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ON SOME EXTENSIONS OF BANACH'S CONTRACTION PRINCIPLE AND APPLICATIONS TO THE CONVERGENCE AND STABILITY OF SOME ITERATIVE PROCESSES

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Abstract. In this paper, we obtain existence and uniqueness results of fixed points of nonlinear operators satisfying the condition of the form (Φ_1, Φ_2) given as a perturbation of Φ_2 contraction by a convenable function Φ_1 in metric and Banach spaces, which enable us to extend the Banach's mapping principle and other results in the literature. Also, the Φ-quasinonexpansive character of our context is shown in order to obtain results of convergence and stability of iterative processes of Mann and Ishikawa.

Keywords: Banach space; fixed point; metric space; iterative process.

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1. Introduction

Let (X,d) be a complete metric space and let $f : X \longrightarrow X$ be a mapping. Recall that f is said to be contractive if there exists a positive constant k with $k < 1$ such that

$$d(f(x), f(y)) \leq kd(x,y), \quad \forall x, y \in X. \tag{1.1}$$

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The classical theorem of Banach that has emerged with the appearance of his work in 1922 [4] says that any self-map f defined on X satisfying an inequality of the form (1.1) (called contraction mapping) has a unique fixed point x_0 in X ($f(x_0) = x_0$).

A fixed point is seen as an invariant point by the transformation f. This point is obtained as a limit of iterative scheme of the form

$$x_n = f^{(n)}(x_0) = \underbrace{(f \circ f \circ .... \circ f)}_{n\ times}(x_0),$$

where x_0 is an arbitrary point in X. Note that, recently, this result has been proved in Palais [33] by using a simple proof based on the triangular inequality.

The Banach Contraction Principle (BCP) is a major tool in functional analysis, nonlinear analysis and differential equations (existence, uniqueness and stability of solutions). Many variants of BCP and some results of its extensions have been established by several authors; see [1, 5, 7, 8, 9, 12, 13, 17, 19, 22, 23, 25, 27, 34, 35, 36, 37, 39, 40, 41, 43] and the references therein.

Recall that the condition k strictly less than 1 ensures the existence and uniqueness of the fixed point as the following example shown: If $X = \{0,1\}$ equipped with the discreet metric, then the map T given by

$$T(x) = \begin{cases} 1 \text{ if } x = 0, \\ 0 \text{ if } x = 1, \end{cases}$$

satisfies that $d(T(0),T(1)) = d(0,1) = 1$ but T has not any fixed point in X. On the other hand, if $S = Id_X$ on any metric space X, then $d(S(x),S(y)) = d(x,y)$ but it is easy to observe that every point of X is a fixed point.

In the case where the metric space (X,d) is compact and the inequality (1.1) is strict with $k = 1$, Edelstein [16] has established the existence and uniqueness of the fixed point of the map f. However, this result is not much used in the case of Banach spaces of infinite dimensional, this is due to the fact that compact sets in theses spaces have empty interiors according to the Riesz Theorem. We also remark here that the mapping T is nonexpansive if and only if $k = 1$ in inequality (1.1).

If X is a Banach space, it was be naturel to ask if nonexpansive self maps defined on convex, closed and bounded sets possess fixed points. The answer to this question is negative as the following example shown due to Kakutani in 1943 and constructed on the closed unit ball of the Banach space c_0.

Example 1.1. Let B the unit ball in c_0 and $T : B \longrightarrow B$ defined by $T(x_1,x_2,....) = (1-\|x\|,x_1,x_2,....)$. It is easy to check that T is nonexpansive and fixed-point free.

After around twenty years, two surprising results due respectively to Browder [10] and Kirk [23] appeared and revolutionized the theory by showing the role played by the geometry of Banach spaces in the existence of fixed points for this class of maps. Since, this direction was the object of several significant contributions, let us quote for example the works of [2, 3, 11, 18, 28].

In this paper, in the case of complete metric spaces, we establish some results of the existence and uniqueness of fixed points for nonlinear maps T satisfying an inequality where the distance between the values $T(x)$ and $T(y)$ are dominated by suitably selected perturbation of a Φ-contraction. Also, the fact that if M is a convex set of a Banach space X, T a self map on M and P is a real polynomial for which the sum of its coefficients is equal to one implies that $P(T)$ is also a self map on M pushed us to study the set of fixed points of $P(T)$ and the possible coincidence with those of T including the framework of iterates as a particular case. In addition, taking into account the recent results of Ruiz [38], we show the convergence of the iterative processes of Mann and Ishikawa and the almost stability of the Picard's process.

2. Main results

We start our results by the principal one given by the following theorem.

Theorem 2.1. *Let (X,d) be a complete metric space and let $T : X \longrightarrow X$ be a continuous map satisfying the following condition:*

$$d(T(x),T(y)) \leq \Phi_1[d(x,T(x)),d(y,T(y)),d(x,y)] + \Phi_2[d(x,y)],$$

for all $x,y \in X$. Here, $\Phi_1 : [0,\infty[\times[0,\infty[\times]0,\infty[\longrightarrow [0,\infty[$ and $\Phi_2 : [0,\infty[\longrightarrow [0,\infty[$ are functions such that

(1) $\Phi_1(t_1,t,t) \leq \widetilde{\Phi_1}(t_1) \qquad \forall t_1 \geq 0$ *and* $\forall t > 0$;

(2) Φ_2 *is nondecreasing and* $\lim_{n\longrightarrow+\infty}\Phi_2^{(n)}(t)=0$;

(3) $(I-\widetilde{\Phi_1})^{-1}$ *exists with* $(I-\widetilde{\Phi_1})^{-1}$ *nondecreasing such that* $\Phi_2(I-\widetilde{\Phi_1})^{-1}\leq(I-\widetilde{\Phi_1})^{-1}\Phi_2$ *and* $\sum_{n=0}^{+\infty}(I-\widetilde{\Phi_1})^{-(n)}\Phi_2^{(n)}(t)<\infty$ *for all* $t\geq 0$.

Then T *has at least a fixed point in* X. *In addition, if* $\Phi_1(0,0,t_3)=0\ \ \forall t_3>0$, *one obtains the uniqueness of the fixed point of* T.

Remark 2.1. Every function Φ which satisfies condition 2 of Theorem 2.1 must check that $\Phi(0)=0$ and $\Phi(t)<t$ for all $t>0$.

Proof of Theorem 2.1. Let x_0 be an arbitrary point of X and let $(x_n)_1^\infty$, where $x_n=T^{(n)}(x_0)$ and n is a positive integer, be the sequence of iterates of T at x_0. If $x_n=x_{n+1}$ for some n then the result is immediate. So let $x_n\neq x_{n+1}$, for all n. Note that

$$\begin{aligned}d(x_{n+1},x_n)&\leq\Phi_1[d(x_n,T(x_n)),d(x_{n-1},T(x_{n-1})),d(x_n,x_{n-1})]+\Phi_2[d(x_n,x_{n-1})]\\&=\Phi_1[d(x_n,x_{n+1}),d(x_{n-1},x_n),d(x_n,x_{n-1})]+\Phi_2[d(x_n,x_{n-1})].\end{aligned}$$

$$d(x_{n+1},x_n)\leq\widetilde{\Phi_1}[d(x_n,x_{n+1})]+\Phi_2[d(x_n,x_{n-1})],$$

which implies that

$$(I-\widetilde{\Phi_1})[d(x_n,x_{n+1})]\leq\Phi_2[d(x_n,x_{n-1})].$$

Since $(I-\widetilde{\Phi_1})$ is invertible and nondecreasing, we find that

$$d(x_n,x_{n+1})\leq(I-\widetilde{\Phi_1})^{-1}[\Phi_2(d(x_n,x_{n-1}))].$$

Moreover, the nondecreasing property of Φ_2, and $(I-\widetilde{\Phi_1})^{-1}$ yields that

$$d(x_n,x_{n+1})\leq(I-\widetilde{\Phi_1})^{-1}[\Phi_2((I-\widetilde{\Phi_1})^{-1}(\Phi_2(d(x_{n-1},x_{n-2}))))].$$

By use of the first part of the assumption (3), we obtain that

$$d(x_n,x_{n+1})\leq(I-\widetilde{\Phi_1})^{-(2)}[\Phi_2^{(2)}(d(x_{n-1},x_{n-2}))].$$

In addition by conduction, we see that

$$d(x_n,x_{n+1})\leq(I-\widetilde{\Phi_1})^{-(n)}[\Phi_2^{(n)}(d(x_1,x_0))].$$

By the triangle inequality, we have, for $m \geq n$,

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+....+d(x_{m-1},x_m) \\ &\leq (I-\widetilde{\Phi_1})^{(-n)}\Phi_2^{(n)}[d(x_1,x_0)]+(I-\widetilde{\Phi_1})^{-(n+1)}\Phi_2^{(n+1)}[d(x_1,x_0)]+... \\ &.....+(I-\widetilde{\Phi_1})^{-(m-1)}\Phi_2^{(m-1)}[d(x_1,x_0)]. \end{aligned}$$

Put $H=(I-\widetilde{\Phi}_1)^{-1}\Phi_2$. Using the second part of the assumption 3, we infer that

$$\begin{aligned} d(x_n,x_m) &\leq (H^{(n)}+H^{(n+1)}+........+H^{(m-1)})d(x_0,x_1) \\ &\longrightarrow 0 \; as \; m,n \longrightarrow +\infty. \end{aligned}$$

This shows that $(x_n)_1^{+\infty}$ is a Cauchy sequence and since X is a complete, there exists $u \in X$ such that $\lim\limits_{n\longrightarrow+\infty} x_n = u$. Further, the continuity of T in X implies $Tu = u$. This implies that u is a fixed point of T in X.

Moreover, assume that the condition $\Phi_1(0,0,t_3)=0, \forall t_3>0$ is satisfied. To show the uniqueness, let $v \neq u$ in X such that $T(v)=v$. Then

$$\begin{aligned} d(u,v) &= d(T(u),T(v)) \\ &\leq \Phi_1[d(u,T(u)),d(v,T(v)),d(u,v)]+\Phi_2[d(u,v)] \\ &= \Phi_1[0,0,d(u,v)]+\Phi_2[d(u,v)] \\ &\leq \Phi_2[d(u,v)] \\ &< d(u,v). \end{aligned}$$

This is a contradiction. Hence u is a unique fixed point of T in X.

Remark 2.2. As a particular cases of Theorem 2.1, we find the following situations

First case: If $\Phi_1 \equiv 0$ and $\Phi_2 = \alpha t$ such that $\alpha \in [0,1[$. We obtain the Banach contraction principle.

Second case: If $\Phi_1 \equiv 0$. We obtain one of the main result of Berinde ([6], Theorem 2).

Third case: If $\Phi_1(t_1,t_2,t_3)=\dfrac{\alpha t_1 t_2}{t_3}$ and $\Phi_2(t)=\beta t$ for some $\alpha,\beta \in [0,1[$ with $\alpha+\beta<1$. We obtain the main result of D. S. Jaggy ([25], Theorem 1).

Remark 2.3. It is easy to observe that the technics of the proof of Theorem 2.1 can be used to establish one of the main result of Imoru *et al.* ([20], Theorem 2.1).

Example 2.1. Let $\Phi_1 : [0,+\infty[\times[0,+\infty[\times]0,+\infty[\longrightarrow[0,+\infty[$ and $\Phi_2 : [0,+\infty[\longrightarrow[0,+\infty[$ given by

$$\Phi_1(t_1,t_2,t_3) = |Sin(t_1)|t_2e^{-t_3}$$

and

$$\Phi_2(t) = \alpha t; (0 \leq \alpha < \frac{e-1}{e}).$$

We can check that Φ_1 and Φ_2 satisfy assumptions of Theorem 2.1. Indeed, we have $|Sin(t_1)| \leq |t_1|$ for all $t_1 \geq 0$ and $te^{-t} \leq \frac{1}{e}$ for $t > 0$, which gives that the assumption 1 is established by taking $\widetilde{\Phi_1}(t_1) = \frac{1}{e}t_1$. Furthermore, we have $\Phi_1(t_1,0,t_3) = 0$. On the other hand, the fact that $(I-\widetilde{\Phi_1})^{-1}\Phi_2 = \Phi_2(I-\widetilde{\Phi_1})^{-1} = \frac{\alpha e}{e-1}t$ implies trivially the convergence of the series $\sum_{n=0}^{+\infty}(I-\widetilde{\Phi_1})^{-(n)}\Phi_2^{(n)}(t) = \sum_{n=0}^{+\infty}(\frac{\alpha e}{e-1})^n t$.

Next, we show the existence of unique common fixed point of two mappings which are commuting.

Lemma 2.1. *Let T_1 and T_2 be two self-maps defined on a metric space (X,d) satisfying the following conditions:*

($\imath$): $T_1 \circ T_2 = T_2 \circ T_1$;

($\imath\imath$): $F(T_i) \subseteq F(T_j)(i \neq j)(i,j = 1,2)$ *, where $F(T_k), k = 1,2$ is the set of fixed points of T_k.*

If T_j has a unique fixed point $x_0 \in X$, then x_0 is the unique fixed point of T_i.

Proof. If x_0 is a fixed point of T_j, then $T_j(x_0) = x_0$. It follows that $T_i[T_j(x_0)] = T_i(x_0)$ Since T_i and T_j are commuting, we have $T_j[T_i(x_0)] = T_i(x_0)$. This shows that $T_i(x_0)$ is a fixed point of T_j. The fact that x_0 is the unique fixed point of T_j, implies that $T_i(x_0) = x_0$. Therefore x_0 is a fixed point of T_i. The uniqueness concerning the fixed point of T_i is trivial.

Corollary 2.1. *Let M be a non empty, convex subset of a Banach space $(X,\|.\|)$ and $T : M \longrightarrow M$ a map (not necessarily continuous) and let P a real polynomial given by:*

$$P(x) = \lambda_0 + \lambda_1 x + ... + \lambda_n x^n$$

with $\sum_{i=0}^{n} \lambda_i = 1$ *and* $\lambda_i \geq 0$ *for* $i = 0, 1, ..., n$. *Then,*

$(\imath)$**:** *If* $x_0 \in M$ *is a fixed point of* T*, then* x_0 *is a fixed point of* $P(T)$ *defined by*

$$P(T) = \lambda_0 I + \lambda_1 T + ... + \lambda_n T^n$$

$(\imath\imath)$**:** *If* $x_0 \in M$ *is a unique fixed point of* $P(T)$ *and* $P(T) \circ T = T \circ P(T)$*, then* x_0 *is a unique fixed point of* T *in* M.

$(\imath\imath\imath)$**:** *If* $x_0 \in M$ *is a unique fixed point of* $P(T)$ *and* $P(T) \circ T \neq T \circ P(T)$. *Then either:*

(a)**:** T *does not have a fixed point in* M*, or*

(b)**:** T *has* x_0 *as a unique fixed point in* M.

Proof.

$(\imath)$**:** It is easy to observe that $P(T) : M \longrightarrow M$. If $x_0 \in M$ is a fixed point of T, then

$$P(T)(x_0) = \{\sum_{i=0}^{n} \lambda_i\} x_0 = x_0 \in M.$$

This shows that $x_0 \in M$ is a fixed point of $P(T)$.

$(\imath\imath)$**:** Follows from Lemma 2.1 since we have $F(T) \subseteq F(P(T))$.

$(\imath\imath\imath)$**:** Assume that $x_1 \neq x_0$ is another fixed point of T in M, the assertion $(\imath)$ shows that x_1 is a fixed point of $P(T)$ which is a contradiction.

Remark 2.4. Let X be a Banach space and K a convex subset of X. A self mapping T on K is called a nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in K$. Let P a real polynomial given as in assertion $(\imath)$ of Corollary 2.1 with $\lambda_1 > 0$. Kirk [24] showed that in this case $F(T) = F(P(T))$, we note also that the condition $\lambda_1 > 0$ is crucial as the following example shows.

Example 2.2. Let $T : [0,1] \longrightarrow [0,1]$ be defined as follows: $T(x) = 1 - x$. It is easy to observe that T is nonexpansive and $x_0 = \frac{1}{2}$ is the unique fixed point of T.

Now consider the polynomial $P(z) = \frac{1}{2}z^2 + \frac{1}{2}$. It follows that

$$P(T) : [0,1] \longrightarrow [0,1],\ x \longmapsto P(T)(x) = x$$

and $P(T) \circ T = T \circ P(T) = 1 - x$. In this case, we observe that the set of fixed points of $P(T)$ is the interval $[0,1]$. Here, we have $F(T) \subsetneq F(P(T))$.

The following theorem generalizes Theorem 2.1 as well.

Theorem 2.2. *Let M be a non empty, closed, convex subset of a Banach space $(X,\|.\|)$ and $T : M \longrightarrow M$ a map (not necessarily continuous). Assume that Φ_1 and Φ_2 satisfy assumptions given in Theorem 2.1. Let P the following real polynomial:*

$$P(x) = \lambda_0 + \lambda_1 x + ... + \lambda_n x^n,$$

where $\sum_{i=0}^{n} \lambda_i = 1.$ and $\lambda_i \geq 0$ for $i = 0,1,...,n$ for which $P(T)$ commutes with T and for all $x, y \in M$,

$$\|P(T)(x) - P(T)(y)\| \leq \Phi_1(\|x - P(T)(x)\|, \|y - P(T)(y)\|, \|x - y\|) + \Phi_2(\|x - y\|).$$

If $P(T)$ is continuous, then T has a unique fixed point.

Proof. Since $P(T)$ satisfies the assumptions given in Theorem 2.1. Then we can deduce that $P(T)$ has a unique fixed point denoted by u. Hence, the result follows immediately from the assertion $(\imath\imath)$ of Corollary 2.1.

Example 2.3. let $X = \mathbb{R}$ and let a map $T : X \longrightarrow X$ defined as follows:

$$T(x) = \begin{cases} 0 \text{ if } x \in \mathbb{Q} \\ 1 \text{ if } x \notin \mathbb{Q} \end{cases}$$

That T is discontinuous can be easily seen. Now, consider the polynomial $P(z) = z^2$. It can be verified that $P(T) \equiv 0$ satisfy the conditions of Theorem 2.2, and 0 is a unique fixed point of $P(T)$ and T.

In the next theorem, we establish a sufficient conditions for the existence of a unique common fixed point of two mappings which are not necessarily continuous or commuting.

Theorem 2.3. *Let T_1 and T_2 be two self-maps defined on a Banach space $(X,\|.\|)$ and P_1 and P_2 be two polynomials defined as in Corollary 2.1 such that:*

- **$(\imath)$:** $\|P_1(T_1)(x) - P_2(T_2)(y)\| \leq \Phi_1(\|x - P_1(T_1)(x)\|, \|y - P_2(T_2)(y)\|, \|x - y\|) + \Phi_2(\|x - y\|)$, *for all $x, y \in X$ with Φ_1 and Φ_2 satisfy assumptions of Theorem 2.1 .*
- **$(\imath\imath)$:** $P_1(T_1) \circ T_1 = T_1 \circ P_1(T_1)$ *and* $P_2(T_2) \circ T_2 = T_2 \circ P_2(T_2)$.
- **$(\imath\imath\imath)$:** $P_1(T_1) \circ P_2(T_2)$ *is continuous.*

Then T_1 and T_2 have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X, we define a sequence x_n as follows

$$x_n = \begin{cases} P_1(T_1)(x_{n-1}) \text{ if n is odd} \\ P_2(T_2)(x_{n-1}) \text{ if n is even} \end{cases}$$

$x_n \neq x_{n-1}$ for all n. It follows that

$$\begin{aligned} \|x_{2n} - x_{2n+1}\| &= \|P_1(T_1)(x_{2n}) - P_2(T_2)(x_{2n-1})\| \\ &\leq \Phi_1(\|x_{2n} - P_1(T_1)(x_{2n})\|, \|x_{2n-1} - P_2(T_2)(x_{2n-1})\|, \|x_{2n} - x_{2n-1}\|) + \\ &\Phi_2(\|x_{2n} - x_{2n-1}\|) \\ &= \Phi_1(\|x_{2n} - x_{2n+1}\|, \|x_{2n-1} - x_{2n}\|, \|x_{2n} - x_{2n-1}\|) + \Phi_2(\|x_{2n} - x_{2n-1}\|), \end{aligned}$$

which implies that:

$$\|x_{2n} - x_{2n+1}\| \leq \widetilde{\Phi_1}(\|x_{2n} - x_{2n+1}\|) + \Phi_2(\|x_{2n-1} - x_{2n}\|).$$

It follows that

$$(I - \widetilde{\Phi_1})(\|x_{2n} - x_{2n+1}\|) \leq \Phi_2(\|x_{2n-1} - x_{2n}\|).$$

Since $(I - \widetilde{\Phi_1})$ is invertible and $(I - \widetilde{\Phi_1})^{-1}$ is nondecreasing, we have

$$\|x_{2n} - x_{2n+1}\| \leq (I - \widetilde{\Phi_1})^{-1}\Phi_2(\|x_{2n-1} - x_{2n}\|).$$

By the properties of $(I - \widetilde{\Phi_1})^{-1}$ and Φ_2, we have

$$\begin{aligned} \|x_{2n} - x_{2n+1}\| &\leq (I - \widetilde{\Phi_1})^{-1}\Phi_2(I - \widetilde{\Phi_1})^{-1}\Phi_2(\|x_{2n-2} - x_{2n-1}\|) \\ &\leq (I - \widetilde{\Phi_1})^{-(2)}\Phi_2^{(2)}(\|x_{2n-2} - x_{2n-1}\|). \end{aligned}$$

In addition, we have

$$\|x_{2n} - x_{2n+1}\| \leq (I - \widetilde{\Phi_1})^{-(2n)}\Phi_2^{(2n)}(\|x_0 - x_1\|).$$

Similarly, we can show that

$$\|x_{2n+1} - x_{2n+2}\| \leq (I - \widetilde{\Phi_1})^{-(2n+1)}\Phi_2^{(2n+1)}(\|x_0 - x_1\|).$$

Now it can be easily seen that (x_n) is a Cauchy sequence. Let $x_n \longrightarrow u$. Then the sequence $x_{n_k} \longrightarrow u$, where $n_k = 2k-1$. Note that

$$\begin{aligned}[P_1(T_1)\circ P_2(T_2)](u) &= [P_1(T_1)\circ P_2(T_2)](\lim_{k\longrightarrow+\infty} x_{n_k})\\ &= \lim_{k\longrightarrow+\infty} x_{n_{k+1}}\\ &= u.\end{aligned}$$

We now show that $P_2(T_2)(u) = u$. If $P_2(T_2)(u) \neq u$, then

$$\begin{aligned}\|P_2(T_2)(u)-u\| &= \|P_2(T_2)(u)-[P_1(T_1)\circ P_2(T_2)](u)\|\\ &\leq \Phi_1(\|u-P_2(T_2)(u)\|, \|P_2(T_2)(u)-[P_1(T_1)\circ P_2(T_2)](u)\|, \|u-P_2(T_2)(u)\|)\\ &+\Phi_2(\|u-P_2(T_2)(u)\|).\end{aligned}$$

Again the assumption on Φ_1 gives

$$\|P_2(T_2)(u)-u\| \leq \widetilde{\Phi}_1(\|u-P_2(T_2)(u)\|)+\Phi_2(\|u-P_2(T_2)(u)\|),$$

which implies

$$(I-\widetilde{\Phi_1})\|u-P_2(T_2)(u)\| \leq \Phi_2(\|u-P_2(T_2)(u)\|).$$

Since $(I-\widetilde{\Phi_1})$ is invertible and $(I-\widetilde{\Phi_1})^{-1}$ is nondecreasing, we have

$$\|P_2(T_2)(u)-u\| \leq (I-\widetilde{\Phi_1})^{-1}\Phi_2(\|u-P_2(T_2)(u)\|).$$

Let $H = (I-\widetilde{\Phi_1})^{-1}\Phi_2$. It is clear that H is nondecreasing and $H(t) < t \; \forall t > 0$. Thus

$$\|P_2(T_2)(u)-u\| < \|P_2(T_2)(u)-u\|.$$

This is a contradiction. Hence $P_2(T_2)(u) = u$. Also

$$\|P_1(T_1)(u)-u\| = \|P_1(T_1)(u)\circ P_2(T_2)(u)-u\| = 0,$$

which shows that $P_1(T_1)(u) = u$. Now, if $\Phi_1(0,0,t_3) = 0\ \forall t_3 > 0$, let $v \neq u \in X$ be such that $P_1(T_1)(v) = v$. It follows that

$$\begin{aligned}\|v-u\| &= \|P_1(T_1)(v) - P_2(T_2)(u)\| \\ &\leq \Phi_1(\|v-P_1(T_1)(v)\|, \|u-P_2(T_2)(u)\|, \|v-u\|) + \Phi_2(\|v-u\|) \\ &= \Phi_1(0,0,\|v-u\|) + \Phi_2(\|v-u\|) \\ &< \|v-u\|,\end{aligned}$$

which is a contradiction. Hence u is a unique fixed point of $P_1(T_1)$. Also, it is easy to check that u is a unique fixed point of $P_2(T_2)$. Finally using the assertion (u) of Corollary 2.1, we deduce that u is a common fixed point of T_1 and T_2 which completes the proof.

3. Applications

We start this section with the concept of φ-quasinonexpansive mappings.

Definition 3.1. Let T be a self-map defined on a metric space (X,d). We say that $T : X \longrightarrow X$ is a φ-quasinonexpansive mapping if $F(T) \neq \emptyset$ and there exists a function $\varphi : [0,+\infty[\longrightarrow [0,+\infty[$ such that

$$d(T(x),z) \leq \varphi(d(x,z)), \quad \forall x \in X, z \in F(T).$$

It is easy to observe that every contraction mapping is φ-quasinonexpansive (here $\varphi(t) = \alpha t, 0 \leq \alpha < 1$ and $t \in [0,+\infty[$) but the converse is false in general as the following example shows:

Example 3.1. Let $\varphi : [0,+\infty[\longrightarrow [0,+\infty[$ satisfying condition (2) given in Theorem 2.1 and let $X = \mathbb{R}$, then the mapping $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$T(x) = \begin{cases} \varphi(|x|)sin(\frac{1}{x}) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

is φ-quasinonexpansive but not a contraction mapping. This implies that the class of φ-quasinonexpansives mappings contains strictly that of contraction mappings. In the case where $\varphi = Id_X$, we obtain the concept of quasinonexpansive mappings introduced by Tricomi [44] and studied by Diaz and Metcalf [14, 15]; for more details on the class of φ-quasinonexpansives, see [29, 31] and the references therein.

Theorem 3.1. *Adding the condition:*

$$\Phi_1(t_1,0,t_3)=0, \forall t_1,t_3>0, \tag{3.1}$$

to the assumptions of Theorem 2.1, then T is Φ_2-quasinonexpansive.

Proof. By the same theorem, we have proven that T has a unique fixed point x_0. Let $x\in X$ with $x\neq x_0$. Then

$$\begin{aligned} d(T(x),x_0) &\leq \Phi_1[d(x,T(x)),0,d(x,x_0)]+\Phi_2(d(x,x_0)) \\ &= 0+\Phi_2(d(x,x_0))=\Phi_2(d(x,x_0)). \end{aligned}$$

Now, we give the definition of convex metric spaces introduced by Takahashi [42] which play an important role in the development of the fixed point theory, in particular, in the case of Banach spaces.

Definition 3.2. A convex metric space $(X,d,\oplus)$ is a metric space (X,d) together with a convexity mapping $\oplus : X\times X\times[0,1]\longrightarrow X$ satisfying

$$d(z,(1-\lambda)x\oplus\lambda y)\leq(1-\lambda)d(z,x)+\lambda d(z,y), \forall x,y,z\in X, \lambda\in[0,1].$$

Example 3.2. Normed spaces, Hilbert balls and $\mathbb{R}$-trees are good examples of convex metric spaces.

In the case of metric and convex metric spaces, several iterative processes have been defined by many mathematicians. Some of them are the following:

Picard iteration: Let (X,d) be a metric space and $T : X\longrightarrow X$ a self mapping. Let $x_0\in X$ be fixed, we define the sequence $\{x_n\}_n$ recursively by

$$x_{n+1}=T(x_n)=T^{n+1}(x_0), \text{for all } n\in\mathbb{N}. \tag{3.2}$$

Mann iteration ([26]*):* If $(X,d,\oplus)$ is a convex metric space, Mann iteration is defined by the following algorithm

$$x_{n+1}=(1-\alpha_n)x_n\oplus\alpha_n T(x_n), \text{for all } n\in\mathbb{N}, \tag{3.3}$$

where $x_0 \in X$ and $\{\alpha_n\}_n \subset [0,1]$.

Ishikawa iteration ([21]*):* Another iterative process of interest in the case of convex metric spaces is the Ishikawa scheme of two steps given as follows: Let $x_0 \in X$ be fixed, consider the sequence $\{x_n\}_n$ defined by

$$\begin{cases} y_n = (1-\beta_n)x_n \oplus \beta_n T(x_n), \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T(y_n), \text{ for all } n \in \mathbb{N}, \end{cases} \tag{3.4}$$

where $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ are sequences in $[0,1]$.

Following the proof of Theorem 2.1, it is easy to establish the following result for the convergence of the Picard's iteration process.

Proposition 3.1. *Let (X,d) be a complete metric space. Under the assumptions of Theorem 2.1, the Picard iterative process* (3.2) *converges to the unique fixed point of T, for any $x_0 \in X$.*

Moreover, by using Theorem 3.1 together with ([38], Theorem 3.7), we obtain the following result for the convergence of the iteratives schemes of Mann and Ishikawa.

Proposition 3.2. *Let $(X,d,\oplus)$ be a convex complete metric space. Let $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ be two real sequences in $[0,1]$ such that $\{\alpha_n\beta_n\}_n$ converges to some positive real number, let $x_0 \in X$. Under the assumptions of Theorem 3.1 with Φ_2 continuous. Then, the Ishikawa sequence given by* (3.4) *converges to the unique fixed point of T. Moreover, if $\{\beta_n\}_n$ is the constant sequence equal to 0, the Mann iteration given by* (3.3) *converges to the same unique fixed point of T.*

For the remainder of our study, we need the following two definitions about stability of a general iterative processes.

Definition 3.3. Let (X,d) be a metric space and $T : X \longrightarrow X$ a self mapping of X. Let $\{x_n\}_n \subset X$ be the sequence generated by an iteration involving T and defined by

$$x_{n+1} = f(T,x_n), \text{for all } n \in \mathbb{N} \tag{3.5}$$

where $x_0 \in X$ and f is some function. Assume that $\{x_n\}_n$ converges to a fixed point z_0 of T. Let $\{y_n\}_n \subset X$ and we define

$$\varepsilon_n := d(y_{n+1}, f(T,y_n)) \text{ for all } n \in \mathbb{N}. \tag{3.6}$$

Then

$(\imath)$ the iteration process (3.5) is said to be T-stable if $\lim_{n\longrightarrow\infty} \varepsilon_n = 0$ implies $\lim_{n\longrightarrow\infty} y_n = z_0$.

$(\imath\imath)$ the iteration process (3.5) is said to be almost T-stable if $\sum_{n\in\mathbb{N}} \varepsilon_n < \infty$ implies $\lim_{n\longrightarrow\infty} y_n = z_0$.

For more informations and interesting comments on these notions of stability, we can see [30]. On the other hand, it is easy to observe that an iterative process (3.5) which is T-stable is almost T-stable but the converse is not true in general; for the counter example, see [32] and the references therein.

In the following result, we establish the almost stability of Picard's iterative process for our context of self mappings.

Corollary 3.1. *Let (X,d) be a complete metric space. Assume that $T : X \longrightarrow X$ is a self mapping of X satisfying the assumptions of Theorem 3.1 with Φ_2 continuous. If z_0 is the unique fixed point of T and $x_0 \in X$ with $x_{n+1} := T(x_n), n \in \mathbb{N}$ be the Picard process and $\{y_n\}_n \subset X$. Define $\{\varepsilon_n\}_n$ by*

$$\varepsilon_n := d(y_{n+1}, T(y_n)) \text{ for all } n \in \mathbb{N}.$$

If $\sum_{n\in\mathbb{N}} \varepsilon_n < \infty$, then $\lim_{n\longrightarrow\infty} y_n = z_0$. In other words, the Picard process is almost T-stable.

Proof. The result is established by combining the fact that T is Φ_2-quasinonexpansive together with Theorem 4.5 in [38]. This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

REFERENCES

[1] R. Agarwal, M. Meehan and D. O'regan, Fixed point theory and applications, Cambridge Univesity Press, 2004.

[2] D. E. Alspach, A fixed point free nonexpansive map, Proc. Amer. Math. Soc. 82 (1981), 423-424.

[3] J. M. Ayerbe, T. Dominguez and G. Lopez, Measure of Noncompactness in Metric Fixed Point Theory, Birkhauser, 1997.

[4] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922), 133-181.

[5] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear. Anal. Forum. 9(1) (2004), 43-53.

[6] V. Berinde, A priori and a posteriori error estimates for a class of φ-contractions, Bull. Appl. Comput. Math. 326 (1999), 183-192.

[7] R. Bianchini, Sun un problema di S. Reich ringuardante la teoria dei punti fissi, Boll. Un. Math. Ital. 5 (1972), 103-108.

[8] D. W. Boyd and J. S. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969), 458-464.

[9] R. M. Brooks and K. Schmitt, The contraction mapping principle and some applications, Elec. J. Diff. Equa. Monograph 09, 2009.

[10] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. USA. 54 (1965), 1041-1044.

[11] W. L. Bynum, Normal structure coefficients for Banach space, Pacific. J. Math. 86 (1980), 427-436.

[12] S. Chatterjea, Fixed point theorems, C. R. Acad. Bulgare. Sci. 25 (1972), 727-730.

[13] L. Ciric, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45 (1974), 267-273.

[14] J. B. Diaz and F. T. Metcalf, On the structure of the set of subsequential limit points of successive approximations, Bull. Amer. Math. Soc. 73 (1967), 516-519.

[15] J. B. Diaz and F. T. Metcalf, On the set of subsequential limit points of successive approximations, Trans. Amer. Math. Soc. 135 (1969), 459-485.

[16] M. Edelstein, An extension of Banach's contraction principle, Proc. Amer. Math. Soc. 12 (1961), 7-10.

[17] B. Florin, Fixed points theorems in metric spaces endowed with a graph, PH. D thesis, North University of Baia Mare (Faculty of Science), 2012.

[18] K. Goebel and W.A. Kirk, Topic in metric fixed point theory, Cambridge University Press, 1990.

[19] [19] G. Hardy and T. Rogers, A generalization of a fixed point theorem of Reich, Canad. Math. Bull. 16 (1973), 201-206.

[20] C. O. Imoru, M. O. Olatinwo, G. Akinbo and A. O. Bossede, On a version of Banach's fixed point theorem, general. Math. 16 (2008), 25-32.

[21] S. Ishikawa, Fixed points and iteration of a nonexpansive mapping in a Banach space, Proc. Amer. Math. Soc. 59 (1976), 65-71.

[22] R. Kannan, Some results on fixed points, Bull. Calcutta. Math. Soc. 60 (1968), 71-76.

[23] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 76 (1965), 1004-1007.

[24] W. A. Kirk, On successive approximations for nonexpansive mappings in Banach spaces, Glasgow. Math. J. 12 (1971), 6-9.

[25] D. S. Jaggi, Some unique fixed point theorems, Indian. J. Pure. Appl. Math. 8 (1977), 223-230.

[26] W. R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953), 506-510.

[27] S. Massa, Generalized contractions metric spaces, Boll. Un. Math. Ital. 4 (1974), 689-694.

[28] Z. Mavar, A fixed point theorem in a reflexive Banach space, Publications de l'institu mathématique, Nouvelle série, tome 33 (1983), 139-141.

[29] M. O. Olatinwo, Some stability results for nonexpansive and quasi-nonexpansive operators in uniformly convex Banach spaces using the Ishikawa iteration process, Carpathian. J. Math. 24 (2008), 82-87.

[30] M. O. Olatinwo, Some stability results for two hybrid fixed point iterative algorithms of Kirk-Ishikawa and Kirk.Mann type, J. Adv. Math. Stud. 1 (2008), 87-96.

[31] M. O. Olatinwo, Convergence and stability results for some iterative schemes, Acta. Univ. Apul. 26 (2011), 225-236.

[32] M. O. Osilike, Stability of the Mann and Ishikawa iteration procedures for ϕ-strong pseudocontractions on nonlinear equation of the ϕ-strongly accretive type, J. Math. Anal. Appl. 227 (1998), 319-334.

[33] R. S. Palais, A simple proof of the Banach contraction principal, J. Fixed Point Theory Appl. 2 (2007), 221-223.

[34] P. Raja and S. M. Vaezpour, Some extension of Banach's principle in complete cone metric spaces, Fixed Point Theory Appl. 2008 (2008), Article ID 768294.

[35] S. Reich, Kannan's fixed point theorem, Boll. Un. Math. Ital. 4 (1971), 1-11.

[36] B. E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226 (1977), 257-290.

[37] C. Rousseau and Y. Saint-Aubin, Mathematics and Technology, SUMAT Series, Springer-Verlag, 2008.

[38] D. A. Ruiz, Convergence and stability of some iterative processes for a class of quasinonexpansive type mappings, J. Nonlinear. Sci. Appl. 5 (2012), 93-103.

[39] I. Rus, Some fixed point theorems in metric spaces, Math. Rev. Trieste.

[40] V. Sehgal, On fixed and periodic points for a class of mappings, J. London Math. Soc. 5 (1972), 571-576.

[41] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136 (2008), 1861-1869.

[42] W. Takahashi, A convexity in metric space and nonexpansive mapping I, Kodai Math. Semp. Rep. 22 (1970), 142-149.

[43] R. Taskovic, A generalization of Banach's contraction principle, Public. Instit. Math. Nouvelle Serie. 23 (1978), 179-191.

[44] F. Tricomi, Un teorema sulla convergenza delle successioni formate delle successive iterate di una funzione di una variabile reale, Giorn. Math. Bataglini. 54 (1916), 1-9.