2\alpha_n\|\gamma f(W_n x_n) - Bp\|\|\rho_n - p\| \\
\leq& \alpha_n\|\gamma f(W_n x_n) - Bp\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| \\
&+ 2\lambda_n\|x_n - \rho_n\|\|Ax_n - Ap\| + 2\alpha_n\|\gamma f(W_n x_n) - Bp\|\|\rho_n - p\|.
\end{aligned}$$

This yields that

$$\lim_{n\to\infty}\|x_n - \rho_n\| = 0. \tag{3.9}$$

Notice that

$$\begin{aligned}
\|\rho_n - W_n\rho_n\| \leq& \|x_{n+1} - W_n\rho_n\| + \|x_n - x_{n+1}\| + \|x_n - \rho_n\| \\
\leq& \alpha_n\|\gamma f(x_n) - BW_n\rho_n\| + \|x_n - x_{n+1}\| + \|x_n - \rho_n\|.
\end{aligned}$$

Using the restriction (a), we find from (3.4) and (3.9) that

$$\lim_{n\to\infty}\|\rho_n - W_n\rho_n\| = 0. \tag{3.10}$$

Since the sequence $\{x_n\}$ is bounded, we see that $\{\rho_n\}$ is also a bounded sequence in C. Without loss of generality, we can assume that there exists a bounded set $K \subset C$ such that $\rho_n \in K$ for all $n \geq 1$. On the other hand, we have

$$\begin{aligned}\|W\rho_n - \rho_n\| &\leq \|W\rho_n - W_n\rho_n\| + \|W_n\rho_n - \rho_n\| \\ &\leq \sup_{\rho\in K}\|W\rho - W_n\rho\| + \|W_n\rho_n - \rho_n\|.\end{aligned}$$

Using Lemma 1.3, we obtain from (3.10) that

$$\lim_{n\to\infty}\|W\rho_n - \rho_n\| = 0. \tag{3.11}$$

Now, we are in a position to show that $x_n \to x^*$ as $n \to \infty$. First, we prove that the uniqueness of the solution of the variational inequality (2.1), which is indeed a consequence of the strong monotonicity of $B - \gamma f$. Suppose that $x^* \in F$ and $x^{**} \in F$ both are solutions to (2.1). Then we have

$$\langle (B-\gamma f)x^*, x^* - x^{**}\rangle \leq 0$$

and

$$\langle (B-\gamma f)x^{**}, x^{**} - x^*\rangle \leq 0.$$

Adding up the two inequalities, we see that

$$\langle (B-\gamma f)x^* - (B-\gamma f)x^{**}, x^* - x^{**}\rangle \leq 0.$$

The strong monotonicity of $B - \gamma f$ implies that $x^* = x^{**}$ and the uniqueness is proved. Let x^* be the unique solution of (2.1). That is, $x^* = P_F(\gamma f + (I - B))x^*$.

Next, we show that

$$\limsup_{n\to\infty}\langle Bx^* - \gamma f(x^*), x^* - x_n\rangle \leq 0. \tag{3.12}$$

To show it, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle Bx^* - \gamma f(x^*), x^* - x_n\rangle = \lim_{i\to\infty}\langle Bx^* - \gamma f(x^*), x^* - x_{n_i}\rangle.$$

Since $\{x_{n_i}\}$ is bounded, it follows that that there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to p. We may assume that, without loss of generality, that $x_{n_i} \rightharpoonup p$. Therefore, we have $p \in F$. Indeed, let us first show that $p \in VI(C,A)$. Put

$$Tw = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v,w) \in G(T)$. Since $w - Av \in N_C v$ and $\rho_n \in C$, we have $\langle v - \rho_n, w - Av \rangle \geq 0$.

On the other hand, from $\rho_n = P_C(I - \lambda_n A)x_n$, we have $\langle v - \rho_n, \rho_n - (I - \lambda_n A)x_n \rangle \geq 0$ and hence $\langle v - \rho_n, \frac{\rho_n - x_n}{\lambda_n} + Ax_n \rangle \geq 0$. It follows that

$$\begin{aligned}
&\langle v - \rho_{n_i}, w \rangle \\
&\geq \langle v - \rho_{n_i}, Av \rangle \geq \langle v - \rho_{n_i}, Av \rangle - \langle v - \rho_{n_i}, \frac{\rho_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i} \rangle \\
&\geq \langle v - \rho_{n_i}, Av - \frac{\rho_{n_i} - x_{n_i}}{\lambda_{n_i}} - Ax_{n_i} \rangle \\
&= \langle v - \rho_{n_i}, Av - A\rho_{n_i} \rangle + \langle v - \rho_{n_i}, A\rho_{n_i} - Ax_{n_i} \rangle - \langle v - \rho_{n_i}, \frac{\rho_{n_i} - x_{n_i}}{\lambda_{n_i}} \rangle \\
&\geq \langle v - \rho_{n_i}, A\rho_{n_i} - Ax_{n_i} \rangle - \langle v - \rho_{n_i}, \frac{\rho_{n_i} - x_{n_i}}{\lambda_{n_i}} \rangle,
\end{aligned}$$

which implies that $\langle v - p, w \rangle \geq 0$. We have $p \in A^{-1}0$ and hence $p \in VI(C,A)$.

Next, let us show $p \in \cap_{i=1}^{\infty} F(T_i)$. Using the Opial's condition, we find that

$$\begin{aligned}
\liminf_{i\to\infty} \|\rho_{n_i} - p\| &< \liminf_{i\to\infty} \|\rho_{n_i} - Wp\| \\
&= \liminf_{i\to\infty} \|\rho_{n_i} - W\rho_{n_i} + W\rho_{n_i} - Wp\| \\
&\leq \liminf_{i\to\infty} \|W\rho_{n_i} - Wp\| \\
&\leq \liminf_{i\to\infty} \|\rho_{n_i} - p\|,
\end{aligned}$$

which is a contradiction. Thus we have $p \in F(W) = \cap_{i=1}^{\infty} F(T_i)$. On the other hand, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle Bx^* - \gamma f(x^*), x^* - x_n \rangle &= \lim_{i\to\infty} \langle Bx^* - \gamma f(x^*), x^* - x_{n_i} \rangle \\
&= \langle Bx^* - \gamma f(x^*), x^* - p \rangle \leq 0.
\end{aligned}$$

Note that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\leq (1-\alpha_n\bar{\gamma})^2\|W_n\rho_n-x^*\|^2+2\alpha_n\langle\gamma f(W_nx_n)-Bx^*,x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n-x^*\|^2+\alpha\gamma\alpha_n(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)\\
&\quad +2\alpha_n\langle\gamma f(x^*)-Bx^*,x_{n+1}-x^*\rangle.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\leq \frac{(1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(x^*)-Bx^*,x_{n+1}-x^*\rangle\\
&=\frac{(1-2\alpha_n\bar{\gamma}+\alpha_n\alpha\gamma)}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2+\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2\\
&\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(x^*)-Bx^*,x_{n+1}-x^*\rangle\\
&\leq [1-\frac{2\alpha_n(\bar{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}]\|x_n-x^*\|^2\\
&\quad +\frac{2\alpha_n(\bar{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}[\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(x^*)-Bx^*,x_{n+1}-x^*\rangle+\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M_3],
\end{aligned}
$$

where M_3 is an appropriate constant such that $M_3\geq\sup_{n\geq1}\|x_n-x^*\|^2$. Put

$$k_n=\frac{2\alpha_n(\bar{\gamma}-\alpha\gamma)}{1-\alpha_n\alpha\gamma},$$

$$d_n=\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(x^*)-Bx^*,x_{n+1}-q\rangle+\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M_3.$$

Hence, we have

$$\|x_{n+1}-x^*\|^2\leq(1-k_n)\|x_n-x^*\|+b_nd_n.$$

It follows that

$$\lim_{n\to\infty}k_n=0,\ \sum_{n=1}^{\infty}k_n=\infty,\ \limsup_{n\to\infty}d_n\leq0.$$

Using (2.7), we conclude the desired conclusion immediately. This completes the proof.

Putting $\gamma=1$ and $B=I$ in Theorem 3.1, we have the following results.

Corollary 3.2. *Let H be a real Hilbert space and let C be be a nonempty closed convex subset of H. Let $A:C\to H$ be a μ-inverse-strongly monotone mapping. Let $f:C\to C$ be a contraction* *with the coefficient α and let $T_1, T_2, \cdots$ be a sequence of nonexpansive self-mappings on C. Let the sequence $\{x_n\}$ be generated by*

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(W_n x_n) + (1-\alpha_n) W_n P_C (I - \lambda_n A) x_n, \quad \forall n \geq 1,$$

where the mapping W_n is defined by (2.3), $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\mu)$. If $F = \cap_{i=1}^{\infty} F(T_i) \cap VI(C,A) \neq \emptyset$ and $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen such that

(a) $\lim_{n\to\infty} \alpha_n = 0$*;*

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(d) $\{\lambda_n\} \subset [u,v]$ *for some* u, v *with* $0 < u < v < 2\mu$*,*

then the sequence $\{x_n\}$ converges strongly to some $x^* \in F$, which uniquely solves the following variation inequality:

$$\langle x^* - \gamma f(x^*), x^* - p \rangle \leq 0, \quad \forall p \in F.$$

Equivalently, we have $x^* = P_F f(x^*)$.

For a single mapping, we have the following.

Corollary 3.3. *Let H be a real Hilbert space and let C be be a nonempty closed convex subset of H such that $C \pm C \subset C$. Let $A : C \to H$ be a μ-inverse-strongly monotone mapping. Let $f : C \to C$ be a contraction with the coefficient α and let T be a nonexpansive self-mappings on C. Let B be a strongly positive linear bounded self-adjoint operator of C into itself with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let the sequence $\{x_n\}$ be generated by*

$$x_1 \in C, \quad x_{n+1} = \alpha_n \gamma f(Tx_n) + (I - \alpha_n B) T P_C (I - \lambda_n A) x_n, \quad \forall n \geq 1,$$

$\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\mu)$. If $F = F(T) \cap VI(C,A) \neq \emptyset$ and $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen such that

(a) $\lim_{n\to\infty} \alpha_n = 0$*;*

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(d) $\{\lambda_n\} \subset [u,v]$ *for some* u, v *with* $0 < u < v < 2\mu$*,* *then the sequence* $\{x_n\}$ *converges strongly to some* $x^* \in F$*, which uniquely solves the following variation inequality:*

$$\langle Bx^* - \gamma f(x^*), x^* - p\rangle \leq 0, \quad \forall p \in F.$$

Equivalently, we have $x^* = P_F(\gamma f + I - B)x^*$*.*

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements

The author was grateful to the reviewer for the useful suggestions which improved the contents of the article.

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