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ORBITAL COMPLETENESS AND FIXED POINTS OF $(\delta,k)-$WEAK CONTRACTIONS

XAVIER UDO-UTUN*

Department of Mathematics and Statistics, University of Maiduguri, Maiduguri - Nigeria

Abstract. Extension of Ciric's (Proc. Amer. Math. Soc. N0. 45 (1974), 267 - 273) asymptotic fixed point theorem for quasi-contractions in $T-$orbitally complete metric spaces is obtained for the class of $(\delta,k)-$weak contractions studied by Berinde (Carpath. J. Math. 19(1):7-22, 2003; Nonlinear Anal. Forum 9(1): 43-53, 2004). Our results are significant extensions of those of Ciric and Berinde mentioned above and numerous others in literature.

Keywords: $(\delta,k)-$weak contraction; almost contractions; fixed points; generalized contractions; metric spaces; orbital completeness; quasi-contractions.

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1. Introduction and Preliminaries

Orbitally complete metric spaces have been investigated by a number of researchers in connection with so many contractive conditions to obtain asymptotic fixed point theorems. In this praxis we prove asymptotic fixed point theorems for the class of $(\delta,k)-$weak contractions as an extension of Ciric [7] result for quasi-contractions in $T-$orbitally complete metric spaces. Fixed point theorems for $(\delta;k)-$weak contractions have been proved by Berinde in the context

*Corresponding author

E-mail address: xavierudoutun@gmail.com

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of complete metric spaces. A complete metric space is $T-$orbitally complete metric space but the converse does not hold, so our results are extensions of those of Berinde as well.

The Banach contraction condition, in a metric space (X,d), given by $d(Tx,Ty) \leq Ld(x,y)$, $0 \leq L < 1$, has so many generalizations - significant among which include the class of generalized contractions defined by Ciric [6] as follows: A mapping $T: X \to X$ is called a generalized contraction if and only if there exist nonnegative numbers q,r,s and t such that:

(1)
$\sup\{q+r+s+2s\} < 1$ and $d(Tx;Ty) \leq qd(x,y)+rd(x,Tx)+sd(y,Ty)+t[d(x,Ty)q+d(y,Tx)]$.

T is called contractive if:

$$d(Tx,Ty) < d(x,y), \tag{2}$$

while T is called a quasi-contraction (Ciric [7]) if there exists $L \in (0,1)$ such that

$$d(Tx,Ty) \leq L\max[d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)]. \tag{3}$$

Ciric [7] observed that the class of quasi-contractions contains the class of generalized contractions as a proper subclass.

It is worth mentioning that the contractive condition (2) restrict applications only to the class of continuous operators while the contractive conditions (1) and (3) accommodates discontinuous operators as well. The search for contractive conditions that do not require continuity of operators culminated in 1969 with the appearance of the Kannan [9] contractive condition below:

$$d(Tx,Ty) \leq a[d(x;Tx)+d(y,Ty), 0 \leq a < \frac{1}{2}. \tag{4}$$

The Chatterjea [5] contractive condition which followed is independent of both the contractive condition (2) and the Kannan condition (4) which in turn is independent of (2). Consequently unlike condition (2) the Kannan condition (4 does not generalize the well known Banach condition above. In a first attempt, the three contractive conditions were combined by Zamfirescu [12] in one theorem to generalize and extend the Banach fixed point theorem. Rhoades [11] noted that the Zamfirescu result is generalized by the Ciric contractive condition (3). There have been numerous generalizations and extensions of the Banach fixed point theorem in literature and they are, basically, modifications of those mentioned above.

Following Ciric [7], Berinde [1, 2] investigated all the contractive conditions which make use of displacements of the forms $d(x,y), d(x,Tx), d(x,Ty), d(Tx,Ty), d(y,Tx)$, and $d(y,Ty)$, to introduce the following the notion of $(\delta,k)-$weak contractions:

Definition 1. [2] *Let X be a metric space, $\delta \in (0,1)$ and $k \geq 0$, then a mapping $T : X \to X$ is called $(\delta,k)-$weak contraction (or a weak contraction) if and only if:*

$$d(Tx;Ty) \leq \delta d(x,y) + kd(y,Tx),\ \textit{for all}\ x,y \in X \tag{5}$$

In [1] Berinde proved the following results:

Theorem 2. [3] *Let (X,d) be a complete metric space and $T : X \to X$ be a $(\delta,k)-$weak contraction. Then*

(1) $Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$.

(2) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{1}$ given by $x_{n+1} = Tx_n,\ n = 0,1,2,...$ converges to some $x^* \in Fix(T)$.

(3) The following estimates:

$$d(x_n,x^*) \leq \frac{\delta^n}{1-\delta} d(x_0,x_1),\ n = 0,1,2,...$$
$$d(x_n,x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1},x_n),\ n = 0,1,2,...$$

hold, where δ is the constant appearing in (5.

(4) Under the additional condition that there exist $\theta \in (0.1)$ and some $k_1 \geq 0$ such that:

$$d(Tx,Ty) \leq \theta.d(x;y) + k_1.d(x,Tx)\ \textit{for all}\ x;y \in X \tag{6}$$

the fixed point x^* is unique and the Picard iteration converges at the rate $d(x_n,x^*) \leq \theta d(x_{n-1},x^*);\ n \in N$.

It is observed in Berinde [1, 2] that most contractive-type conditions including those of Edelstein [8], Kannan [9], Zamfirescu [12], Ciric [7] and several others can be and are being studied and investigated under the concept of $(\delta,k)-$weak contractions. An excellent compilation and comparison of different contractive definitions and fixed point results, prior to introduction of $(\delta,k)-$weak contractions, is treated in Rhoades [11].

Following Ciric [7] we employ the following notations: For any subset $A \subset X$, $D(A) = \sup\{d(a,b) : a,b \in A\}$ and for each $x \in X$, $\mathscr{O}(x,n) = \{x, Tx, ...T^n x\}$, $n = 1,2...$ while $\mathscr{O}(x,\infty) = \{x, Tx, ...T^n x, ...\}$. Below are other terms and results needed in the sequel:

Definition 3. *Let (X,d) be a metric space and $T : X \to X$ a self-map. Then X is called $T-$orbitally complete if and only if every Cauchy sequence which is contained in the orbit $\mathscr{O}(x,\infty)$ of T at $x \in X$ (where $\mathscr{O}(x,\infty) = \{x, Tx, ...T^n x, ...\} = \cup_{n=0}^{\infty}\{T^n x\}$) converges in X.*

Theorem 4. *Let T be a self-map of a metric space (X,d) which is $T-$orbitally complete. Suppose, for every pair $x,y \in X$ and $q \in [0,1)$, T satisfies the following contractive condition:*

$$d(Tx;Ty) \leq q \max\{d(x,y), d(x,Tx), d(x,Ty), d(y,Tx), d(x,Ty)\}. \tag{7}$$

Then,

(a) T has a fixed point $p \in X$,

(b) $\lim_{n\to\infty} T^n x = p$, and

(c) for every $x \in X$ we have

$$d(T^n x, p) \leq \frac{q^n}{1-q} d(x,Tx). \tag{8}$$

2. Main Results

To prove the main results, we prove the following analogues of lemmas in Ciric [7]

Lemma 5. *Let (X,d) be a metric space, $T : X \to X$ a $(\delta,k)-$weak contraction and $n \in N$ then, for each $x \in X$ we have*

$$d(T^j x, T^k x) \leq \min\{1, [\delta + L]\} D[\mathscr{O}(x,n)],\ 1 \leq j,k \leq n \text{ and} \tag{9}$$

$$D[\mathscr{O}(x,\infty)] \leq \frac{1}{1-\delta} d(x,Tx). \tag{10}$$

PROOF

Let $1 \leq j \leq k \leq n \in \mathbb{N}$ and for any $x \in X$ we have:

$$\begin{aligned} d(T^j x, T^k x) = d(TT^{j-1}x, TT^{k-1}x) &\leq \delta d(T^{j-1}x, T^{k-1}x) + Ld(T^{k-1}x, T^j x) \\ &\leq [\delta + L]D[\mathscr{O}(x,n)] \end{aligned}$$

$$\implies d(T^j x, T^k x) \leq \min\{1; [\delta + L]\} D[\mathscr{O}(x,n)]\}$$

This proves (9). To prove (10), we observe that for any $n \in N$ and for a given $x \in X$ we have, using (9):

$$\sup_{1 \leq j,k \leq n} d(T^j x, T^k x) = \min\{1; [\delta + L]\} D[\mathscr{O}(x,n)], 1 \leq j,k \leq n]$$

$$\implies D[\mathscr{O}(x,n)] = d(T^{j_*}x; T^{k_*}x) \tag{11}$$

for some j_* and k_* satisfying $0 \leq j_* \leq k_* \leq n$. Now, if $D[\mathscr{O}(x,n)] = d(x, T^{k_*}x)$ (i.e $j_* = 0$) then $D[\mathscr{O}(x,n)] \leq \frac{1}{1-\delta} d(x,Tx)$ (see [7]) and the proof is done. Suppose $j_* \neq 0$ then $D[\mathscr{O}(x,n)] = d(T^{j_*}x, T^{k_*}x),\ j_* \geq 1$. In this case setting $p = k_* - j_*$ we make use of the fact that $d(T^n x, T^{n+1}x) \leq \delta^n d(x,Tx)$ (see [1, 3]) to obtain:

$$\begin{aligned} D[O(x,n)] &= d(T^{j_*}x, T^{k_*}x) = d(T^{j_*}x, T^{j_*+p}x) \\ &\leq d(T^{j_*}x, T^{j_*+1}x) + d(T^{j_*+1}x, T^{j_*+2}x) + ... + d(T^{j_*+p-1}x; T^{j_*+p}x) \\ &= d(T^{j_*}x, T^{j_*+1}x) + d(T^{j_*+1}x, T^{j_*+2}x) + ... + d(T^{k_*-1}x; T^{k_*}x) \\ &= \delta^{j_*} d(x,Tx) + \delta^{j_*+1} d(x,Tx) \delta^{j_*+2} d(x,Tx) + ::: + \delta^{j_*+p} d(x,Tx) \\ &= \delta^{j_*}[1 + \delta\delta^2 + ... + \delta^p] d(x,Tx) \\ &= \frac{\delta^{j_*}[1-\delta^p]}{1-\delta} d(x,Tx) \qquad (12) \end{aligned}$$

$$\therefore\ D[O(x,n)] \leq \frac{1}{1-\delta} d(x,Tx). \tag{13}$$

By induction, since the last inequality holds for all n we have that the desired result holds viz, $D[O(x,\infty)] \leq \frac{1}{1-\delta} d(x,Tx)$. End of proof. □

Theorem 6. *Let T be a self-mapping of a metric space (X,d) which is $T-$orbitally complete. Suppose T is an almost contraction mapping ($(\delta, L)-$weak contraction) i.e T satisfies (5) stated* *below:*

$$d(Tx,Ty) \leq \delta d(x,y) + Ld(y,Tx)$$

for all pair $x,y \in X$, *some* $\delta \in [0,1)$ *and* $L \geq 0$. *Then,*

(a) T *has a fixed point* $x^* \in X$,

(b) $\lim_{n\to\infty} T^n x = x^*$, *and*

(c)

$$d(T^n x, x^*) \leq \frac{\delta^n}{1-\delta} d(x,Tx) \tag{14}$$

for every $x \in X$.

PROOF

It follows from equation (12) that $d(T^n x, T^m x) \leq \frac{\delta^n[1-\delta^{n-m}]}{1-\delta} d(x,Tx)$ for all $n,m \in \mathbb{N}$, where $n > m$. This means that $\{T^n x\}_{n=0}^{\infty}$ is a Cauchy sequence which, considering the fact that X is a $T-$orbitally complete metric space, converges to a limit x^* in X. That x^* is a fixed point of T (i.e $x^* = Tx^*$) follows from (13) since for all $x \in X$ we have $d(x^*, Tx^*) \leq D[O(x,\infty)]$ $\leq \frac{1}{1-\delta} d(x,Tx); n,m \in \mathbb{N}$. This implies that $d(x^*,Tx^*) \leq \frac{1}{1-\delta} d(x^*,Tx^*)$ yielding:

$$\begin{aligned} d(x^*,Tx^*) &\leq \frac{1}{1-\delta}[d(x^*,T^{n+1}x) + d(TT^n x, Tx^*)] \\ &\leq \frac{1}{1-\delta}[d(x_{n+1},x^*) + \delta d(x_n,x^*) + Ld(T^{n+1}x,x^*)] \\ &\leq \frac{1}{1-\delta}[(L+1)d(x_{n+1},x^*) + \delta d(x_n,x^*). \qquad (15) \end{aligned}$$

We observe that equation (15) holds for all $n \in \mathbb{N}$ and $d(x_n,x^*) \longrightarrow 0$ as $n \longrightarrow 0$. Therefore $x^* = Tx^*$ since $d(x^*,Tx^*)$ can be made arbitrary small with arbitrarily large n. This proves (a) and (b). The proof of (c) follows from that $x^* \in Fix(T)$ and from equation (15) viz:

$$\begin{aligned} d(T^n x, x^*) = d(T^n x, T^{n-1}x^*) &\leq \frac{\delta^n[1-\delta^{n-n+1}]}{1-\delta} d(x,Tx) \\ &\leq \frac{\delta^n}{1-\delta} d(x,Tx). \end{aligned}$$

This completes the proof. □

3. Conclusion

We conclude by making the following highlights:

(1) Uniqueness is not ensured for all $(\delta, L)-$weak contractions except in the case of a subclass for which condition (6) of Berinde's [1] result holds.

(2) Equation (9) stresses the fact that whereas the orbit-size $D[O(x,\infty)]$ of a quasi-contraction depends on an initial guess $x \in X$, the orbit size $D[O(x,\infty)]$ of a $(\delta,k)-$weak contraction may not depend on an initial guess $x \in X$, it depends on some images $T^k x$, $k = 0,1,2,...$ of the initial guess x. This is in the case where $\delta + L \geq 1$. On the other hand, when $\delta + L < 1$ the size of the orbits of a $(\delta,k)-$weak contractive operator like the quasi-contractive counterpart depend on an initial guess x and the $(\delta,k)-$weak contractions in this case behave like quasi-contractions in so many respects.

(3) Following from above uniqueness of fixed points of $(\delta,k)-$weak contractions is guaranteed if $\delta + L < 1$ otherwise, particularly for $L = 1$, uniqueness is guaranteed by the condition of Berinde mentioned above.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] V. Berinde *On the approximation of fixed point of weak contraction mappings,* Carpathian J. Math. 19 (2003), 7 - 22.

[2] V. Berinde, *Approimating fixed point of weak contractions using Picard's iteration,* Nonlinear Anal. Forum, 9 (2004), 43 - 53.

[3] V. Berinde, *Iterative approximation of fixed points,* Lecture Notes in Mathematics, Springer- VCerlag Berlin Heidelberg (2007).

[4] D. W. Boyd, J. S. W. Wong, *On Nonlinear contractions,* Proc. Amer. Math. Soc. 20 (1969), 458 - 464.

[5] S. K. Chatterjea, *Fixed-point theorems,* C.R. Acad. Bulgare. Sc. 25 (1972), 727 - 730.

[6] Lj. Ciric, *Generalized contractions and fixed point theorems,* Publ. Inst. Math. 12(26) (1971), 19 - 26.

[7] Lj. Ciric, *Generalzation of Banach's contraction principle,* Proc. Amer. Math. Soc. 45 (1974), 267 - 273.

[8] M. Edelstein, *On fixed and periodic points under contractive mappings* J. London Math. Soc. 37 (1962), 74 -79.

[9] R. Kannan *Some result on fixed points II,* Amer. Math. Monthly 76 (1969), 405 - 406.

[10] E. Rakotch, *A note on contractive mappings,* Proc. Amer. Math. Soc. 13 (1962), 459 - 465.

[11] B. E. Rhoades, *A comparison of various definitions of contractive mappings,* Trans. Amer. Math. Soc. 226 (1977), 257 - 290.

[12] T. Zamfirescu, *Fixed point theorem in metric spaces,* Arch. Math. (Basel), 23 (1992), 292 - 298.