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SOME COMMON FIXED POINT THEOREMS IN CONE METRIC SPACES

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Abstract. In this paper, some new coincidence and common fixed point theorems are obtained for four mappings, which are assumed to satisfy certain weak inequalities, in the framework of cone metric spaces. The results presented in this paper generalize and unify the corresponding fixed point theorems in [2]. Examples are also provided to support the main results.

Keywords: coincidence point; fixed point; cone metric space; weakly compatible maps.

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1. Introduction

Huang and Zhang [1]generalized the concept of metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. Since 2007, fixed point results for cone metric spaces were studied by many other authors; see [2-10] and the references therein.

The weak contraction principle was first given for Hilbert spaces and subsequently extend to metric spaces.Also the weak contraction principle was extended by many authors.

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In this paper, some new coincidence and common fixed point theorems are obtained for four mappings, which are assumed to satisfy certain weak inequalities, in the framework of cone metric spaces. The results presented in this paper generalize and unify the corresponding fixed point theorems in [2]. Examples are also provided to support the main results.

2. Preliminaries

First, we recall the definitions of cone metric spaces, the notion of convergence and other results that will be needed in the sequel.

Definition 2.1. [1] Let E be a real Banach space. A subset P of E is called a cone if and only if:
(a) P is closed,non-empty and $P \neq \{0\}$;
(b) $\forall a,b \in R, a,b \geq 0, \forall x,y \in P$ imply that ax+$by \in P$;
(C) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y-x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stand for $y-x \in intP$, $intP$ denotes the interior of P.

Definition 2.2. [1] A cone P is said to be normal if there exists a constant $K > 0$ such that for all $x,y \in E$,$0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$.
The least position number satisfying the above inequality is called the normal constant of P.

Definition 2.3. [1] Let X be a nonempty set,suppose the mapping $d : X \times X \to E$ satisfy:
(d1) $d(x,y) \geq 0$ for all $x,y \in X$and $d(x,y) = 0$ if and only if $x = y$;
(d2) $d(x,y) = d(y,x)$ for all $x,y \in X$;
(d3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.
Then d is called a cone metric on X and (X,d) is called a cone metric space,the concept of a cone metric space is more general than that of a metric space.

Definition 2.4. [1] Let (X,d) be a cone metric space,$\{x_n\}$ be a sequence in X and $x \in X$.For every $c \in E$ with $0 \ll c$,we say that $\{x_n\}$ is
(i) a Cauchy sequence if there is an N such that for all $n,m > N$,$d(x_n,x_m) \ll c$.

(ii) a convergent sequence if there is an N such that for all $n > N, d(x_n, x) \ll c$ for some x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 2.5. [2] Let (X,d) be a cone metric space,E be a real Banach space and cone $P \subset E$, $intP \neq \phi$,$\{x_n\}$ be a sequence in X. We have

(i)$\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$;

(ii)$\{x_n\}$ is Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 2.6. [2] Let (X,d) be a cone metric space,cone $P \subset E$ and $intP \neq \phi$. $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.

Definition 2.7. [2] Let $\Psi : P \to P$ be a function.

(i) Ψ is strongly monotonic increasing if for $x, y \in P, x \leq y \Leftrightarrow \Psi(x) \leq \Psi(y)$;

(ii) Ψ is said to be continuous at $x_0 \in P$ for any sequence $\{x_n\} \subseteq P, x_n \to x_0 \Rightarrow \Psi(x_n) \to \Psi(x_0)$;

(iii) Ψ is a subadditive function, for all $x, y \in P, \Psi(x+y) \leq \Psi(x) + \Psi(y)$.

Definition 2.8. [3] Let f and g be self-maps on a set X. If $w = fx = gx$ for some x in X, then x is called coincidence point of f and g, where w is called a point of coincidence of f and g.

Definition 2.9. [3] Let f and g be self-maps on a set X. Then f and g are said to be weakly compatible if they commute at every coincidence point.

Lemma 2.10. [3] Let f, g, S and T be self-maps on a cone metric space X with cone P having non-empty interior,satisfying $f(X) \subset T(X)$ and $g(X) \subset S(X)$. Define $\{x_n\}$ and $\{y_n\}$ by $y_{2n+1} = fx_{2n} = Tx_{2n+1}, y_{2n+2} = gx_{2n+1} = Sx_{2n+2}$, $n \geq 0$. Suppose that there exist a $\lambda \in [0,1)$ such that $d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n)$ for each $n \geq 1$. Then either

(a) $\{f,S\}$ and $\{g,T\}$ have coincidence points,and $\{y_n\}$ converges,or

(b) $\{y_n\}$ is Cauchy.

Lemma 2.11. [3] *If E is a real Banach space with a cone P and if $a \leq ha$, where $a \in P$ and $h \in [0,1)$, then $a = 0$.*

Lemma 2.12. [3] *If $0 \leq u \ll c$ for each $0 \ll c$ then $u = 0$.*

3. Main results

Theorem 3.1. *Let (X,d) be a complete cone metric space with regular cone P such that $d(x,y) \in intP$,for all $x,y \in X$ with $x \neq y$. Let f, g, S and T be self-maps of a cone metric space X satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$ and*

$$\Psi(d(fx,gy)) \leq \Psi(\alpha d(Sx,Ty) + \beta d(fx,Sx) + \gamma d(gy,Ty) + \lambda d(fx,Ty) + \kappa d(gy,Sx)),$$

where $\alpha, \beta, \gamma, \lambda, \kappa \geq 0$ and $\alpha + \beta + \gamma + 2\lambda + 2\kappa < 1$. $\Psi : P \to P$ is a continuous function with the following properties:
(i) Ψ is strongly monotonic increasing;
(ii) $\Psi(t) = 0$ if and only if $t = 0$;
(iii) $\Psi(x+y) \leq \Psi(x) + \Psi(y)$ for all $x,y \in P$.
If one of $f(X)$, $g(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then $\{f,S\}$ and $\{g,T\}$ have a common point of coincidence in X. Moreover if $\{f,S\}$ and $\{g,T\}$ are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. For any arbitrary point x_0 in X, construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n-1} = fx_{2n-2} = Tx_{2n-1} \text{ and } y_{2n} = gx_{2n-1} = Sx_{2n}.$$

It follows that

$$\begin{aligned}
\Psi(d(y_{2n+1}, y_{2n+2})) &= \Psi(d(fx_{2n}, gx_{2n+1})) \\
&\leq \Psi(\alpha d(Sx_{2n}, Tx_{2n+1}) + \beta d(fx_{2n}, Sx_{2n}) + \gamma d(gx_{2n+1}, Tx_{2n+1}) \\
&\quad + \lambda d(fx_{2n}, Tx_{2n+1}) + \kappa d(gx_{2n+1}, Sx_{2n})) \\
&= \Psi(\alpha d(y_{2n}, y_{2n+1}) + \beta d(y_{2n+1}, y_{2n}) + \gamma d(y_{2n+2}, y_{2n+1}) \\
&\quad + \lambda d(y_{2n+1}, y_{2n+1}) + \kappa d(y_{2n+2}, y_{2n})) \\
&\leq \Psi(\alpha d(y_{2n}, y_{2n+1}) + \beta d(y_{2n+1}, y_{2n}) + \gamma d(y_{2n+2}, y_{2n+1}) \\
&\quad + \kappa d(y_{2n+2}, y_{2n+1}) + \kappa d(y_{2n+1}, y_{2n})).
\end{aligned}$$

From the above and the properties of Ψ, we see that

$$d(y_{2n+1}, y_{2n+2}) \leq (\alpha + \beta + \kappa) d(y_{2n+1}, y_{2n}) + (\gamma + \kappa) d(y_{2n+1}, y_{2n+2}),$$

which implies that $d(y_{2n+1}, y_{2n+2}) \leq h_1 d(y_{2n+1}, y_{2n})$, where $h_1 = \frac{\alpha+\beta+\kappa}{1-(\gamma+\kappa)}$. Note that $h_1 < 1$. Similarly it can be shown that $d(y_{2n}, y_{2n+1}) \leq h_2 d(y_{2n}, y_{2n-1})$, where $h_2 = \frac{\alpha+\gamma+\lambda}{1-(\beta+\lambda)}$ and $h_2 < 1$. Let $h = max\{h_1, h_2\}$. It follows that $h \in (0,1)$. And we know that $d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n)$ for each $n \geq 1$. Hence lemma 2.10 is satisfied.

Now we show that $\{f,S\}$ and $\{g,T\}$ have coincidence points in X. Without loss of generality, we may assume that $y_n \neq y_{n+1}$ for any n. If we have equality for some n, then (a) of lemma 2.10 applies. In view of lemma 2.10, we find that $\{y_n\}$ is a Cauchy sequence. Suppose that $S(X)$ is complete. Then there exists a u in $S(X)$ such that $Sx_{2n} = y_{2n} \to u$ as $n \to \infty$. Consequently, we can find a v in X such that $Sv = u$. we claim that $fv = u$. To this end, we consider

$$\begin{aligned}
\Psi(d(fv,u)) &\leq \Psi(d(fv, gx_{2n-1}) + d(gx_{2n-1}, u)) \\
&\leq \Psi(d(fv, gx_{2n-1})) + \Psi(d(gx_{2n-1}, u)) \\
&\leq \Psi(\alpha d(Sv, Tx_{2n-1}) + \beta d(fv, Sv) + \gamma d(gx_{2n-1}, Tx_{2n-1}) \\
&\quad +\lambda d(fv, Tx_{2n-1}) + \kappa d(gx_{2n-1}, Sv)) + \Psi(d(gx_{2n-1}, u)).
\end{aligned}$$

Taking the limit as $n \to \infty$ gives

$$\begin{aligned}
\Psi(d(fv,u)) &\leq \Psi(\alpha d(u,u) + \beta d(fv,u) + \gamma d(u,u) + \lambda d(fv,u) + \kappa d(u,u)) + \Psi(d(u,u)) \\
&= \Psi((\beta+\lambda) d(fv,u)).
\end{aligned}$$

Using (i), we have $d(fv,u) \leq (\beta+\lambda) d(fv,u)$, which is a contradiction since $\alpha, \beta, \gamma, \lambda, \kappa \geq 0$ and $\alpha+\beta+\gamma+2\lambda+2\kappa < 1$. It follows that $d(fv,u) = 0$, $fv = Sv = u$. Since $u \in f(X) \subset T(X)$ there exists a $w \in X$ such that $Tw = u$. Now we are in a position to show that $gw = u$. Consider

$$\begin{aligned}
\Psi(d(gw,u)) &\leq \Psi(d(gw, fx_{2n}) + d(fx_{2n}, u)) \\
&\leq \Psi(d(gw, fx_{2n})) + \Psi(d(fx_{2n}, u)) \\
&\leq \Psi(\alpha d(Sx_{2n}, Tw) + \beta d(fx_{2n}, Sx_{2n}) + \gamma d(gw, Tw) \\
&\quad +\lambda d(fx_{2n}, Tw) + \kappa d(gw, Sx_{2n})) + \Psi(d(fx_{2n}, u)).
\end{aligned}$$

Letting $n \to \infty$, we have $d(gw,u) \leq (\gamma+\kappa) d(gw,u)$. So we obtain $gw = u = Tw$. Thus $\{f,S\}$ and $\{g,T\}$ have a common point of coincidence in X. If $\{f,S\}$ and $\{g,T\}$ are weakly compatible,

$fu = fSv = Sfv = Su = w_1$ (say) and $gu = gTw = Tgw = Tu = w_2$ (say).

$$\begin{aligned}\Psi(d(w_1,w_2)) &= \Psi(d(fu,gu)) \\ &\le \Psi(\alpha d(Su,Tu)+\beta d(fu,Su)+\gamma d(gu,Tu)+\lambda d(fu,Tu)+\kappa d(gu,Su)) \\ &= \Psi(\alpha d(w_1,w_2)+\beta d(w_1,w_1)+\gamma d(w_2,w_2)+\lambda d(w_1,w_2)+\kappa d(w_2,w_1)) \\ &= \Psi((\alpha+\lambda+\kappa)d(w_1,w_2)).\end{aligned}$$

From the properties of Ψ, we have $\Psi(d(w_1,w_2)) \le \Psi((\alpha+\lambda+\kappa)d(w_1,w_2))$. Therefore $d(w_1,w_2) \le (\alpha+\lambda+\kappa)d(w_1,w_2)$, which implies that $w_1 = w_2$ and hence $fu = gu = Su = Tu$. Next, we prove that $u = gu$.

$$\begin{aligned}\Psi(d(u,gu)) &= \Psi(d(fv,gu)) \\ &\le \Psi(\alpha d(Sv,Tu)+\beta d(fv,Sv)+\gamma d(gu,Tu)+\lambda d(fv,Tu)+\kappa d(gu,Sv)) \\ &= \Psi(\alpha d(u,gu)+\beta d(u,u)+\gamma d(gu,gu)+\lambda d(u,gu)+\kappa d(gu,u)) \\ &= \Psi((\alpha+\lambda+\kappa)d(u,gu)).\end{aligned}$$

Thus $d(u,gu) \le (\alpha+\lambda+\kappa)d(u,gu)$, which implies that $gu = u$, and u is a common fixe point of f, g, S and T. For uniqueness: Suppose that u^* is also a fixed point of f, g, S and T. It follows that

$$\begin{aligned}\Psi(d(u,u^*)) &= \Psi(d(fu,gu^*)) \\ &\le \Psi(\alpha d(Su,Tu^*)+\beta d(fu,Su)+\gamma d(gu^*,Tu^*)+\lambda d(fu,Tu^*)+\kappa d(gu^*,Su)) \\ &= \Psi(\alpha d(u,u^*)+\beta d(u,u)+\gamma d(u^*,u^*)+\lambda d(u,u^*)+\kappa d(u^*,u)) \\ &= \Psi((\alpha+\lambda+\kappa)d(u,u^*)),\end{aligned}$$

which is possible only if $u = u^*$. The proofs for the cases in which $g(X)$,$f(X)$ or $T(X)$ is complete are similar, and are therefore omitted.

Corollary 3.2. *Let (X,d) be a complete cone metric space with regular cone P such that $d(x,y) \in intP$ for all $x,y \in X$ with $x \ne y$. Let f, g, S and T be self-maps of a cone metric space*

X satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$ and for some $m,n \in N$,

$$\begin{aligned}\Psi(d(f^m x, g^n y)) \;\; &\leq \;\; \Psi(\alpha d(S^m x, T^n y) + \beta d(f^m x, S^m x) + \gamma d(g^n y, T^n y) \\ &\quad + \lambda d(f^m x, T^n y) + \kappa d(g^n y, S^m x)),\end{aligned}$$

where $\alpha, \beta, \gamma, \lambda, \kappa \geq 0$ and $\alpha + \beta + \gamma + 2\lambda + 2\kappa < 1$. $\Psi : P \to P$ is a continuous function with the following properties:

(i) Ψ is strongly monotonic increasing;

(ii) $\Psi(t) = 0$ if and only if $t = 0$;

(iii) $\Psi(x+y) \leq \Psi(x) + \Psi(y)$ for all $x, y \in P$.

If one of $f^m(X)$,$g^n(X)$,$S^m(X)$ or $T^n(X)$ is a complete subspace of X, then $\{f^m, S^m\}$ and $\{g^n, T^n\}$ have a common point of coincidence in X. Moreover if $\{f^m, S^m\}$ and $\{g^n, T^n\}$ are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. It follows from Theorem 3.1 that $\{f^m, S^m\}$and $\{g^n, T^n\}$ have a unique common fixed point p. Now $f(p) = f(f^m(p)) = f^{m+1}(p) = f^m(f(p))$ and $S(p) = S(S^m(p)) = S^{m+1}(p) = S^m(S(p))$ implies that $f(p)$ and $S(p)$ are also fixed points for f^m and S^m. Hence $f(p) = S(p) = p$. By using the same argument, we obtain $g(p) = T(p) = p$.

Corollary 3.3. *Let (X,d) be a complete cone metric space with regular cone P such that $d(x,y) \in intP$ for all $x, y \in X$ with $x \neq y$. Let f, g and T are self-maps of a cone metric space X satisfying $f(X) \bigcup g(X) \subset T(X)$ and*

$$\Psi(d(fx, gy)) \leq \Psi(\alpha d(Tx, Ty) + \beta d(fx, Tx) + \gamma d(gy, Ty) + \lambda d(fx, Ty) + \kappa d(gy, Tx)),$$

where $\alpha, \beta, \gamma, \lambda, \kappa \geq 0$ and $\alpha + \beta + \gamma + 2\lambda + 2\kappa < 1$. $\Psi : P \to P$ is a continuous function with the following properties:

(i) Ψ is strongly monotonic increasing;

(ii) $\Psi(t) = 0$ if and only if $t = 0$;

(iii) $\Psi(x+y) \leq \Psi(x) + \Psi(y)$ for all $x, y \in P$.

If one of $f(X)$, $g(X)$ or $T(X)$ is a complete subspace of X, then $\{f, T\}$ and$\{g, T\}$ have a common point of coincidence in X. Moreover if $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then f, g and T have a unique common fixed point.

Proof. By taking $S=T$ in Theorem 3.1, we find the desired result immediately.

Theorem 3.4. *Let (X,d) be a complete cone metric space with regular cone P such that $d(x,y)\in intP$ for all $x,y\in X$ with $x\neq y$. Let f, g, S and T be self-maps of a cone metric space X satisfying $f(X)\subset T(X)$, $g(X)\subset S(X)$ and*

$$\begin{aligned}\Psi(d(fx,gy)) &\leq \Psi(\alpha(x,y)d(Sx,Ty)+\beta(x,y)d(fx,Sx)+\gamma(x,y)d(gy,Ty)\\ &\quad +\lambda(x,y)d(fx,Ty)+\kappa(x,y)d(gy,Sx)),\end{aligned}$$

where $\alpha(x,y),\beta(x,y),\gamma(x,y),\lambda(x,y),\kappa(x,y)\geq 0$ and

$$\sup_{(x,y)\in X}[\alpha(x,y)+\beta(x,y)+\gamma(x,y)+2\lambda(x,y)+2\kappa(x,y)]\leq\eta<1.$$

$\Psi:P\to P$ is a continuous function with the following properties:

(i) Ψ is strongly monotonic increasing;

(ii) $\Psi(t)=0$ if and only if $t=0$;

(iii) $\Psi(x+y)\leq\Psi(x)+\Psi(y)$ for all $x,y\in P$.

If one of $f(X)$, $g(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then $\{f,S\}$ and $\{g,T\}$ have a common point of coincidence in X. Moreover if $\{f,S\}$ and $\{g,T\}$ are weakly compatible,then f, g, S and T have a unique common fixed point.

Proof. For any arbitrary point x_0 in X, construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n-1}=fx_{2n-2}=Tx_{2n-1}\text{ and }y_{2n}=gx_{2n-1}=Sx_{2n}.$$

So we have

$$\begin{aligned}\Psi(d(y_{2n+1},y_{2n+2})) &= \Psi(d(fx_{2n},gx_{2n+1}))\\ &\leq \Psi(\alpha(x_{2n},x_{2n+1})d(Sx_{2n},Tx_{2n+1})+\beta(x_{2n},x_{2n+1})d(fx_{2n},Sx_{2n})\\ &\quad +\gamma(x_{2n},x_{2n+1})d(gx_{2n+1},Tx_{2n+1})+\lambda(x_{2n},x_{2n+1})d(fx_{2n},Tx_{2n+1})\\ &\quad +\kappa(x_{2n},x_{2n+1})d(gx_{2n+1},Sx_{2n}))\\ &\leq \Psi(\alpha(x_{2n},x_{2n+1})d(y_{2n},y_{2n+1})+\beta(x_{2n},x_{2n+1})d(y_{2n+1},y_{2n})\\ &\quad +\gamma(x_{2n},x_{2n+1})d(y_{2n+2},y_{2n+1})+\kappa(x_{2n},x_{2n+1})d(y_{2n+2},y_{2n+1})\\ &\quad +\kappa(x_{2n},x_{2n+1})d(y_{2n+1},y_{2n})).\end{aligned}$$

From the above and the properties of Ψ, we find that

$$\begin{aligned} d(y_{2n+1},y_{2n+2}) &\leq (\alpha(x_{2n},x_{2n+1})+\beta(x_{2n},x_{2n+1})+\kappa(x_{2n},x_{2n+1}))d(y_{2n+1},y_{2n}) \\ &\quad +(\gamma(x_{2n},x_{2n+1})+\kappa(x_{2n},x_{2n+1}))d(y_{2n+1},y_{2n+2}), \end{aligned}$$

which implies that $d(y_{2n+1},y_{2n+2}) \leq h_1 d(y_{2n+1},y_{2n})$. Letting

$$h_1 = \sup_{(x,y)\in X} \frac{\alpha(x,y)+\beta(x,y)+\kappa(x,y)}{1-\gamma(x,y)-\kappa(x,y)},$$

we see that $h_1 < 1$. If not, assume $h_1 \geq 1$. From the properties of supremum, we have $\{x_n\},\{y_n\} \subset X$. Let

$$\lim_{n\to\infty} \frac{\alpha(x_n,y_n)+\beta(x_n,y_n)+\kappa(x_n,y_n)}{1-\gamma(x_n,y_n)-\kappa(x_n,y_n)} = h_1.$$

Because $\eta < 1$ for $\varepsilon_0 = \frac{1-\eta}{2}$, when $N > 0$ and $n > N$, we have

$$h_1 - \frac{1-\eta}{2} < \frac{\alpha(x_n,y_n)+\beta(x_n,y_n)+\kappa(x_n,y_n)}{1-\gamma(x_n,y_n)-\kappa(x_n,y_n)} < h_1 + \frac{1-\eta}{2}$$
$$h_1 - \frac{1-\eta}{2} = \frac{2h_1+\eta-1}{2} = \frac{h_1+\eta+(h_1-1)}{2} \geq \frac{h_1+\eta}{2} > \eta.$$

It follows that

$$\alpha(x_n,y_n)+\beta(x_n,y_n)+\kappa(x_n,y_n) > \eta - \eta(\gamma(x_n,y_n)+\kappa(x_n,y_n)).$$

Hence, we arrive at

$$\begin{aligned} \eta &< \alpha(x_n,y_n)+\beta(x_n,y_n)+\eta\gamma(x_n,y_n)+(1+\eta)\kappa(x_n,y_n) \\ &\leq \alpha(x_n,y_n)+\beta(x_n,y_n)+\gamma(x_n,y_n)+2\kappa(x_n,y_n)+2\lambda(x_n,y_n) \\ &\leq \sup_{(x,y)\in X} [\alpha(x,y)+\beta(x,y)+\gamma(x,y)+2\kappa(x,y)+2\lambda(x,y)], \end{aligned}$$

which is a contradiction by the condition in Theorem 3.4. Similarly, we find that $d(y_{2n},y_{2n+1}) \leq h_2 d(y_{2n},y_{2n-1})$, where

$$h_2 = \sup_{(x,y)\in X} \frac{\alpha(x,y)+\gamma(x,y)+\lambda(x,y)}{1-\beta(x,y)-\lambda(x,y)}$$

and $h_2 < 1$. Letting $h = max\{h_1,h_2\}$, one has $h \in (0,1)$. And we know that $d(y_n,y_{n+1}) \leq hd(y_{n-1},y_n)$ for each $n \geq 1$. Hence lemma 2.10 is satisfied. Now we show that $\{f,S\}$ and $\{g,T\}$ have coincidence points in X. Without loss of generality, we may assume that $y_n \neq y_{n+1}$

for any n for if we have equality for some n, then (a) of lemma 2.10 applies. Now from lemma 2.10, we know $\{y_n\}$ is a Cauchy sequence. Suppose that $S(X)$ is complete. Then there exists a u in $S(X)$ such that $Sx_{2n} = y_{2n} \to u$ as $n \to \infty$. Consequently, we can find a v in X such that $Sv = u$. Next, we claim that $fv = u$. For this, we consider

$$\begin{aligned}
\Psi(d(fv,u)) &\leq \Psi(d(fv, gx_{2n-1}) + d(gx_{2n-1}, u)) \\
&\leq \Psi(d(fv, gx_{2n-1})) + \Psi(d(gx_{2n-1}, u)) \\
&\leq \Psi(\alpha(v,x_{2n-1})d(Sv, Tx_{2n-1}) + \beta(v,x_{2n-1})d(fv, Sv) \\
&\quad +\gamma(v,x_{2n-1})d(gx_{2n-1}, Tx_{2n-1}) + \lambda(v,x_{2n-1})d(fv, Tx_{2n-1}) \\
&\quad +\kappa(v,x_{2n-1})d(gx_{2n-1}, Sv)) + \Psi(d(gx_{2n-1}, u)) \\
&\leq \Psi[(\alpha(v,x_{2n-1}) + \lambda(v,x_{2n-1}))d(Sv, Tx_{2n-1}) \\
&\quad +(\beta(v,x_{2n-1}) + \lambda(v,x_{2n-1}))d(fv, Sv) + \gamma(v,x_{2n-1})d(gx_{2n-1}, Tx_{2n-1}) \\
&\quad +\kappa(v,x_{2n-1})d(gx_{2n-1}, Sv)] + \Psi(d(gx_{2n-1}, u)) \\
&\leq \Psi[\eta d(Sv, Tx_{2n-1}) + \eta d(fv, Sv) + \eta d(gx_{2n-1}, Tx_{2n-1}) \\
&\quad +\eta d(gx_{2n-1}, Sv)] + \Psi(d(gx_{2n-1}, u)).
\end{aligned}$$

Taking the limit as $n \to \infty$, we find that $\Psi(d(fv,u)) \leq \Psi(\eta d(fv,u))$. Using (i), we have $d(fv,u) \leq \eta d(fv,u)$. Since $\eta < 1$, we find from lemma 2.11 that $d(fv,u) = 0$, $fv = Sv = u$. Since $u \in f(X) \subset T(X)$, there exists a $w \in X$ such that $Tw = u$. Next, we show that $gw = u$. Consider

$$\begin{aligned}
\Psi(d(gw,u)) &\leq \Psi(d(gw, fx_{2n}) + d(fx_{2n}, u)) \\
&\leq \Psi(d(gw, fx_{2n})) + \Psi(d(fx_{2n}, u)) \\
&\leq \Psi(\alpha(x_{2n}, w)d(Sx_{2n}, Tw) + \beta(x_{2n}, w)d(fx_{2n}, Sx_{2n}) \\
&\quad +\gamma(x_{2n}, w)d(gw, Tw) + \lambda(x_{2n}, w)d(fx_{2n}, Tw) \\
&\quad +\kappa(x_{2n}, w)d(gw, Sx_{2n})) + \Psi(d(fx_{2n}, u)) \\
&\leq \Psi(\eta d(Sx_{2n}, Tw) + \eta d(fx_{2n}, Sx_{2n}) + \eta d(gw, Tw) \\
&\quad +\eta d(fx_{2n}, Tw)) + \Psi(d(fx_{2n}, u)).
\end{aligned}$$

Letting $n \to \infty$, we have $d(gw,u) \le \eta d(gw,u)$. It follows that $gw = u = Tw$. Thus $\{f,S\}$ and $\{g,T\}$ have a common point of coincidence in X. If $\{f,S\}$ and $\{g,T\}$ are weakly compatible, $fu = fSv = Sfv = Su = w_1$ (say) and $gu = gTw = Tgw = Tu = w_2$ (say).

$$\begin{aligned}
\Psi(d(w_1,w_2)) &= \Psi(d(fu,gu)) \\
&\le \Psi(\alpha(u,u)d(Su,Tu) + \beta(u,u)d(fu,Su) + \gamma(u,u)d(gu,Tu) \\
&\quad +\lambda(u,u)d(fu,Tu) + \kappa(u,u)d(gu,Su)) \\
&= \Psi(\alpha(u,u)d(w_1,w_2) + \beta(u,u)d(w_1,w_1) + \gamma(u,u)d(w_2,w_2) \\
&\quad +\lambda(u,u)d(w_1,w_2) + \kappa(u,u)d(w_2,w_1)) \\
&= \Psi[(\alpha(u,u) + \lambda(u,u) + \kappa(u,u))d(w_1,w_2)] \\
&\le \Psi(\eta d(w_1,w_2)).
\end{aligned}$$

From the properties of Ψ, we have $d(w_1,w_2) \le \eta d(w_1,w_2)$, which implies that $w_1 = w_2$ and hence $fu = gu = Su = Tu$. Now we are in a position to show that $u = gu$. Note that

$$\begin{aligned}
\Psi(d(u,gu)) &\le \Psi(\alpha(v,u)d(Sv,Tu) + \beta(v,u)d(fv,Sv) + \gamma(v,u)d(gu,Tu) \\
&\quad +\lambda(v,u)d(fv,Tu) + \kappa(v,u)d(gu,Sv)) \\
&= \Psi(\alpha(v,u)d(u,gu) + \beta(v,u)d(u,u) + \gamma(v,u)d(gu,gu) \\
&\quad +\lambda(v,u)d(u,gu) + \kappa(v,u)d(gu,u)) \\
&= \Psi[(\alpha(v,u) + \lambda(v,u) + \kappa(v,u))d(u,gu)] \\
&\le \Psi(\eta d(u,gu)).
\end{aligned}$$

Thus $d(u,gu) \le \eta d(u,gu)$, which implies that $gu = u$ and u is a common fixe point of f, g, S and T. For uniqueness, let us suppose that u^* is also a fixed point of f, g, S and T. Then

$$\begin{aligned}
\Psi(d(u,u^*)) &\le \Psi(\alpha(u,u^*)d(Su,Tu^*) + \beta(u,u^*)d(fu,Su) + \gamma(u,u^*)d(gu^*,Tu^*) \\
&\quad +\lambda(u,u^*)d(fu,Tu^*) + \kappa(u,u^*)d(gu^*,Su)) \\
&= \Psi[(\alpha(u,u^*) + \lambda(u,u^*) + \kappa(u,u^*))d(u,u^*)] \\
&\le \Psi(\eta d(u,u^*)),
\end{aligned}$$

which is possible only if $u = u^*$. The proofs for the cases in which $g(X)$, $f(X)$ or $T(X)$ is complete are similar, and are therefore omitted.

4. Example

Let $X = [0,1]$, $E = R^2$ with the usual norm, be a real Banach space. Define $P = \{(x,y) \in E, x,y \geq 0\}$. Let $d : X \times X \to E$ be given as

$$d(x,y) = (|x-y|, |x+y|).$$

Then (X,d) is a cone metric space. Let $\alpha = \beta = \gamma = \iota = \kappa = 1/8$, $\Psi(t) = (\frac{x}{1+x^2}, \frac{y}{1+y^2})$ Then Ψ has the properties mentioned in Theorem 3.1. Let f, g, S and T:$X \to X$ be defined as

$$f(X) = 2^x - 1,\ T(X) = 4^x - 1,\ g(X) = x,\ S(X) = 2x$$

Then f, g, T and S have the properties in Theorem 3.1. Also f, g, T and S satisfy inequalities 3.1. Hence the conditions of Theorem 3.1 are satisfied. Here it is that 0 is the common point of coincidence and also the unique common fixed point of f, g, T and S.

Conflict of Interests

The authors declare that there is no conflict of interests.

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