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FIXED POINT THEOREMS FOR M-CONTRACTION TYPE MAPS IN PARTIALLY ORDERED METRIC SPACES AND APPLICATIONS

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Abstract. This paper presents hybrid fixed point theorems of Krasnosel'skii type and some nonlinear alternatives of Leray-Schauder type involving sums of two operators in a partially ordered normed linear spaces. The results are applied to functional integral equations that have been considered in Ntouyas and Tsamatos [21] for proving the existence of solutions under certain monotonic conditions blending with the existence of upper or lower solution. Applications are also given to some nonlinear functional initial and boundary value problems of ordinary differential equations for proving the existence results.

Keywords: hybrid fixed point theorem; partially ordered normed linear space; fractional integral equation; functional integral equation; existence theorem.

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1. Introduction and preliminaries

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The fixed point theory is one of the most rapidly growing topic of nonlinear functional analysis. It is a vast and inter-disciplinary subject whose study belongs to several mathematical domains such as classical analysis, differential and integral equations, functional analysis, operator theory, topology and algebraic topology etc. Most important nonlinear problems of applied mathematics reduce to finding solutions of nonlinear functional equations (e.g., nonlinear integral equations, boundary value problems for nonlinear ordinary or partial differential equations, the existence of periodic solutions of nonlinear partial differential equations). They can be formulated in terms of finding fixed points of a given nonlinear mapping on an infinite dimensional function space into itself; see Amann [1], Heikkllä and Lakshmikantham [15], Zeidler [27] and the references therein.

Nonlinear functional integral equations have also been discussed in the literature, see, for example, Subramanyam and Sundersanam [25], Ntouyas and Tsamatos [21] etc. Recently, Ran and Reurings [23] initiated the study of hybrid fixed point theorems in partially ordered sets which is further continued in Nieto and Rodríguez-López [19] and [20].

In this paper, we prove some Krasnosel'skii type hybrid fixed point theorems in partially ordered complete normed linear spaces and discuss some of their applications.

The notions of partially ordered linear spaces or vector spaces are given as follows.

Let X be a vector space or linear space. We introduce a partial order $\preceq$ in X as follows. A relation $\preceq$ in X is said to be partial order iff it satisfies following properties:

1. Reflexivity: $a \preceq a$ for all $a \in X$,
2. Antisymmetry: $a \preceq b$ and $b \preceq a$ implies $a = b$,
3. Transitivity: $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ and
4. Order linearity: $x_1 \preceq y_1$ and $x_2 \preceq y_2 \Rightarrow x_1 + x_2 \preceq y_1 + y_2$ and $x \preceq y \Rightarrow tx \preceq ty$ for $t \geq 0$.

The linear space X together with a partial order $\preceq$ becomes a partially ordered vector space or a linear space. Two elements x and y in a partially ordered linear space X are called comparable if either the relation $x \preceq y$ or $y \preceq x$ holds. We introduce a norm $\|.\|$ in partially ordered linear space X so that X becomes now a partially ordered normed linear space. If X is complete w.r.t. the metric d defined through the above norm, then it is called a partially ordered complete normed linear space.

The following definitions are useful in the subsequent part of this paper.

Definition 1.1. A mapping $T : X \to X$ is called monotone nondecreasing if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in X$.

Definition 1.2. A mapping $T : X \to X$ is called monotone nonincreasing if $x \preceq y$ implies $Tx \succeq Ty$ for all $x, y \in X$.

Definition 1.3. A mapping $T : X \to X$ is called monotonic function if it is either monotone nonincreasing or monotone nondecreasing.

Definition 1.4. A mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a monotone dominating function or, in short, M-function if it is an upper or lower semi-continuous and monotonic nondecreasing or nonincreasing function satisfying $\varphi(0) = 0$.

Definition 1.5. (Bedre *et al.* [4]) Given a partially ordered normed linear space E, a mapping $Q : E \to E$ is called partially M-Lipschitz or partially nonlinear M-Lipschitz if there is a M-function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Qx - Qy\| \leqq \varphi(\|x - y\|)$$

for all comparable elements $x, y \in E$. The function is called a M-function of Q on E. If $\varphi(r) = kr,\ k > 0$, then Q is called partially M-Lipschitz with the Lipschitz constant k. In particular, if $k < 1$, then Q is called a partially M-contraction on X with the contraction constant k. Further, if $\varphi(r) < r$ for $r > 0$, then Q is called a partially nonlinear M-contraction with a M-function φ of Q on X.

It is obvious that every contraction is a nonlinear contraction and every nonlinear contraction is a nonlinear M-contraction but the converse assertion does not hold.

There do exist M-functions and the commonly used M-functions are $\varphi(r) = kr$ and $\varphi(r) = \frac{r}{1+r}$, etc. These M-functions can be used in the theory of nonlinear differential and integral equations for proving the existence results via fixed point methods.

It is clear that M-contraction is a weaker than D-contaction or nonlinear contraction in literature.

Another notions that we used are the following definitions.

Defnition 1.6. [12] An operator Q on a normed linear space E into itself is called compact if $Q(E)$ is a relatively compact subset of E. Q is called totally bounded if for any bounded subset S of E, $Q(S)$ is a relatively compact subset of E. If Q is continuous and totally bounded, then it is called completely continuous on E.

Defnition 1.7. [12] An operator Q on a normed linear space E into itself is called partially compact if $Q(C)$ is a relatively compact subset of E for all totally ordered set or chain C in E. Q is called partially totally bounded if for any totally ordered and bounded subset C of E, $Q(C)$ is a relatively compact subset of E. If Q is continuous and partially totally bounded, then it is called partially completely continuous on E.

Recently, Nieto and Rodríguez-López [20] proved the following hybrid fixed point theorems for the monotone mappings in partially ordered metric spaces using the mixed arguments from algebra and geometry.

Theorem 1.1. (Nieto and Rodríguez-López [20]). *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \to X$ be a monotone function (nondecreasing or nonincreasing) such that there exists $k \in [0,1)$ with $d(T(x),T(y)) \leqq kd(x,y)$, for all $x \geqq y$. Suppose that either T is continuous or X is such that if $x_n \to x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \leqq T(x_0)$ or $x_0 \geqq T(x_0)$, then T has a fixed point which is further unique if "every pair of elements in X has a lower and an upper bound".*

The hybrid fixed point theorem of Heikillä and Lakshmikantham [15] for the monotone mappings in ordered metric spaces is as follows.

Theorem 1.2. (Heikkillä and Lakshmikantham [15]). *Let $[a,b]$ be an order interval in a subset Y of the ordered metric space X and let $G : [a,b] \to [a,b]$ be a nondecreasing mapping. If the sequence $\{Gx_n\}$ converges in Y whenever $\{x_n\}$ is a monotone sequence in $[a,b]$, then the well ordered chain of G-iterations of a has a maximum x^* which is a fixed point of G. Moreover, $x^* = max\{y \in [a,b] \mid y \leqq Gy\}$.*

The convergence of the sequence in Heikillä and Lakshmikantham [15] is straight forward whereas the convergence of the sequence in Nieto and Rodríguez-López [20] is due mainly to the metric condition of contraction. The main advantage of Theorem 1.1 is that the uniqueness of fixed point of the monotone mappings is obtained under certain additional conditions on the domain space such as lattice structure of the partially ordered space under consideration and these fixed point results are useful in establishing the uniqueness of the solution of nonlinear differential and integral equations.

The slight generalization of Theorem 1.1 is as follows.

Theorem 1.3. (Bedre *et al.* [4]) *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \to X$ be a monotone function (nondecreasing or nonincreasing) such that there exists a M-function φ with*

$$d(T(x),T(y)) \leqq \varphi_T(d(x,y)) \tag{1.1}$$

for all comparable elements $x,y \in X$ and satisfying $\varphi_T(r) < r$ for $r > 0$. Assume that either T is continuous on X or X is such that if $x_n \to x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \leqq T(x_0)$ or $x_0 \geqq T(x_0)$, then T has a fixed point which is further unique if "every pair of elements in X has a lower and an upper bound".

As a special case of Theorem 1.3, we obtain Theorem 1.1 of Nieto and Rodríguez-López [20] for partially contraction mappings in partially ordered complete metric spaces.

Sometimes it is possible that a mapping T is not a nonlinear M-contraction, but some iterations of it is a nonlinear M-contraction on X . Therefore following hybrid fixed point theorem is noteworthy.

Corollary 1.4. (Bedre *et al.* [4]) *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \to X$ be a monotone function (nondecreasing or nonincreasing) such that there exists a M-function φ and a positive integer p such that*

$$d(T^p(x),T^p(y)) \leqq \varphi_T(d(x,y)) \tag{1.6}$$

for all comparable elements $x, y \in X$ and satisfying $\varphi_T(r) < r$ for $r > 0$. Suppose that either T is continuous or X is such that if $x_n \to x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \leqq T(x_0)$ or $x_0 \geqq T(x_0)$, then T has a fixed point which is further unique if "every pair of elements in X has a lower and an upper bound".

Definition 1.8. The order relation $\preceq$ and the metric d in a non-empty set X are said to be compatible if $\{x_n\}$ is a monotone sequence in X and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x_0 implies that the whole sequence $\{x_n\}$ converges to x_0. similarly, given a partially ordered normed linear space $(X, \preceq, \|.\|)$, the ordered relation $\preceq$ and the norm $\|.\|$ are said to be compatible if $\preceq$ and the metric d define through the norm are compatible.

Clearly, the set $\mathbb{R}$ with usual order relation $\leqq$ and the norm defined by absolute value function has this property. Similarly the space $C(J, \mathbb{R})$ with usual order relation defined by $x \leqq y$ if and only if $x(t) \leqq y(t)$ for all $t \in J$ or $x \leqq y$ if and only if $x(t) \geqq y(t)$ for all $t \in J$ and the usual standard supremum norm $\|.\|$ are compatible.

Now we state more basic hybrid fixed point theorem.

Theorem 1.5. (Bedre *et al.* [4]) *Let X be a partially ordered linear space and suppose that there is a norm in X such that X is a normed linear space. Let $T : X \to X$ be a monotonic (nondecreasing or nonincreasing), partially compact and continuous mapping. Further if the order relation $\preceq$ or $\succeq$ and the norm $\|.\|$ in X are compatible and if there is an element $x_0 \in X$ satisfying $x_0 \leqq Tx_0$ or $x_0 \geqq Tx_0$, then T has a fixed point.*

The following version of Krasnosel'skii's fixed point is known in the literature.

Theorem 1.6. (Krasnosel'skii [16]). *Let S be a closed convex and bounded subset of a Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators satisfying the following conditions:*

(a) *A is nonlinear contraction;*

(b) *B is completely continuous;*

(c) *$Ax + By = x$ for all $y \in S$ implies $x \in S$.*

Then the operator equation $Ax + Bx = x$ has a solution.

Theorem 1.6 is useful and applied to linear perturbations of differential and integral equations by several authors in the literature for proving the existence of the solutions. Here we do not need any order structure of the Banach space under consideration.

2. Krasnosel'skii type hybrid fixed point theorems

We now obtain another version of Krasnosel'skii type hybrid fixed point theorems, some non-linear alternative of leray-schauder type theorems in partially ordered complete normed linear spaces under weaker conditions and discuss some of their applications to Nonlinear functional integral equations of mixed type.

Theorem 2.1. *Let $(X, \preceq, \|.\|)$ be a partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|.\|$ in X are compatible. Let $A, B : X \to X$ be two monotone operators (nondecreasing or nonincreasing) satisfying the following conditions:*
(a) *A is continuous and partially nonlinear M-contraction;*
(b) *B is continuous and partially compact;*
(c) *There exists an x_0 such that $x_0 \preceq Ax_0 + By$ or $x_0 \succeq Ax_0 + By$ for all $y \in X$;*
(d) *Every pair of elements $x, y \in X$ has a lower and an upper bound in X.*
Then the operator equation $Ax + Bx = x$ has a solution.

Proof. Define an operator $T : X \to X$ by

$$T = (I - A)^{-1}B. \tag{2.1}$$

Clearly the operator T is well defined. To see this, let $y \in X$ be fixed and define a mapping $A_y : X \to X$ by

$$A_y(x) = Ax + By. \tag{2.2}$$

A_y is monotonic and by hypothesis (c), there is a point $x_0 \in X$ such that $x_0 \leqq Tx_0$ or $x_0 \geqq Tx_0$. Now, for any two comparable elements $x_1, x_2 \in X$, one has

$$\|A_y(x_1) - A_y(x_2)\| = \|Ax_1 - Ax_2\| \leqq \varphi_A(\|x_1 - x_2\|) \tag{2.3}$$

where, A is a M-function of T on X so A_y is partially nonlinear M-contraction on X. Hence, by an application of a fixed point Theorem 1.3, A_y has a unique fixed point, say $x^* \in X$. Thus we have an unique element $x^* \in X$ such that $A_y(x^*) = Ax^* + By = x^*$ which implies that $(I - A)^{-1}By = x^*$ or, $Ty = x^*$. Thus the mapping $T : X \to X$ is well defined. Now define a sequence $\{x_n\}$ of iterates of T, that is, $x_{n+1} = Tx_n$ for $n \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}$. From hypothesis (c), we see that the mapping T is monotonic on X. So we have that

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots x_n \preceq \cdots \tag{2.4}$$

or

$$x_0 \succeq x_1 \succeq x_2 \succeq \cdots x_n \succeq \cdots . \tag{2.5}$$

Since B is compact and $(I - A)^{-1}$ is continuous, the composition mapping $T = (I - A)^{-1}B$ is partially compact and continuous on X into X. Therefore the sequence $\{x_n\}$ has a convergent subsequence converging to some point say $x^* \in X$ and from the compatibility of the order relation and the norm it follows that the whole sequence $\{x_n\}$ converges to a point x^* in X. Hence, an application of Theorem 1.5 implies that T has a fixed point. This further implies that $(I - A)^{-1}Bx^* = x^*$ or $Ax^* + Bx^* = x^*$. This completes the proof.

Theorem 2.2. *Let $(X, \preceq, \|.\|)$ be a partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|.\|$ in X are compatible. Let $A, B : X \to X$ be two monotone mappings (nondecreasing or nonincreasing) satisfying the follwing conditions:*

(a) *A is linear and bounded and A^p is partially nonlinear M-contraction for some positive integer p,;*

(b) *B is continuous and partially compact;*

(c) *There exists an x_0 such that $x_0 \preceq Ax_0 + By$ or $x_0 \succeq Ax_0 + By$ for all $y \in X$;*

(d) *Every pair of elements $x, y \in X$ has a lower and an upper bound in X. Then the operator equation $Ax + Bx = x$ has a solution.*

Proof. Define an operator T on X by

$$T = (I - A)^{-1}B. \tag{2.6}$$

Now the mapping $(I - A)^{-1}$ exists in view of the relation

$$(I - A)^{-1} = (I - A^p)^{-1} \sum_{j=1}^{p-1} A^j \tag{2.7}$$

where $\sum_{j=1}^{p-1} A^j$ is bounded and $(I-A^p)^{-1}$ exists in view of a corollary 1.4. Hence, $(I-A)^{-1}$ exists and is continuous on X. Next, the operator T is well defined. To see this, let $y \in X$ be fixed and define a mapping $A_y : X \to X$ by

$$A_y(x) = A_x + B_y. \tag{2.8}$$

Clearly A_y is monotonic on X into itself. Again by hypothesis (c) there is a point $x_0 \in X$ such that

$$x_0 \leqq A_y x_0 \leqq A_y^2 x_0 \leqq \dots \leqq A_y^p x_0$$

or

$$x_0 \geqq A_y x_0 \geqq A_y^2 x_0 \geqq \dots \geqq A_y^p x_0$$

Then for any two comparable elements $x_1, x_2 \in X$, one has

$$\|A_y^p(x_1) - A_y^p(x_2)\| \leqq \varphi_A(\|x_1 - x_2\|).$$

Hence by Corollary 1.4, there exists an unique element x^* such that $A_y^p(x^*) = A^p(x^*) + By = x^*$. This further implies that $A_y(x^*) = x^*$ and x^* is a unique fixed point of A_y. Thus we have $A_y(x^*) = x^* = Ax^* + By$ or, $(I-A)^{-1}By = x^*$. As a result, $Ty = x^*$ and so T is well defined. The rest of the proof is similar to Theorem 2.1 and we omit the details. The proof is completed.

As a special case of our Theorem 2.1 and 2.2 we obtain Theorems of Dhage [12] for partially contraction mappings in partially ordered complete metric spaces.

It is known since long time that the fixed point theory has some nice applications to nonlinear differential and integral equations for proving the existence and uniqueness theorems. An interesting topological fixed point result that has been widely used while dealing with the nonlinear equations is the following variant of nonlinear alternative due to Leray and Schauder [6].

Theorem 2.3. *Let U and $\bar{U}$ denote respectively the open and closed subset of a convex set K of a normed linear space X such that $0 \in U$ and let $N : \bar{U} \to K$ be a compact and continuous operator. Then either*

(i) *the equation $x = Nx$ has a solution in $\bar{U}$, or*

(ii) *there exists a point $u \in \partial U$ such that $u = \lambda N u$ for some $\lambda \in (0,1)$, where ∂U is a boundary of U.*

We now obtain another version of Leray and Schauder type fixed point theorems in partially-ordered complete normed linear spaces under weaker conditions, which improve Theorem Theorem 2.3.

Theorem 2.4. *Let U and $\bar{U}$ denotes respectively the open and closed subset of a partially ordered complete normed linear space $(X,\preceq,\|.\|)$ such that the order relation $\preceq$ and the norm $\|.\|$ in X are compatible and $0\in U$. Let $A : X \to X$ and $B : \bar{U} \to X$ be two monotone operators (nondecreasing or nonincreasing) satisfying the following conditions:*

(a) *A is continuous and partially nonlinear M-contraction;*

(b) *B is continuous and partially compact;*

(c) *There exists an x_0 such that $x_0 \preceq Ax_0 + By$ or $x_0 \succeq Ax_0 + By$ for all $y \in X$;*

(d) *Every pair of elements $x, y \in X$ has a lower and an upper bound in X.*

Then either

(i) *the operator equation $Ax + Bx = x$ has a solution in $\bar{U}$, or*

(ii) *there exist an $u \in \partial u$ such that $\lambda A(\frac{u}{\lambda}) + \lambda Bu = u$ for $\lambda \in (0,1)$ where ∂U is the boundary of $\bar{U}$.*

Proof. Define an operator $T : X \to X$ by

$$T = (I - A)^{-1}B$$

Clearly the operator T is well defined. To see this, let $y \in X$ be fixed and define a mapping $A_y : X \to X$ by

$$Ay(x) = Ax + By.$$

A_y is monotonic and by hypothesis (c), there is a point $x_0 \in X$ such that $x_0 \leqq Tx_0$ or $x_0 \geqq Tx_0$. Now, for any two comparable elements $x_1, x_2 \in X$, one has

$$\|A_y(x_1) - A_y(x_2)\| = \|Ax_1 - Ax_2\| \leqq \varphi_A(\|x_1 - x_2\|)$$

where, A is a M-function of T on X so A_y is partially nonlinear M-contraction on X. Hence, by an application of a fixed point Theorem 1.3, A_y has a unique fixed point, say $x^* \in X$. Thus we have a unique element $x^* \in X$ such that $A_y(x^*) = Ax^* + By = x^*$, which implies that $(I - A)^{-1}By = x^*$ or, $Ty = x^*$. Thus the mapping $T : X \to X$ is well defined. Now define a sequence

$\{x_n\}$ of iterates of T, that is, $x_{n+1} = Tx_n$ for $n \in \mathbb{N}_0 := \{0,1,2,\cdots\}$. From hypothesis (c) it follows that the mapping T is monotonic on X. So we have that

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots x_n \preceq \cdots$$

or

$$x_0 \succeq x_1 \succeq x_2 \succeq \cdots x_n \succeq \cdots$$

Since B is compact and $(I-A)^{-1}$ is continuous, the composition mapping $T = (I-A)^{-1}B$ is partially compact and continuous on X into X. Therefore the sequence $\{x_n\}$ has a convergent subsequence converging to some point say $x^* \in X$ and from the compatibility of the order relation and the norm it follows that the whole sequence $\{x_n\}$ converges to a point x^* in X. Hence by an application of Theorem 2.3 yields that either the operator equation $(I-A)^{-1}Bx = x$ has a solution in $\bar{U}$, or the operator equation $\lambda(I-A)^{-1}Bx = x$ has a solution on the boundary $\partial\bar{U}$ of $\bar{U}$ for some $\lambda \in (0,1)$. This further implies that either (i) the operator equation $Ax + Bx = x$ has a solution in $\bar{U}$, or (ii) there exist $u \in \partial u$ such that $\lambda A(\frac{u}{\lambda}) + \lambda Bu = u$ for $\lambda \in (0,1)$ where ∂U is the boundary of $\bar{U}$.

An interesting corollaries to Theorem 2.4 are as follows.

Corollary 2.5. *Let $B(0,r)$ and $B[0,r]$ denotes respectively the open and closed balls in a partially ordered complete normed linear space $(X,\preceq,\|.\|)$ such that the order relation $\preceq$ and the norm $\|.\|$ in X are compatible and $0 \in U$. Let $A : X \to X$ and $B : \bar{U} \to X$ be two monotone operators (nondecreasing or nonincreasing) satisfying the following conditions:*

(a) *A is continuous and partially contraction;*

(b) *B is continuous and partially compact;*

(c) *There exists an x_0 such that $x_0 \preceq Ax_0 + By$ or $x_0 \succeq Ax_0 + By$ for all $y \in X$;*

(d) *Every pair of elements $x, y \in X$ has a lower and an upper bound in X.*

Then either

(i) *the operator equation $Ax + Bx = x$ has a solution in $B[0,r]$, or*

(ii) *there exist an $u \in X$ with $\|u\| = r$ such that $\lambda A(\frac{u}{\lambda}) + \lambda Bu = u$ for $\lambda \in (0,1)$.*

Corollary 2.6. *Let U and $\bar{U}$ denotes respectively the open and closed subset of a partially ordered complete normed linear space $(X,\preceq,\|.\|)$ such that the order relation $\preceq$ and the norm*

$\|.\|$ *in* X *are compatible and* $0 \in U$. *Let* $A : X \to X$ *and* $B : \bar{U} \to X$ *be two monotone operators (nondecreasing or nonincreasing) satisfy*

(a) *A is linear and bounded and A^p is partially nonlinear M-contraction for some positive integer p;*

(b) *B is continuous and partially compact;*

(c) *There exists an x_0 such that $x_0 \preceq Ax_0 + By$ or $x_0 \succeq Ax_0 + By$ for all $y \in X$;*

(d) *Every pair of elements $x, y \in X$ has a lower and an upper bound in X.*

Then either

(i) *the operator equation $Ax + Bx = x$ has a solution in $\bar{U}$, or*

(ii) *there exist an $u \in \partial u$ such that $\lambda A(\frac{u}{\lambda}) + \lambda Bu = u$ for $\lambda \in (0,1)$ where ∂U is the boundary of $\bar{U}$.*

Proof. The desired conclusion of the Corollary follows by a direct application of Theorem 2.3 to the mapping $T : \bar{U} \to X$ defined by $T = (I-A)^{-1}B$. Now $(I-A)^{-1} = (I-A^p)^{-1}\sum_{j=1}^{p-1} A^j$. By hypothesis (a), A^p is a partially nonlinear M-contraction, so by Theorem 2.3, the mapping $(I-A^p)^{-1}$ exists on X and consequently the mapping $(I-A)^{-1}$ exists on X. Moreover from hypothesis (a) it follows that $(I-A)^{-1}$ is continuous on X. The rest of the proof is similar to Theorem 2.2 We omit the details.

A special case to Corollary 2.6 in its applicable form is as follows.

Corollary 2.7. *Let $B(0,r)$ and $B[0,r]$ denotes respectively the open and closed balls in a partially ordered complete normed linear space $(X, \preceq, \|.\|)$ centered at origin and of radius $r > 0$ such that the order relation $\preceq$ and the norm $\|.\|$ in X are compatible and $0 \in U$. Let $A : X \to X$ and $B : \bar{U} \to X$ be two monotone operators (nondecreasing or nonincreasing) satisfying the follwing conditions:*

(a) *A is linear and bounded and A^p is partially contraction for some positive integer p;*

(b) *B is continuous and partially compact;*

(c) *There exists an x_0 such that $x_0 \preceq Ax_0 + By$ or $x_0 \succeq Ax_0 + By$ for all $y \in X$;*

(d) *Every pair of elements $x, y \in X$ has a lower and an upper bound in X.*

Then either

(i) *the operator equation* $Ax+Bx=x$ *has a solution in* $B[0,r]$, *or*

(ii) *there exist an* $u \in X$ *with* $\|u\| = r$ *such that* $\lambda A(\frac{u}{\lambda}) + \lambda Bu = u$ *for* $\lambda \in (0,1)$.

Remark 2.1. *The hypothesis (d) of Theorems 2.1 and 2.2 holds if the partially ordered set X is a lattice. Furthermore, the space* $C(J,\mathbb{R})$ *of continuous real-valued functions on the closed and bounded interval* $J=[a,b]$ *is a lattice, where the order relation* $\leqq$ *is defined as follows. For any* $x,y \in C(J,\mathbb{R})$, $x \leqq y$ *if and only if* $x(t) \leqq y(t)$ *for all* $t \in J$. *The real variable operations shows that min*(x,y) *and max*(x,y) *are respectively the lower and upper bounds for the pair of elements x and y in X.*

In this section we apply the hybrid fixed point theorems proved in the preceding sections to some nonlinear functional integral equations.

3. Nonlinear functional integral equations

Let $M(J,\mathbb{R})$ and $B(J,\mathbb{R})$ respectively denote the spaces of measurable and bounded real-valued functions on J. We shall seek the solution of

$$x(t) = q(t) + \int_0^{\mu(t)} k(t,s) f(s,x(\theta(s)))ds + \int_0^{\sigma(t)} v(t,s) g(s,x(\eta(s)))ds \quad (t \in J) \tag{3.1}$$

where $q : J \to \mathbb{R}$, $k, \nu : J \times J \to \mathbb{R}$, $f,g : J \times \mathbb{R} \to \mathbb{R}$ and $\mu,\theta,\sigma,\eta : J \times J \to J$.

$BM(J,\mathbb{R})$ denotes all bounded and measurable real-valued functions on J.

We consider the following hypotheses:

(H_{10}) The functions $\mu,\theta,\sigma,\eta : J \to J$ are continuous.

(H_{11}) The function $q : J \to \mathbb{R}$ is bounded and measurable.

(H_{12}) The functions $\kappa, \nu : J \times J \to J$ are continuous.

(H_{13}) The function $f(t,x)$ is continuous, nondecreasing and there exists a function $\alpha \in L_1(J,\mathbb{R})$ such that $\alpha(t) > 0$, $a.e.\ t \in J$ and

$$|f(t,x) - f(t,y)| = \alpha(t)|x-y|,\ a.e.\ t \in J$$

for all $x, y \in \mathbb{R}$.

(H_{14}) The function $f(t,x)$ is continuous and satisfies

$$f(t,x) - f(t,y) \leqq \frac{L(x-y)}{K+(x-y)} \ a.e.\ t \in J$$

for all $x, y \in \mathbb{R}$ where $K = \sup_{t,s\in J} |k(t,s)|$

(H_{15}) The function $g(t,x)$ is nondecreasing and L^1-Carathéodory.

(H_{16}) There exists a continuous and nondecreasing function $\Omega : [0,\infty) \to [0,\infty)$ and a function $\phi \in L^1(J,\mathbb{R})$ such that $\phi(t) > 0,\ a.e.\ t \in J$ and

$$|g(t,x)| \leqq \phi(t)\Omega(|x|),\ a.e.\ t \in J$$

for all $x \in \mathbb{R}$.

(H_{17})

$$u_0 \leqq q(t) + \int_0^{\mu(t)} k(t,s) f(s, x(\theta(s)))ds + \int_0^{\sigma(t)} v(t,s) g(s, x(\eta(s)))ds \quad (t \in J)$$

Theorem 3.1. *Assume that the hypotheses* (H_{10}) *through* (H_{13}) *and* (H_{15}) *through* (H_{17}) *hold. Suppose that there exists a real number* $r > 0$ *such that*

$$r > \frac{\|q\| + KF + V\|\phi\|_{L^1}\Omega(r)}{1 - K\|\alpha\|_{L^1}} \quad (K\|\alpha\|_{L^1} < 1) \tag{3.2}$$

where $K = \sup_{t,s\in J} |k(t,s)|$, $V = \sup_{t,s\in J} |v(t,s)|$ *and* $F = \int_0^{\alpha} |f(s,0)|ds$. *Then the FIE (3.1) has a solution on* J.

Proof. Define an open ball $B(0,r)$ in the Banach space $X = BM(J,\mathbb{R})$ centered at the origin and of radius $r > 0$, where r satisfies the inequality (3.1). Now consider the two mappings A and B on $BM(J,\mathbb{R})$ defined by

$$Ax(t) = q(t) + \int_0^{\mu(t)} k(t,s) f(s, x(\theta(s)))ds \quad (t \in J) \tag{3.3}$$

and

$$Bx(t) = \int_0^{\sigma(t)} v(t,s) f(s, x(\eta(s)))ds \quad (t \in J). \tag{3.4}$$

Since the function μ and σ are continuous in view of the hypotheses (H_{10}) through (H_{15}), A and B define the operators $A, B : BM(J,\mathbb{R}) \to BM(J,\mathbb{R})$. We shall show that the operators A and

B satisfy all the conditions of Theorem 2.4. First show that A is a contraction on $BM(J,\mathbb{R})$. Let $x,y \in BM(J,\mathbb{R})$. Then by (H_{14}),

$$\begin{aligned}|Ax(t)-Ay(t)| &\leqq \int_0^{\mu(t)} |k(t,s)||f(s,x(\theta(s)))-f(s,y(\theta(s)))|ds \\ &\leqq K\int_0^1 \alpha(s)\|x-y\|ds \\ &\leqq K\|\alpha\|_{L^1}\|x-y\|.\end{aligned}$$

Taking the maximum over t, we see that $|Ax-Ay| \leqq K\|\alpha\|_{L^1}\|x-y\|$ for all $x,y \in BM(J,\mathbb{R})$, where $K\|\alpha\|_{L^1} < 1$. So A is a contraction on $BM(J,\mathbb{R})$. Next we show that B is completely continuous on $BM(J,\mathbb{R})$. Using the standard arguments as in Granas *et al.* [13], it is shown that B is a continuous operator on $BM(J,\mathbb{R})$. Let S be a bounded set in $BM(J,\mathbb{R})$ and let $\{x_n\}$ be a sequence in S. Then there exists a constant $r>0$ such that $\|x_n\| < r$ for all $n \in \mathbb{N}$. Now by (H_{16}),

$$\begin{aligned}|Bx_n(t)| &\leqq \int_0^{\sigma(t)} |v(t,s)||g(s,x_n(\eta(s)))|ds \\ &\leqq \int_0^{\sigma(t)} V\phi(s)\Omega(|x_n(\eta(s))|)ds \\ &\leqq V\|\phi\|_{L^1}\Omega(r)\end{aligned}$$

i.e., $\|Bx_n\| \leqq M$ for all $n \in \mathbb{N}$, where $M = V\|\phi\|_{L^1}\Omega(r)$. This shows that $\{Bx_n\}$ is a uniformly bounded sequence in $BM(J,\mathbb{R})$. Now we show $\{Bx_n\}$ is an equi-continuous set. Let $t_1,t_2 \in J$. Then by (3.3), we have

$$\begin{aligned}|Bx_n(t_1)-Bx_n(t_2)| &\leqq \left|\int_0^{\sigma(t_1)} v(t_1,s)g(s,x_n(\eta(s)))ds - \int_0^{\sigma(t_2)} v(t_2,s)g(s,x_n(\eta(s)))ds\right| \\ &\leqq \left|\int_0^{\sigma(t_1)} v(t_1,s)g(s,x_n(\eta(s)))ds - \int_0^{\sigma(t_1)} v(t_2,s)g(s,x_n(\eta(s)))ds\right| \\ &\quad + \left|\int_0^{\sigma(t_1)} v(t_2,s)g(s,x_n(\eta(s)))ds - \int_0^{\sigma(t_2)} v(t_2,s)g(s,x_n(\eta(s)))ds\right| \\ &\leqq \left|\int_0^{\sigma(t_1)} [v(t_1,s)-v(t_2,s)]g(s,x_n(\eta(s)))ds\right| \\ &\quad + \left|\int_{\sigma(t_2)}^{\sigma(t_1)} v(t_2,s)g(s,x_n(\eta(s)))ds\right|\end{aligned}$$

$$
\begin{aligned}
&\leqq \int_0^1 |v(t_1,s)-v(t_2,s)|h_r(s)ds+V|p(t_1)-p(t_2)| \\
&\leqq \|h_r\|_L^1 \int_0^1 |v(t_1,s)-v(t_2,s)|ds+V|p(t_1)-p(t_2)|,
\end{aligned}
$$

where $p(t)=\int_0^{\sigma(t)} h_r(s)ds$. From the above inequality it follows that $|Bx_n(t_1)-Bx_n(t_2)| \to 0,\ as\ t_1 \to t_2$.

Hence $\{Bx_n\}$ is equi-continuous and consequently $\{Bx_n\}$ is compact by Arzela-Ascoli theorem . Thus every sequence $\{Bx_n\}$ in $B(S)$ has a convergent subsequence. Therefore $B(S)$ is partially compact and B is completely continuous operator on $BM(J,\mathbb{R})$.

Thus all the conditions of Theorem 2.4 are satisfied and by direct application of it yields that either conclusion (i), or conclusion (ii) of Theorem 2.4 holds. We show that the conclusion (ii) is not possible. Let $u \in X$ be such that $\|u\|=r$. Then we have for any $\lambda \in (0,1)$,

$$
\begin{aligned}
u(t) &= \lambda A(\frac{u}{\lambda})(t)+\lambda Bu(t) \\
&= \lambda\left(q(t)+\int_0^{\mu(t)} k(t,s)f(s,\frac{u}{\lambda}(\theta(s)))ds\right)+\lambda\left(\int_0^{\sigma(t)} v(t,s)f(s,u(\eta(s)))ds\right) \\
&= \lambda q(t)+\lambda\int_0^{\mu(t)} k(t,s)[f(s,\frac{u}{\lambda}(\theta(s)))-f(s,0)]ds \\
&+\lambda\int_0^{\mu(t)} k(t,s)f(s,0)ds+\lambda\int_0^{\sigma(t)} v(t,s)f(s,u(\eta(s)))ds.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
u(t) &\leqq |q(t)|+\lambda\int_0^{\mu(t)} |k(t,s)||f(s,\frac{u}{\lambda}(\theta(s)))-f(s,0)|ds \\
&+\int_0^{\mu(t)} |k(t,s)||f(s,0)|ds+\lambda\int_0^{\sigma(t)} |v(t,s)||f(s,u(\eta(s)))|ds \\
&\leqq \|q\|+K\int_0^1 \alpha(s)|u(\theta(s))|ds+KF+V\int_0^1 \phi(s)\Omega(\|u\|)ds \\
&= \|q\|+K\|\alpha\|_{L^1}\|u\|+KF+V\|\phi\|_{L^1}\Omega(\|u\|)
\end{aligned}
$$

Taking the maximum over t, we have $\|u\|=\frac{\|q\|+KF+V\|\phi\|_{L^1}\Omega(\|u\|)}{1-K\|\alpha\|_{L^1}}$ or $r \leqq \frac{\|q\|+KF+V\|\phi\|_{L^1}\Omega(r)}{1-K\|\alpha\|_{L^1}}$, which is a contradiction to the inequality in (3.2). Therefore the conclusion (i) holds and consequently the FIE (3.1) has a solution on J. This completes the proof.

Theorem 3.2. *Assume that the hypotheses* (H_{10}) *through* (H_{12}) *and* (H_{14}) *through* (H_{17}) *hold. Suppose that there exists a real number* $r > 0$ *such that*

$$r > \|q\| + K(1+F) + V\|\phi\|_{L^1}\Omega(r), \tag{3.5}$$

where $K = \sup_{t,s\in J}|k(t,s)|$, $V = \sup_{t,s\in J}|v(t,s)|$ *and* $F = \int_0^\alpha |f(s,0)|ds$. *Then the FIE (3.1) has a solution on J.*

Proof. Define an open ball $B(0,r)$ in the Banach space $X = BM(J,\mathbb{R})$ centered at the origin and of radius $r > 0$, where r satisfies the inequalities in (3.5). Now consider the two mappings A and B on $BM(J,\mathbb{R})$ defined by (3.3) and (3.4) respectively. We shall show that A and B satisfy all the conditions of Theorem 2.4 on $U = B[0,r]$. First we shall show that A is a nonlinear contraction on X. Let $x, y \in X$. By hypothesis (H_{14}),

$$\begin{aligned}
|Ax(t) - Ay(t)| &\leqq \int_0^{\mu(t)} |k(t,s)||f(s,x(\theta(s))) - f(s,y(\theta(s)))|ds \\
&\leqq \int_0^{\mu(t)} |k(t,s)|\left(\frac{|x(\theta(s)) - y(\theta(s))|}{K + |x(\theta(s)) - y(\theta(s))|}\right)ds \\
&\leqq K\left(\frac{L\|x-y\|}{K + \|x-y\|}\right).
\end{aligned}$$

Taking the supremum over t, we get $\|A(x) - B(x)\| \leqq \varphi(\|x-y\|)$, where $\varphi(r) = \frac{KLr}{K+r},\ r > 0$. This shows that A is a nonlinear M-contraction on X. Again by giving the arguments as in the proof of Theorem 3.1 it is proved that the operator B is completely continuous on $B[0,r]$. Thus all the conditions Theorem 2.4 are satisfied and a direct application of it yields that either conclusion (i), or conclusion (ii) of Theorem 2.4 holds. We show that the conclusion (ii) is not possible. Let $u \in X$ be such that $\|u\| = r$. Then we have for any $\lambda \in (0,1)$,

$$\begin{aligned}
u(t) &= \lambda A(\frac{u}{\lambda})(t) + \lambda Bu(t) \\
&= \lambda\left(q(t) + \int_0^{\mu(t)} k(t,s)f(s,\frac{u}{\lambda}(\theta(s)))ds\right) + \lambda\left(\int_0^{\sigma(t)} v(t,s)f(s,u(\eta(s)))ds\right) \\
&= \lambda q(t) + \lambda\int_0^{\mu(t)} k(t,s)f\left(s,\frac{u}{\lambda}(\theta(s))\right)ds - \lambda\int_0^{\mu(t)} k(t,s)f(s,0)ds \\
&+ \lambda\int_0^{\mu(t)} k(t,s)f(s,0)ds + \lambda\int_0^{\sigma(t)} v(t,s)f(s,u(\eta(s)))ds
\end{aligned}$$

$$
\begin{aligned}
&= \lambda q(t) + \lambda \int_0^{\mu(t)} k(t,s) \left[f\left(s, \frac{u}{\lambda}(\theta(s))\right) - f(s,0) \right] ds \\
&+ \lambda \int_0^{\mu(t)} k(t,s) f(s,0) ds + \int_0^{\sigma(t)} v(t,s) f(s,u(\eta(s))) ds.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
u(t) &\leqq |q(t)| + \lambda \int_0^{\mu(t)} |k(t,s)||f(s, \frac{u}{\lambda}(\theta(s))) - f(s,0)| ds \\
&+ \int_0^{\mu(t)} |k(t,s)||f(s,0)| ds + \lambda \int_0^{\sigma(t)} |v(t,s)||f(s,u(\eta(s)))| ds \\
&\leqq \|q\| + K + KF + V \int_0^1 \phi(s) \Omega(\|u\|) ds \\
&= \|q\| + K + KF + V \|\phi\|_{L^1} \Omega(\|u\|).
\end{aligned}
$$

Taking the maximum over t, we see that $\|u\| = \|q\| + K + KF + V\|\phi\|_{L^1}\Omega(\|u\|)$ or $r \leqq \|q\| + K + KF + V\|\phi\|_{L^1}\Omega(r)$, which is a contradiction to the inequality in (3.5). Therefore the conclusion (i) holds and consequently the FIE (3.1) has a solution on J. This completes the proof.

4. Applications

In this section we shall discuss some applications of the main result of the previous section to the initial and boundary value problem of nonlinear ordinary differential equations.

4.1. Initial value problems

Consider the initial value problem of first order ordinary functional differential equations (in short FIVP) (4.1)

$$
\begin{cases}
x'(t) = f(t, x(\theta(t))) + g(t, x(\theta(t))), \ a.e. \ t \in J, \\
x(0) = x_0 \in \mathbb{R}.
\end{cases}
\tag{4.1}
$$

where $\theta, \eta : J \to J$ are continuous and $f, g : J \times \mathbb{R} \to \mathbb{R}$.

By the solution the FIVP (4.1) we mean a function $x \in AC(J, \mathbb{R})$ that satisfies the equations in (4.1), where $AC(J, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on J.

Theorem 4.1. *Assume that the hypotheses* $(\mathrm{H}_{13}),(\mathrm{H}_{15})$ *through* (H_{17}) *hold. Further if there exists a real number* $r > 0$ *such that*

$$r > \frac{\|x_0\| + F + \|\phi\|_{L^1}\Omega(r)}{1 - \|\alpha\|_{L^1}} \quad (|\alpha\|_{L^1} < 1) \tag{4.2}$$

where $F = \int_0^1 |F(t,0)|ds$*. Then the FIVP(4.1) has a solution on J.*

Proof. The FIVP (4.1) is equivalent to the FIE

$$x'(t) = x_0 + \int_0^t f(s,x(\theta(s)))ds + \int_0^t g(s,x(\eta(s)))ds \tag{4.3}$$

for $t \in J$. Now the desired conclusion follows by an application of Theorem 3.1 with $q(t) = x_0$, $\mu(t) = t = \sigma$ for all $t \in J$ since $AC(J,\mathbb{R}) \subset BM(J,\mathbb{R})$.

Next we consider the functional IVP of neutral type, namely

$$\begin{cases} (x(t) - f(t,x(\theta(t))))' = g(t,x(\eta(t))), \ a.e.\ t \in J, \\ x(0) = x_0 \in \mathbb{R}. \end{cases} \tag{4.4}$$

where $\theta, \eta : J \to J$ are continuous with $\theta(0) = 0$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$. By the solution of the IVP (4.4) is a function $x \in AC(J,\mathbb{R})$ and that satisfies the equations in (4.4).

Theorem 4.2. *Assume that the hypotheses* (H_{15}) *through* (H_{17}) *hold. Further suppose that there exists a real number* $r > 0$ *such that*

$$r > \frac{D + F + \|\phi\|_{L^1}\Omega(r)}{1 - \|\alpha\|} \quad (\|\alpha\| < 1) \tag{4.5}$$

where $D = |x_0 - f(0,x_0)| + \sup_{t \in J}|f(s,0)|$*. Then the functional IVP (4.4) has a solution on J.*

Proof. Now the FIVP (4.4) is equivalent to the FIE

$$x(t) = x_0 - f(0,x_0) + f(t,x(\theta(t)))ds + \int_0^t g(s,x(\eta(s)))ds \tag{4.6}$$

for $t \in J$. If we define a function $k : J \times \mathbb{R} \to \mathbb{R}$ by $k(t,x(\theta(t))) = x_0 - f(0,x_0) + f(t,x(\theta(t)))$, then FIE (4.6) reduces to FIE (3.7) with $\sigma(t) = t,\ t \in J$. Now the desired conclusion follows by an application of Theorem 3.2. The proof is complete.

4.2. Boundary value problems

Consider the functional two point boundary value problems (in short BVPs) of second order differential equations

$$\begin{cases} -x''(t) = f(t,x(\theta(t))) + g(t,x(\eta(t))),\ a.e.\ t \in J, \\ x(0) = 0 = x(1). \end{cases} \tag{4.7}$$

and

$$\begin{cases} -x''(t) = f(t,x(\theta(t))) + g(t,x(\eta(t))),\ a.e.\ t \in J, \\ x(0) = 0 = x'(1) \end{cases} \tag{4.8}$$

where $\theta, \eta : J \to J$ are continuous and $f, g : J \times \mathbb{R} \to \mathbb{R}$.

By the solution of the BVP (4.7) or (4.8) we mean a function $x \in AC^1(J,\mathbb{R})$ that satisfies the equations in (4.7) and (4.8), where $AC^1(J,\mathbb{R})$ is the space of all continuous real-valued functions on J whose first derivative exists and is absolutely continuous on J.

Theorem 4.3. *Assume that the hypotheses* (H_{13}) *and* (H_{15}) *through* (H_{17}) *hold. Further suppose that there exists a real number* $r > 0$ *such that*

$$r > \frac{F + \|\phi\|_{L^1}\Omega(r)}{4 - \|\alpha\|_{L^1}}, \tag{4.9),}$$

$\|\alpha\|_{L^1} < 4$, *and* $F = \int_0^1 |f(t,0)|ds$. *Then the FBVP (4.7) has a solution on* J.

Proof.

$$x(t) = \int_0^1 G(t,s)f(s,x(\theta(s)))ds + \int_0^1 G(t,s)g(s,x(\eta(s)))ds \tag{4.10}$$

for all $t \in J$, where $G(t,s)$ is a Greens function associated with the linear homogeneous BVP

$$\begin{cases} -x''(t) = 0,\ a.e.\ t \in J, \\ x(0) = 0 = x(1). \end{cases} \tag{4.11}$$

and given by

$$G(t,s) = \begin{cases} s(1-t) & \text{if } 0 \leqq s \leq t \leqq 1, \\ t(1-s) & \text{if } 0 \leqq t \leq s \leqq 1. \end{cases} \tag{4.12}$$

It is clear that the function $G(t,s)$ is continuous and nonnegative on $J\times J$ and satisfies the inequality

$$|G(t,s)| = G(t,s) \leqq \frac{1}{4} \tag{4.13}$$

for all $t,s \in J\times J$. See Bailey *et al.* [2].

Now the functions involved in (4.10) satisfy all the conditions of Theorem 3.1 with $q(t)=0$ on J, $\mu(t)=1=\sigma(t)$ for all $t\in J$ and $k(t,s)=G(t,s)=v(t,s)$ for all $t,s\in J$. Hence an application of it yields that FBVP (4.7) has a solution on J.

Theorem 4.4. *Assume that the hypotheses* (H_{13}) *and* (H_{15}) *through* (H_{17}) *hold. Further suppose that there exists a real number* $r>0$ *such that*

$$r > \frac{F+\|\phi\|_{L^1}\Omega(r)}{1-\|\alpha\|_{L^1}}, \tag{4.14}$$

$\|\alpha\|_{L^1}<1$, *and* $F=\int_0^1 |f(t,0)|ds$. *Then the FBVP (4.8) has a solution on J.*

Proof.

$$x(t) = \int_0^1 H(t,s)f(s,x(\theta(s)))ds + \int_0^1 H(t,s)g(s,x(\eta(s)))ds \tag{4.15}$$

for all $t\in J$, where $H(t,s)$ is a Greens function associated with the linear homogeneous BVP

$$\begin{cases} -x''(t)=0,\ a.e.\ \ t\in J,\\ x(0)=0=x'(1). \end{cases} \tag{4.16}$$

and given by

$$H(t,s)=\begin{cases} s\ \text{ if } 0\leqq s\leqq t\leqq 1,\\ t\ \text{ if } 0\leqq t\leqq s\leqq 1. \end{cases} \tag{4.17}$$

It is clear that the function $H(t,s)$ is continuous and nonnegative on $J\times J$ and satisfies the inequality $|H(t,s)|=H(t,s)\leqq 1$. Now an application of Theorem 3.1 with $q(t)=0, k(t,s)=H(t,s)=v(t,s)$ for all $t,s\in J$ and $\mu(t)=1=\sigma(t)$ for all $t\in J$ yields that the FIE (4.15) has a solution on J. Clearly this solution belongs to the space $AC^1(J,\mathbb{R})$. Consequently the FBVP (4.8) has a solution on J. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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