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SOME NONUNIQUE RANDOM FIXED POINT THEOREMS IN POLISH SPACES

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Abstract. In this paper, we prove the existence of some nonunique random fixed point theorems for random mappings in the context of separable complete G-metric spaces. Our study includes the special cases of orbitally complete G-metric spaces, G-metric spaces with two metrics and the G-metric spaces satisfying the minimal class condition.

Keywords: PPF dependence; orbitally complete G-metric space; random mapping; fixed point theorem.

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1. Introduction and preliminaries

The notion of G-metric spaces was introduced by Mustafa and Sims [10] as a generalization of the notion of metric spaces. Many other authors also studied fixed point results in G-metric spaces; see [1], [9], [13] and the references therein. In fact the study of common fixed points

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of mappings satisfying certain contractive conditions has been at the center of strong research activity. The following definition is introduced by Mustafa and Sims [10].

Definition 1.1. [10] Let X be a nonempty set and let $G : X \times X \times X \to \mathbb{R}$ be a function satisfying the following properties:

(G_1) $G(x,y,z) = 0$ if $x = y = z$;

(G_2) $0 < G(x,x,y)$, for all $x,y \in X$ with $x \neq y$;

(G_3) $G(x,x,y) \leq G(x,y,z)$, for all $x,y,z \in X$ with $y \neq z$;

(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = ...$, symmetry in all three variables;

(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 1.2. [10] Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, and we say that the sequence $\{x_n\}$ is G-convergent to x or $\{x_n\}$ G-converges to x. Thus, $x_n \to x$ in a G-metric space (X,G) if for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x,x_n,x_m) < \varepsilon$ for all $m,n \geq k$.

Proposition 1.3. [10] Let (X,G) be a G-metric space. Then the following are equivalent:

(1) $\{x_n\}$ is G-convergent to x;

(2) $G(x_n,x_n,x) \to 0$ as $n \to \infty$;

(3) $G(x_n,x,x) \to 0$ as $n \to \infty$;

(4) $G(x_n,x_m,x) \to 0$ as $n,m \to \infty$.

Definition 1.4. Let (X,G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$, for all $n,m,l \geq k$, that is $G(x_n,x_m,x_l) < 0$ as $n,m,l \to +\infty$.

Proposition 1.5. [10] Let (X,G) be a G-metric space. Then the following are equivalent:

(1) the sequence $\{x_n\}$ is G-Cauchy;

(2) for every $\in> 0$, there is $k \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$, for all $n,m \geq k$.

Definition 1.6. [10] Let (X,G) and (X',G') be G-metric spaces and let $f:(X,G)\to(X',G')$ be a function. Then f is said to be G-continuous at a point $a\in X$ if and only if for every $\varepsilon>0$, there is $\delta>0$ such that $x,y\in X$ and $G(a,x,y)<\delta$ implies $G'(f(a),f(x),f(y))<\varepsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a\in X$.

Proposition 1.7. [10] Let (X,G) be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables. Every G-metric on X will define a metric d_G on X by

$$d_G(x,y)\le G(x,y,y)+G(y,x,x),\quad \forall\, x,y\in X. \tag{1.1}$$

For a symmetric G-metric space,

$$d_G(x,y)=2G(x,y,y),\quad \forall\, x,y\in X. \tag{1.2}$$

However, if G is not symmetric, then the following inequality holds:

$$\frac{3}{2}G(x,y,y)\le d_G(x,y)\le 3G(x,y,y),\quad \forall x,y\in X. \tag{1.3}$$

The following are examples of G-metric spaces.

Example 1.8. [10] Let $(\mathbb{R},d)$ be the usual metric space. Define G_S by

$$G_s(x,y,z)\le d(x,y)+d(y,z)+d(x,z),\quad \forall x,y,z\in\mathbb{R}. \tag{1.4}$$

Then it is clear that $(\mathbb{R},G_s)$ is a G-metric space.

Example 1.9. [10] Let $X=\{a,b\}$. Define G on $X\times X\times X$ by

$$G(a,a,a)=G(b,b,b)=0,$$

$$G(a,a,b)=1,\ G(a,b,b)=2 \tag{1.5}$$

and extend G to $X\times X\times X$ by using the symmetry in the variables. Then it is clear that (X,G) is a G-metric space.

Definition 1.10. Let $(X,\le)$ be a partially ordered set, (X,G) be a G-metric space. A partially ordered G-metric space $(X,G,\le)$ is called ordered complete if for each convergent sequence $\{x_n\}_{n=1}^{\infty}\subset X$, the following conditions hold:

i) if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x^*$, then $x_n \leq x^*$ $\forall n \in N$,

ii) if $\{y_n\}$ is a non-increasing sequence in X such that $y_n \to y^*$, then $y^* \leq y_n$ $\forall n \in N$.

Definition 1.11. [10] A G-metric space (X,G) is called G-complete if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

Let (X,d) be a G-metric space and let $T : X \to X$ be a mapping. Given an element $x \in X$, we define an orbit $\mathscr{O}(x;T)$ of T at x by

$$\mathscr{O}(x;T) = \{x, Tx, T^2x, ..., T^nx, \ldots\}. \tag{1.6}$$

Then T is called T-orbitally continuous on X if for any sequence $\{x_n\} \subseteq \mathscr{O}(x;T)$, we have that $x_n \to x^*$ implies $Tx_n \to Tx^*$ for each $x \in X$. The G-metric space X is called T-orbitally complete if every Cauchy sequence $\{x_n\} \subseteq \mathscr{O}(x;T)$ converges to a point x^* in X. Notice that continuity implies that T-orbitally continuity and completeness implies T-orbitally completeness of a G-metric space X, but the converse may not be true.

Ćirić [5] proved the following nonunique fixed point theorem for T-orbitally continuous mappings in T-orbitally complete metric spaces.

Theorem 1.12. [5] *Let $T : X \to X$ be a mapping satisfying*

$$\begin{aligned} &\min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\} \\ &- \min\{d(x,Ty), d(y,Tx)\} \leq qd(x,y) \end{aligned} \tag{1.7}$$

for all $x, y \in X$, where $0 \leq q < 1$. Further, if T is T-orbitally continuous and X is T-orbitally complete, then T has a fixed point.

The purpose of the present paper is to prove the nonunique fixed point theorems of above type for random mappings in a polish space in different directions.

2. Random mappings with a nonunique random fixed point

Throughout the rest of the paper, let X denote a polish space, *i.e.*, a complete, separable G-metric space with a metric d. Let $(\Omega, \mathscr{A})$ denote a measurable space with σ-algebra $\mathscr{A}$. A function $x : \Omega \to X$ is said to be a random variable if it is measurable. A mapping $T : \Omega \times X \to X$

is called random mapping if $T(.,x)$ is measurable for each $x \in X$. A random mapping on a G-metric space X is denoted by $T(\omega,x)$ or simply $T(\omega)x$ for $\omega \in \Omega$ and $x \in X$. A random mapping $T(\omega)$ is said to be continuous on X into itself if the mapping $T(\omega,\cdot)$ is continuous on X for each $\omega \in \Omega$. A measurable function $x : \Omega \to X$ is called a random fixed point of the random mapping $T(\omega)$ if $T(\omega)x(\omega) = x(\omega)$ for all $\omega \in \Omega$. Given a random variable $x : \Omega \to X$, by a $T(\omega)$-orbit of $T(\omega)$ at x, we mean a set

$$\mathscr{O}(x;T(\omega)) = \big\{x(\omega), T(\omega)x(\omega), T^2(\omega)x(\omega), \dots\big\}, \tag{2.1}$$

for $\omega \in \Omega$. A random mapping $T : \Omega \times X \to X$ is called $T(\omega)$-orbitally continuous, if a sequence $\{x_n\}$ of measurable functions in $\mathscr{O}(x;T(\omega))$ converses to x implies that $T(\omega)x_n \to T(\omega)x$ for each $\omega \in \Omega$. The G-metric space X is called $T(\omega)$-orbitally complete if every Cauchy sequence of measurable functions $\{x_n\}$ in $\mathscr{O}(x;T(\omega))$ converges to a measurable function x on Ω into X.

The following theorem is essential and frequently used in the theory of random equations and random fixed point theory for random operators in Polish spaces.

Theorem 2.1. *Let X be a Polish space, that is, a complete and separable metric space. Then, the following statements hold in X.*

(a) *If $\{x_n(\omega)\}$ is a sequence of random variables converging to $x(\omega)$ for all $\omega \in \Omega$, then $x(\omega)$ is also a random variable.*

(b) *If $T(\omega,\cdot)$ is continuous for each $\omega \in \Omega$ and $x : \Omega \to X$ is a random variable, then $T(\omega)x$ is also a random variable.*

Theorem 2.2. *Let $T(\omega)$ be a $T(\omega)$-orbitally continuous random mapping on $T(\omega)$-orbitally complete and seperable G-metric spaces X into itself. Satisfying for each $\omega \in \Omega$*

$$\begin{aligned}
&\min\big\{G(T(\omega)x,T(\omega)y,T(\omega)z), G(x,T(\omega)x,T(\omega)y), G(y,T(\omega)y,T(\omega)z), G(z,T(\omega)z,T(\omega)x)\big\}\\
&-\min\big\{G(x,T(\omega)z,T(\omega)z), G(T(\omega)x,y,T(\omega)x), G(T(\omega)y,T(\omega)y,z)\big\}\\
&\le q(\omega)G(x,y,z)
\end{aligned} \tag{2.2}$$

for all $x,y,z \in X$, where $q : \Omega \to \mathbb{R}^+$ is measurable function satisfying $0 \le q(\omega) \le 1$. Then $T(\omega)$ has a random fixed point.

Proof. Let $x:\Omega\to X$ be an arbitrary measurable function and consider the sequence $\{x_n\}$ of successive iterates of $T(\omega)$ at x defined by

$$x=x_0,\ x_1=T(\omega)x_0,\ \ldots .\ x_n=T(\omega)x_{n-1} \tag{2.3}$$

for each $n\in\mathbb{N}$. Clearly, $\{x_n\}$ is a sequence of measurable functions on Ω into X. We shall show that $\{x_n\}$ is a cauchy sequence in X. Taking $x=x_0,\ y=x_1$ and $z=x_2$ in (2.2), we obtain

$$\begin{aligned}&\min\big\{G(T(\omega)x_0,T(\omega)x_1,T(\omega)x_2),G(x_0,T(\omega)x_0,T(\omega)x_1),\\&G(x_1,T(\omega)x_1,T(\omega)x_2),G(x_2,T(\omega)x_2,T(\omega)x_0)\big\}\\&-\min\big\{G(x_0,T(\omega)x_2,T(\omega)x_2),G(T(\omega)x_0,x_1,T(\omega)x_0),\\&G(T(\omega)x_1,T(\omega)x_1,x_2)\big\}\\&\le q(\omega)G(x_0,x_1,x_2),\end{aligned}$$

which further gives

$$\min\big\{G(x_1,x_2,x_3),G(x_0,x_1,x_2),G(x_1,x_2,x_3),G(x_2,x_3,x_1)\big\}$$

$$-\min\big\{G(x_0,x_3,x_3),G(x_1,x_1,x_1),G(x_2,x_2,x_2)\big\}\le qG(x_0,x_1,x_2)$$

$$\min\big\{G(x_1,x_2,x_3),G(x_0,x_1,x_2)\big\}\le qG(x_0,x_1,x_2).$$

Since $G(x_0,x_1,x_2)\le qG(x_0,x_1,x_2)$ is not possible in view of $q<1$, one has $G(x_1,x_2,x_3)\le qG(x_0,x_1,x_2)$. Proceeding in this way by induction, we see that

$$G(x_n,x_{n+1},x_{n+2})\le qG(x_{n-1},x_n,x_{n+1}) \tag{2.4}$$

for each $n\in\mathbb{N}$. From (2.4) it follows that

$$\begin{aligned}G(x_n,x_{n+1},x_{n+2})&\le qG(x_{n-1},x_n,x_{n+1})\\&\le q^2G(x_{n-2},x_{n-1},x_n)\\&\ \vdots\\&\le q^nG(x_0,x_1,x_2).\end{aligned} \tag{2.5}$$

Now, we obtain by triangle inequality

$$
\begin{aligned}
G(x_n,x_m,x_m) &\leq G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_{n+2},x_{n+2}) \\
&\quad + G(x_{n+2},x_{n+3},x_{n+3}) + \ldots + G(x_{m-1},x_m,x_m) \\
&\leq (q^n + q^{n+1} + \ldots\ldots + q^{m-1})G(x_0,x_1,x_2) \\
&\leq \frac{q^n}{1-q} G(x_0,x_1,x_1).
\end{aligned}
\tag{2.6}
$$

Letting $n,m \to \infty$, we find that $G(x_n,x_m,x_m) \to 0$. This shows that $\{x_n\}$ is a cauchy sequence in X. The metric space X being $T(\omega)$-orbitally complete, there is a measurable function $x^* : \omega \to X$ in X such that $\lim_{n\to\infty} x_n = x^*$ Again as $T(\omega)$ is $T(\omega)$-orbitally continuous we have

$$
T(\omega)x^*(\omega) = \lim_{n\to\infty} T(\omega)x_n(\omega) = \lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega), \quad \forall \omega \in \Omega.
$$

Thus x^* is a random fixed point of a random mapping $T(\omega)$ on X into itself. This completes the proof.

Corollary 2.3. *Let $T(\omega)$ be a $T(\omega)$-orbitally continuous random mapping on a $T(\omega)$-orbitally complete and separable G-metric space X into itself satisfying for each $\omega \in \Omega$,*

$$
G(T(\omega)x, T(\omega)y, T(\omega)z) \leq q(\omega)\, G(x,y,z)
$$

for all $x,y,z \in X$, where $q : \Omega \to \mathbb{R}^+$ is a measurable function satisfying $0 \leq q(\omega) < 1$. Then $T(\omega)$ has a random fixed point.

When $T(\omega)x = Tx$ for all $\omega \in \Omega$ in Corollary 2.3, we obtain the corresponding results in Mustafa [9] as a corollary which again includes the famous Banach fixed point theorem for contraction mappings on a metric space X into X.

Theorem 2.4. *Let $T(\omega)$ be a $T(\omega)$-orbitally continuous random selfmapping of a $T(\omega)$-orbitally complete and separable G-metric space X satisfying for each $\omega \in \Omega$,*

$$\begin{aligned}
&\min\big\{[G(T(\omega)x,T(\omega)y,T(\omega)z)]^2, G(T(\omega)x,T(\omega)y,T(\omega)z)G(x,y,z),\\
&\quad d(x,T(\omega)x,T(\omega)y)G(y,T(\omega)y,T(\omega)z)\big\}\\
&-\min\big\{G(x,T(\omega)x,T(\omega)y)G(y,T(\omega)y,T(\omega)z)G(z,T(\omega)z,T(\omega)x),\\
&\quad G(x,T(\omega)z,T(\omega)z)G(y,T(\omega)x,T(\omega)x)G(z,T(\omega)y,T(\omega)y)\big\}\\
&\leq q(\omega)G(x,T(\omega)x,T(\omega)y)G(y,T(\omega)y,T(\omega)z)G(z,T(\omega)z,T(\omega)x)
\end{aligned} \tag{2.7}$$

for all $x,y,z \in X$, where $q : \Omega \to \mathbb{R}^+$ is a measurable function satisfying $0 \leq q(\omega) < 1$. Then $T(\omega)$ has a random fixed point.

Proof. The proof is similar to Theorem 2.2. Therefore, we omit the details here.

As a consequence of Theorem 2.2 we obtain the following corollary.

Corollary 2.5. *Let T be a T-orbitally continuous selfmapping of a T-orbitally complete metric space X satisfying*

$$\begin{aligned}
&min\big\{[G(T(\omega)x,T(\omega)y,T(\omega)z)]^2, G(x,T(\omega)x,T(\omega)y)G(y,T(\omega)y,T(\omega)z)G(z,T(\omega)z,\\
&\quad T(\omega)x), G(T(\omega)x,T(\omega)y,T(\omega)z)G(x,y,z)\big\}\\
&-min\big\{G(x,T(\omega)z,T(\omega)z)G(y,T(\omega)x,T(\omega)x)G(z,T(\omega)y,T(\omega)y),\\
&\quad G(x,T(\omega)x,T(\omega)y)G(y,T(\omega)y,T(\omega)z), G(z,T(\omega)z,T(\omega)x)\big\}\\
&\leq q(\omega)G(x,T(\omega)x,T(\omega)y)G(y,T(\omega)y,T(\omega)z)G(z,T(\omega)z,T(\omega)x)
\end{aligned} \tag{2.8}$$

for all $x,y \in X$, where $0 \leq q(\omega) < 1$. Then T has a fixed point.

Sometimes it possible that a metric space may be complete w.r.t. a metric but may not be complete w.r.t. another metric defined on it. Therefore, it is interesting to obtain the fixed point theorems in such situation. Next we prove a couple of nonunique random fixed point theorem in a metric space with two metrics defined on it.

Theorem 2.6. *Let X be a G-metric space with two metrics G_1 and G_2. Let (Ω, A) be a measurable space and $T : \Omega \times X \to X$ be a random mapping satisfying the condition (2.2) w.r.t. G_2 for* *each $\omega \in \Omega$. Further suppose that*

(i) $G_1(x,y,z) \leq G_2(x,y,z)$ for all $x,y \in X$;

(ii) $T(\omega)$ is a $T(\omega)$-orbitally continuous w.r.t.G_1;

(iii) X is $T(\omega)$-orbitally complete w.r.t.G_1 and

(iv) X is separable G-metric space.

Then $T(\omega)$ has random fixed point.

Proof. Let $x : \Omega \to X$ be an arbitrarily measurable function and consider the sequence $\{x_n\}$ of successive iterates of $T(\omega)$ at x defined by

$$x = x_0,\ x_1 = T(\omega)x_0,\ ....\ x_n = T(\omega)x_{n-1}$$

for each $n \in \mathbb{N}$. clearly $\{x_n\}$ is a sequence of measurable functions on Ω into X. Now proceeding as in the proof of theorem (2.1) we obtain $G_2(x_n,x_m,x_m) \leq \frac{q^n}{1-q}$. Now by hypothesis (i) we have

$$G_1(x_n,x_m,x_m) \leq G_2(x_n,x_m,x_m) \leq \frac{q^n}{1-q} G_2(x_0,x_1,x_1).$$

Letting $n,m \to \infty$, we find that $G_1(x_n,x_m,x_m) \to 0$. This shows that $\{x_n\}$ is a cauchy sequence in X w.r.t. G_1. The G-metric space (X,G_1) being $T(\omega)$-orbitally complete. There is a measurable function $x^* : \Omega \to X$ such that $\lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega)$. For each $\omega \in \Omega$, from the above limit, it follows that

$$T(\omega)x^*(\omega) = \lim_{n\to\infty} T(\omega)x_n(\omega) = \lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega)$$

for each $\omega \in \Omega$. Thus $T(\omega)$ has random fixed point and the proof of Theorem 2.3 is completed.

Example 2.4. *Let $X = \{0,1,2,3\}$ and G be a G-metric on X given by $G(x,y,z) \leq d(x,y) + d(y,z) + d(x,z)$ for all $x,y,z \in X$. Define $T(\omega) : X \to X$ by $T_0 = T_1 = T_2 = 0$ and $T_3 = 1$. Since (X,G) is complete and separable G-metric space, then it is $T(\omega)$-orbitally complete and separable G-metric space. Obviously $T(\omega)$ is continuous with respect to G, so it is orbitally* *continuous. An easy computation shows that*

$$\begin{aligned}
& min\{G(T_1(\omega),T_2(\omega),T_3(\omega)),G(1,T_1(\omega),T_2(\omega)),\\
&\quad G(2,T_2(\omega),T_3(\omega)),G(3,T_3(\omega)),T_1(\omega))\}\\
&-min\{G(1,T_3(\omega),T_3(\omega)),G(T_1(\omega),2,T_1(\omega)),\\
&\quad G(T_2(\omega),T_2(\omega),3)\}\\
&= min\{G(0,0,1),G(1,0,0),G(2,0,1),G(3,1,0)\}\\
&-min\{G(1,1,1),G(0,2,0),G(0,0,3)\}\\
&\le min\{2,2,4,6\}-min\{0,4,6\}\\
&\le 2\\
&\le q(\omega)\{4\}\\
&= q(\omega)G(x_0,x_1,x_2)
\end{aligned}$$

for all $x,y,z \in X$. So the conditions of Theorem 2.2 are satisfied.

3. Nonunique PPF dependant random fixed point theory

The fixed point theory of nonlinear operators with PPF dependence which is depending upon past, present and future data was developed in Bernfield *et. al*. [3]. The domain space of the nonlinear operator was taken as $C(I,E)$, $I=[a,b]\subset\mathbb{R}$ and the range space as E, a Banach space. An important example of such a nonlinear operator is a delay differential equation. The PPF dependent fixed point theorems are applied to ordinary nonlinear functional differential equations for proving the existence of solutions. Random fixed point theory for random operators in separable Banach spaces is initiated by Hans [6] and Spacek [15] and further developed by several authors in the literature. A brief survey of such random fixed point theorems appears in Joshi and Bose [7].

In the present section, we obtain a successful fusion of above two ideas and prove some PPF dependent random fixed point theorems for random mappings in a separable G-metric space. In the PPF dependent classical fixed point theory, the Razumikkin or minimal class of functions plays a significant role both in proving existence as well as uniqueness of PPF dependent fixed points. Let E be a metric space and let I be a given closed and bounded interval in $\mathbb{R}$, the set of real numbers. Let $E_0 = C(I,E)$ denote the class of continuous mappings from I to E. We equip the class $C(J,E)$ with metric d_0 defined by $d_0(x,y) = \sup_{t\in J} d(x(t),y(t))$.

Lemma 3.1. *If (E,G) is complete then the G-metric space (E_0,G_0) is also complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence. Then for $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $G_0(\phi_m,\phi_n,\phi_n) < \varepsilon$ for all $m > n \geq n_0$. Since $G(\phi_m(t),\phi_n(t),\phi_n(t)) < \varepsilon$ for all $m,n \geq n_0$, we have $\{\phi_n(t)\}$ is a Cauchy sequence in E. So there exists a function $\phi^* \in E_0$ such that $\lim_{n\to\infty} \phi_n(t) = \phi^*(t)$ for all $t \in J$. Now $\lim_{n\to\infty} G(\phi_n,\phi,\phi) = \lim_{n\to\infty} \sup_{t\in J} G(\phi_n(t),\phi^*(t),\phi^*(t)) = 0$. Hence $\phi_n \to \phi$ in E_0 and the proof of the lemma is completed.

When E is a Banach space and let $E_0 = C(J,E)$ be a space of continuous E-valued functions defined on J Then the minimal class of functions related to a fixed $c \in J$ is defined as

$$\mathscr{M}_c = \left\{\phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E\right\}.$$

Now we are in a position to state and prove our random fixed point results concerning the existence of random fixed points with PPF dependence.

Theorem 3.2. *Let $(\Omega,\mathscr{A})$ be a measurable space and E, be a separable complete G-metric space. Let $T : \Omega \times E_0 \to E$ be a continuous random mapping satisfying for each $\omega \in \Omega$*

$$\begin{aligned}
&\min\big\{G(T(\omega)\phi,T(\omega)\psi,T(\omega)\xi),G(\phi(c,w),T(\omega)\phi,T(\omega)\psi),\\
&\qquad G(\psi(c,w),T(\omega)\psi,T(\omega)\xi),G(\xi(c,w),T(\omega)\xi,T(\omega)\phi)\big\}\\
&-\min\big\{G(\phi(c,w),T(\omega)\xi),T(\omega)\xi),G(T(\omega)\phi,\psi(c,w),T(\omega)\phi),\\
&\qquad G(T(\omega)\psi,T(\omega)\psi,\xi(c,w))\big\}\\
&\leq q(\omega)G(\phi,\psi,\xi)
\end{aligned}\tag{3.1}$$

for all $\phi,\psi,\xi \in E_0$, where $q : \Omega \to \mathbb{R}^+$ is a measurable function satisfying $0 \leq q(\omega) < 1$ for all $\omega \in \Omega$ and $c \in I$ is a fixed point. Then $T(\omega)$ has a random fixed point with PPF dependence.

Proof. Let $\phi_0 : \Omega \to E_0$ be an arbitrary measurable function and define a sequence $\{x_n\}$ in E_0 as follows. Suppose that $T(\omega)\phi_0 = x_1$ for some $x_1 \in E$. choose $\phi_1 \in E_0$ such that $\phi_1(c,\omega) = x_1$

and then choose $\phi_2(c,\omega) = x_2$ for some fixed $c \in I$ and

$$G_0(\phi_0,\phi_1,\phi_2) = G(\phi_0(c,w),\phi_1(c,w),\phi_2(c,w))$$

for all $\omega \in \Omega$. Again let $T(\omega)\phi_1 = x_2$ for some $x_2 \in E$. Choose $\phi_2(c,w) = x_2$ and $\phi_3(c,w) = x_3$ for the fixed $c \in I$ and

$$G_0(\phi_1,\phi_2,\phi_3) = G(\phi_1(c,w),\phi_2(c,w),\phi_3(c,w))$$

for all $\omega \in \Omega$. Proceeding in this way, we obtain a sequence $\{\phi_n\}$ of points in E of iterations of $T(\omega)$ at ϕ_0 as

$$T(\omega)\phi_{n-1} = x_n = \phi_n(c,\omega) \tag{3.2}$$

with

$$G_0(\phi_{n-2},\phi_{n-1},\phi_n) = G(\phi_{n-2}(c,w),\phi_{n-1}(c,w),\phi_n(c,w)) \tag{3.3}$$

for all $\omega \in \Omega$. Clearly, $\{\phi_n\}$ is a sequence of measurable functions from Ω into E. Consequently $\{\phi_n(c)\}$ is a sequence of measurable functions from Ω into E. Consequently $\{\phi_n(c)\}$ is a measurable function from Ω into E. We show that $\{\phi_n(c,\omega)\}$ is a cauchy sequence in E. Taking $\phi = \phi_0$, $\psi = \phi_1$ and $\xi = \phi_2$ in inequality (3.1) we obtain

$$\begin{aligned}
&\min\big\{G(T(\omega)\phi_0,T(\omega)\phi_1,T(\omega)\phi_2),G(\phi_0(c,w),T(\omega)\phi_0,T(\omega)\phi_1),\\
&\qquad G(\phi_1(c,w),T(\omega)\phi_1,T(\omega)\phi_2),G(\phi_2(c,w),T(\omega)\phi_2,T(\omega)\phi_0)\big\}\\
&-\min\big\{G(\phi_0(c,w),T(\omega)\phi_2),T(\omega)\phi_2),G(T(\omega)\phi_0,\phi_1(c,w),T(\omega)\phi_0),\\
&\qquad G(T(\omega)\phi_0,T(\omega)\phi_1,\phi_2(c,w))\big\}\\
&\le q(\omega)G(\phi_0,\phi_1,\phi_2),
\end{aligned} \tag{3.4}$$

which further gives

$$\begin{aligned}
&\min\big\{G_0(\phi_1,\phi_2,\phi_3),G_0(\phi_0,\phi_1,\phi_2)\big\}\\
&=\min\big\{G(\phi_1(c,\omega),\phi_2(c,\omega),\phi_3(c,\omega)),G_0(\phi_0(c,\omega),\phi_1(c,\omega),\phi_2(c,\omega))\big\}\\
&\le qG_0(\phi_0,\phi_1,\phi_2).
\end{aligned} \tag{3.5}$$

Since $G_0(\phi_0,\phi_1,\phi_2) \le qG_0(\phi_0,\phi_1,\phi_2)$ $(q<1)$ is not possible, one has $G_0(\phi_1,\phi_2,\phi_3) \le qG_0(\phi_0,\phi_1,\phi_2)$. Proceeding in this way by induction, one has

$$G_0(\phi_n,\phi_{n+1},\phi_{n+2}) \le qG_0(\phi_{n-1},\phi_n,\phi_{n+1}) \tag{3.6}$$

for $n \in \mathbb{N} := \{1,2,3,\cdots\}$. By a repeated application of the inequality we get

$$\begin{aligned} G_0(\phi_n,\phi_{n+1},\phi_{n+2}) &\le qG_0(\phi_{n-1},\phi_n,\phi_{n+1}) \\ &\vdots \\ &\le q^n G_0(\phi_0,\phi_1,\phi_2). \end{aligned} \tag{3.7}$$

Now by triangle inequality, one has

$$\begin{aligned} G_0(x_n,x_m,x_m) &\le G_0(x_n,x_{n+1},x_{n+1}) + G_0(x_{n+1},x_{n+2},x_{n+2}) \\ &\quad + G_0(x_{n+2},x_{n+3},x_{n+3}) + \ldots + G_0(x_{m-1},x_m,x_m) \\ &\le (q^n + q^{n+1} + \ldots\ldots + q^{m-1})G_0(x_0,x_1,x_2) \\ &\le \frac{q^n}{1-q}G_0(x_0,x_1,x_2) \\ &\to 0. \end{aligned} \tag{3.8}$$

Since $G(\phi_n(c,\omega),\phi_{n+1}(c,\omega),\phi_{n+2}(c,\omega)) = G_0(\phi_n(\omega),\phi_{n+1}(\omega),\phi_{n+2}(\omega))$ for all $\omega \in \Omega$, we have that $\{T(\omega)\phi_n\}$ is also a cauchy sequence in E. As E is a complete G-metric space, there exists a measurable function $\phi^* : \Omega \to E_0$ such that $\phi_n \to \phi^*$ and $T(\omega)\phi_n = \phi_n(c,\omega) \to \phi^*(c,\omega)$ as $n \to \infty$. To prove that ϕ^* is a PPF dependent random fixed point of $T(\omega)$, we first observe that since $T(\omega)$ is continuous on E_0, $T(\omega)$ is a continuous at ϕ^* . Hence for $\varepsilon > 0$, there exists a $\delta > 0$ such that $G_0(\phi_{n+1},\phi^*,\phi^*) < \delta \Rightarrow G(T\phi_{n+1},T\phi^*,T\phi^*) < \frac{\varepsilon}{2}$. Also since $T(\omega)\phi_n \to \phi^*(c,\omega)$ for $\gamma = \min\{\frac{\varepsilon}{2},\delta\}$ there exist $n_0 \in \mathbb{N}$ such that $G(T(\omega)\phi_n,\phi^*(c,\omega),\phi^*(c,\omega)) < \gamma$ for $n \ge n_0$. Thus, one has

$$\begin{aligned} &G(T(\omega)\phi_n,\phi^*(c,\omega),\phi^*(c,\omega)) \\ &\le G(T(\omega)\phi^*,T(\omega)\phi_n,T(\omega)\phi_n) + G(T(\omega)\phi_n,\phi^*(c,\omega),\phi^*(c,\omega)) \\ &< \frac{\varepsilon}{2} + \gamma \\ &< \varepsilon. \end{aligned} \tag{3.9}$$

Since ε is arbitrary, we have $T(\omega)\phi^*(\omega) = \phi^*(c,\omega)$, for all $\omega \in \Omega$. This completes the proof.

4. Random fixed points mappings in ordered G-metric spaces

We define an order relation $\leq$ in X which is a reflexive, antisymmetric and transitive relation in X. The metric space X together with the order relation $\leq$ becomes a partially ordered metric space. A random mapping $T : \Omega \times X \to X$ is called non-decreasing if for any $x, y \in X$ with $x \leq y$ we have that $T(\omega)x \leq T(\omega)y$ for all $\omega \in \Omega$. Similarly random mapping $T : \Omega \times X \to X$ is called non-increasing if for any $x, y \in X$, $x \leq y$ implies $T(\omega)x \geq T(\omega)y$ for all $\omega \in \Omega$. A monotone random mapping which is either non-decreasing or non-increasing on X.

The investigation of the existence of fixed points in a partially ordered metric space was first considered in Ran and Reuriungs [12]. This study was continued in Nieto and Rodríguez-López [11] by assuming the existence of only lower solution instead of usual approach where both the lower and upper solutions are assumed to exist for the nonlinear equation. These fixed point theorems are then applied to obtain existence and uniqueness results for nonlinear ordinary differential equations in the same paper. A further extension of this idea was considered in Bhaskar and Lakshmikanthan [4], Shatanawi [14], Luong and Thuan [8] and Sachin Bedre *et al*. [2] for the coupled fixed point theorems in partially ordered metric spaces. Below we prove some nonunique random fixed point theorems for monotone random mappings in separable and complete G-metric spaces.

Theorem 4.1. *Let $(\Omega, \mathscr{A})$ be a measurable space and let X be a separable and complete partially ordered G-metric space. Let $T : \Omega \times X \to X$ be a monotone non-decreasing random mapping satisfying the contraction condition (2.2) for all comparable elements x, y and z in X. Further if $T(\omega)$ is G-continuous and if there exists an element $x_0 \in X$ such that $x_0 \leq T(\omega)x_0$ for all $\omega \in \Omega$, then the random mapping $T(\omega)$ has a random fixed point. Further, if every triplet of elements $x, y, z \in X$ has a lower bound and an upper bound, then $T(\omega)$ has a unique random fixed point.*

Proof. Let $x : \Omega \to X$ be an arbitrary measurable function and define a sequence $\{x_n\}$ of successive approximations of $T(\omega)$ by $x_{n+1} = T(\omega)x_n$. Clearly, $\{x_n\}$ is a sequence of measurable functions from Ω into X such that

$$x_0 \le x_1 \le \ldots \le x_n \le \ldots. \tag{4.1}$$

We show that $\{x_n\}$ is a Cauchy sequence in X. Taking $x = x_0$, $y = x_1$ and $z = x_2$ in (2.2) we obtain $G(x_1,x_2,x_3) \le qG(x_0,x_1,x_2)$. Processing in this way, by induction, we see that $G(x_n,x_{n+1},x_{n+2}) \le qG(x_{n-1},x_n,x_{n+1})$ for each $n \in \mathbb{N} := \{1,2,3,\cdots\}$. Then, by repeated application of the above inequality, we obtain $G(x_n,x_{n+1},x_{n+2}) \le q^n G(x_0,x_1,x_2)$.

Now, by triangle inequality, we get

$$\begin{aligned} G_0(x_n,x_m,x_m) &\le G_0(x_n,x_{n+1},x_{n+1}) + G_0(x_{n+1},x_{n+2},x_{n+2}) \\ &\quad + G_0(x_{n+2},x_{n+3},x_{n+3}) + \ldots + G_0(x_{m-1},x_m,x_m) \\ &\le \frac{q^n}{1-q} G_0(x_0,x_1,x_1) \\ &\to 0. \end{aligned} \tag{4.2}$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. The ordered G-metric space X being complete, there is a measurable function $x^* : \Omega \to X$ such that $\lim_{n\to\infty} x_n = x^*$. From the continuity of the random mapping $T(\omega)$ it follows that

$$x^*(\omega) = \lim_{n\to\infty} x_{n+1}(\omega) = \lim_{n\to\infty} T(\omega)x_n(\omega) = T(\omega)\lim_{n\to\infty} x_n(\omega) = T(\omega)x^*(\omega) \tag{4.3}$$

for all $\omega \in \Omega$. Thus x^* is a random fixed point of the random mapping $T(\omega)$ on X. If every triplet of elements $x,y,z \in X$ has a lower bound and an upper bound, then it can be shown $\lim\limits_{n\to\infty} T^n(\omega)x = x^*(\omega)$ for all measurable functions $x : \Omega \to X$, where $x^* = \lim\limits_{n\to\infty} T^n(\omega)x_0$. Thus $T(\omega)$ has a unique random fixed point and the proof of the theorem is completed.

Corollary 4.2. *Let $(\Omega, \mathscr{A})$ be a measurable space and let X be a separable and complete partially ordered G-metric space. Let $T : \Omega \times X \to X$ be a monotone nondecreasing random mapping satisfying*

$$G(T(\omega)x, T(\omega)y, T(\omega)z) \le G(x,y,z) \tag{4.4}$$

for all comparable elements $x,y,z \in X$. Further if $T(\omega)$ is continuous and if there exists an element $x_0 \in X$ such that $x_0 \le T(\omega)x_0$ for all $\omega \in \Omega$, then the random mapping $T(\omega)$ has a *random fixed point. Further, if every triplet of elements $x,y,z \in X$ has a lower bound and an upper bound, then $T(\omega)$ has a unique random fixed point.*

Corollary 4.3. *Let X be a partially ordered G-complete metric space and let $T : X \to X$ be a monotone nondecreasing mapping satisfying the contraction condition (2.7) for all comparable elements $x,y,z \in X$. Further if T is continuous and if there exists an element $x_0 \in X$ such that $x_0 \leq Tx_0$, then the mapping T has a fixed point.*

Corollary 4.4. *Let X be a complete metric space and let $T : X \to X$ be a monotone non-decreasing mapping satisfying*

$$G(Tx,Ty,Tz) \leq qG(x,y,z) \tag{4.5}$$

for comparable elements $x,y,z \in X$, where $0 \leq q < 1$. Further if T is continuous and if there exists an element $x_0 \in X$ such that $x_0 \leq Tx_0$, then the mapping T has a fixed point.

Theorem 4.5. *Let $(\Omega,\mathscr{A})$ be a measurable space and let (X,G) be a partially ordered complete G-separable metric space. Let $T : \Omega \times X \to X$ be a random mapping satisfying for each $\omega \in \Omega$,*

$$\begin{aligned}
&\min\big\{[G(T(\omega)x,T(\omega)y,T(\omega)z)]^2, G(x,T(\omega)x,T(\omega)y)G(y,T(\omega)y,T(\omega)z)G(z,T(\omega)z,T(\omega)x),\\
&\qquad G(T(\omega)x,T(\omega)y,T(\omega)z)G(x,y,z)\big\}\\
&-\min\big\{G(x,T(\omega)z,T(\omega)z)G(y,T(\omega)x,T(\omega)x)G(z,T(\omega)y,T(\omega)y),\\
&\qquad G(x,T(\omega)x,T(\omega)y)G(y,T(\omega)y,T(\omega)z), G(z,T(\omega)z,T(\omega)x)\big\}\\
&\quad \leq q(\omega)G(x,T(\omega)x,T(\omega)y)G(y,T(\omega)y,T(\omega)z)G(z,T(\omega)z,T(\omega)x)
\end{aligned} \tag{4.6}$$

for all comparable elements $x,y,z \in X$, where $q : \Omega \to \mathbb{R}^+$ is a measurable function satisfying $0 \leq q(\omega) < 1$ for all $\omega \in \Omega$. Further if there exists an element $x_0 \in X$ such that $x_0 \leq T(\omega)x_0$, then $T(\omega)$ has a fixed point.

Proof. The proof is similar to Theorem 4.1 and therefore, we omit the details.

Theorem 4.6. *Let $(\Omega,\mathscr{A})$ be a measurable space and let (X,G) be a partially ordered separable G-metric space. Let $T : \Omega \times X \to X$ be a continuous random mapping satisfying for each $\omega \in \Omega$,*

$$
\begin{aligned}
&\min\big\{G(T(\omega)x,T(\omega)y,T(\omega)z),G(x,T(\omega)x,T(\omega)y),\\
&\quad G(y,T(\omega)y,T(\omega)z),G(z,T(\omega)z,T(\omega)x)\big\}\\
&-\min\big\{G(x,T(\omega)z,T(\omega)z),G(y,T(\omega)x,T(\omega)x),G(z,T(\omega)y,T(\omega)y)\big\} \qquad (4.7)\\
&\le p(\omega)\min\big\{G(x,T(\omega)x,T(\omega)y),G(y,T(\omega)y,T(\omega)z),G(z,T(\omega)z,T(\omega)x)\big\}\\
&+q(\omega)G(x,y,z)
\end{aligned}
$$

for all comparable elements $x,y,z\in X$, where $p,q:\Omega\to\mathbb{R}^+$ are measurable functions such that $0\le p(\omega)+q(\omega)<1$ for all $\omega\in\Omega$. If there exists an element $x_0\in X$ such that $x_0\le T(\omega)x_0$ for each $\omega\in\Omega$, then $T(\omega)$ has a random fixed point.

Proof. The proof is simple and can be obtained by closely observing the proof of Theorem 4.1. Hence we omit the details.

Next, we deal with the case of a G-metric space X with two metrics G_1 and G_2 defined on it and prove some nonunique random fixed point theorems on separable partially ordered G-metric spaces.

Theorem 4.7. *Let $(\Omega,\mathscr{A})$ be a measurable space and let X be an partially ordered G-metric space with two metrics G_1 and G_2. Let $T:\Omega\times X\to X$ be a non-decreasing random mapping satisfying the contractive condition on (2.2) for all comparable elements $x,y,z\in X$. Suppose that the following conditions hold in X.*

(i) *$G_1(x,y,z)\le G_2(x,y,z)$ for all $x,y,z\in X$.*
(ii) *$T(\omega)$ is continuous w.r.t. G_1.*
(iii) *X is Polish space w.r.t. G_1.*

Furthermore, if there exists an element $x_0\in X$ such that $x_0\le T(\omega)x_0$ for all $\omega\in\Omega$, then $T(\omega)$ has a random fixed point.

Proof. Consider the sequence $\{x_n\}$ of successive iterations of $T(\omega)$ at x_0 defined by $x_{n+1}=T(\omega)x_n$. Clearly, $\{x_n\}$ is a sequence of measurable functions from Ω into X w.r.t. the metric G_1 such that $x_0\le x_1\le\dots\le x_n\le\dots$. Then, it can be shown as in the proof of Theorem 4.1 that $\{x_n\}$ is a Cauchy sequence in X w.r.t. the metric G_2, that is, for any positive integer $m>$

n, $G_2(x_m,x_n,x_n) \leq \frac{q^n}{1-q}G_2(x_0,x_1,x_1)$. From the hypothesis (i), it follows that $G_1(x_m,x_n,x_n) \leq \frac{q^n}{1-q}G_2(x_0,x_1,x_1) \to 0$ as $n \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence w.r.t. the metric G_1. The metric space (X,G_1) being complete and separable, there exists a measurable function $x^* : \Omega \to X$ such that $\lim_{n\to\infty} x_n(\omega) = x^*(\omega)$ for each $\omega \in \Omega$. From the continuity of $T(\omega)$ w.r.t. G_1, it follows that

$$T(\omega)x^*(\omega) = \lim_{n\to\infty} T(\omega)x_n(\omega) = \lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega)$$

for all $\omega \in \Omega$. This proves that $T(\omega)$ has a random fixed point in X. This completes the proof.

Remark 4.1. The conclusion of Theorem 4.7 also remains true if we replace the condition (2.2) with those of (4.6) and (4.7).

Conclusion. In this paper, we prove nonunique random fixed point theorems in polish spaces. However, more general random fixed point theorems under weaker conditions may be proved along the similar lines with appropriate modifications. Also, these results may be extended to two, three and four mappings to prove the random common fixed point theorems in Polish spaces along the similar lines with appropriate modifications.

Conflict of Interests

The authors declare that there is no conflict of interests.

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