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FIXED POINT THEOREMS OF MULTI-VALUED AND SINGLE-VALUED MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. Matthews (1994) introduced the concept of nonzero self-distance called a partial metric and extended the Banach contraction principle in the context of partial metric spaces. This was followed by Aydi *et al.* (2012) by extending Nadler's fixed point theorem to partial metric spaces and introducing the concept of partial Hausdorff metric. In this paper, we prove some fixed point theorems in the context of partial metric spaces endowed with partial ordering using partial Hausdorff metric and a notion of monotone multivalued mappings. Moreover, an example is provided to illustrate the usability of our results.

Keywords: fixed point; multivalued mapping; partially ordered set; partial metric space.

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1. Introduction

Fixed point theorems of multivalued mappings play fundamental roles in economics and engineering, control theory, convex optimization, and game theory. The first well known theorem

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for multivalued contraction mapping was given by Nadler [1] in 1967, which is a generalization of Banach's contraction principle [2].

Theorem 1.1 [1] *Let (X,d) be a complete metric space, and $F : X \to CB(X)$ is a multivalued mapping, where $CB(X)$ is the set of all nonempty closed bounded subsets of X. Assume that there exists $\alpha \in [0,1)$ such that $H(Fx,Fy) \leq \alpha d(x,y)$ for all $x,y \in X$. Then F has a fixed point.*

The Nadler's fixed point theorem has been generalized in many ways. One generalization of Nadler's fixed point theorem was given by Reich in 1972 [3], which was followed with a relaxed condition by Mizoguchi and Takahashi in 1989 [4] where they used the concept of $\mathscr{MT}-$function ($\mathscr{R}-$function).

Definition 1.1 A function $\varphi : [0,\infty) \to [0,1)$ is said to be an $\mathscr{MT}-$function if $\limsup_{r \to t^+} \varphi(r) < 1$ for all $t \in [0,\infty)$.

Theorem 1.2 [3] *Let (X,d) be a complete metric space, and $F : X \to Comp(X)$ is a multivalued mapping, where $Comp(X)$ is the set of all nonempty compact subsets of X. Assume that $H(Fx,Fy) \leq \varphi(d(x,y))d(x,y)$ for all $x,y \in X$, where φ is an $\mathscr{MT}-$function. Then F has a fixed point.*

Theorem 1.3 [4] *Let (X,d) be a complete metric space, and $F : X \to CB(X)$ is a multivalued mapping. Assume that $H(Fx,Fy) \leq \varphi(d(x,y))d(x,y)$ for all $x,y \in X$, where φ is an $\mathscr{MT}-$function. Then F has a fixed point.*

Recently, many fixed point theorems have been extended to partially ordered spaces. Some single valued fixed point theorems for partially ordered metric spaces were proved by Ran and Reurings [5], and Nieto and Lopez and applied their results to study a problem of ordinary differential equation. Moreover, fixed point theorems of multivalued mappings in partially ordered metric spaces were established by Beg and Butt [7], and Gregorio and Macansantos [8], and references therein.

Theorem 1.4 [5] *Let $(X,\preceq)$ be a partially ordered set, and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Furthermore, suppose that every pair $x,y \in X$*

has a lower bound and an upper bound. If f is a continuous monotone (either order-preserving or order-reversing) map from X into X such that

(i). there exists $\alpha \in (0,1)$ such that $d(fx, fy) \leq \alpha d(x,y)$ for all $x \preceq y$

(ii). there exists $x_0 \in X$ such that $x_0 \preceq fx_0$ or $fx_0 \preceq x_0$

Then f has a unique fixed point.

On the other hand, Matthews in 1994 [9] introduced the concept of partial metric, where self-distance need not be equal to zero, as part of the study of denotational semantics of dataflow networks. He gave a modified version of Banach's contraction principle in partial metric spaces. For multivalued mappings, Aydi *et al.* [10] and Macansantos [11] established an analogue of Nadler's fixed point theorem in the context of partial metric spaces.

Theorem 1.5 [9] *Let (X, p) be a complete partial metric space. If f is a mapping from X into itself such that there exists a real number $\alpha \in [0,1)$ satisfying $p(fx, fy) \leq \alpha p(x,y)$ for all $x, y \in X$. Then f has a unique fixed point.*

Theorem 1.6 [10] *Let (X, p) be a complete partial metric space. If $F : X \to CB^p(X)$ is a multivalued mapping, where $CB^p(X)$ is the set of all nonempty closed and bounded subsets of X, assume that for all $x, y \in X$, we have $H_p(Fx, Fy) \leq \alpha p(x,y)$ where $\alpha \in (0,1)$. Then F has a fixed point.*

Motivated by these works, we are going to combine the techniques employed by Mizoguchi and Takahashi [4] and Ran and Reurings [5] in generalizing and extending Nadler's fixed point theorem and the fixed point theorem established by Aydi *et al.* [10] in the context of ordered partial metric spaces.

2. Preliminaries

We recall some definitions and important results on partial metric spaces and partial Hausdorff metric from [9], [10], and [12]. The notations $\mathbb{R}, \mathbb{R}^+$ and $\mathbb{N}$ denote the set of all real numbers, the set of all nonnegative real numbers, and the set of all positive integer numbers, respectively.

Definition 2.1 Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for any $x,y,z \in X$, the following conditions hold:

(P1). $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;

(P2). $p(x,x) \leq p(x,y)$;

(P3). $p(x,y) = p(y,x)$;

(P4). $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X, p) is then called a partial metric space.

Remark 2.1 If $p(x,y) = 0$, then P1 and P2 imply that $x = y$. But the converse does not always hold.

Example 2.1 A trivial example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $p(x,y) = \max\{x,y\}$.

Example 2.2 If $X = \{[a,b] : a,b \in \mathbb{R}, a \leq b\}$, then $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$ defines a partial metric p on X.

Definition 2.2 A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$, with respect to τ_p, if and only if $p(x,x) = \lim\limits_{n\to\infty} p(x,x_n)$.

Definition 2.3 If p is a partial metric on X, then the mapping $p^s : X \times X \to \mathbb{R}^+$ given by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, defines a metric on X. Furthermore, a sequence $\{x_n\}$ converges in (X, p^s) to a point $x \in X$ if and only if $\lim\limits_{n,m\to\infty} p(x_n,x_m) = \lim\limits_{n\to\infty} p(x_n,x) = p(x,x)$.

Definition 2.4 Let (X, p) be a partial metric space.

(i). A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim\limits_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

(ii). (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $\lim\limits_{n\to\infty} p(x_n,x) = p(x,x)$.

Lemma 2.1 *Let (X, p) be a partial metric space, then*

(i). A sequence $\{x_n\}$ in X is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in metric space (X, p^s).

(ii). A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete.

Let $CB^p(X)$ be the set of all nonempty, closed and bounded subsets of the partial metric space (X,p). Closedness is in the context of (X,τ_p), where τ_p is the topology induced by p, and boundedness is defined as follows: A is a bounded set in (X,p) if there exists $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0,M)$, that is, $p(x_0,a) < p(a,a)+M$.

For $A,B \in CB^p(X)$ and $x \in X$, define

(i). $p(x,A) = \inf\{p(x,a), a \in A\}$

(ii). $\delta_p(A,B) = \sup\{p(a,B) : a \in A\}$

(iii). $\delta_p(B,A) = \sup\{p(b,A) : b \in B\}$

Note that if $p(x,A) = 0$ then $p^s(x,A) = 0$ where $p^s(x,A) = \{\inf p^s(x,a) : a \in A\}$.

From these natural extensions of partial metric p on X, if (X,p) is a partial metric space and $A,B \in CB^p(X)$, define $H_p : CB^p(X) \times CB^p(X) \to \mathbb{R}^+$ as $H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}$. We call H_p as the partial Hausdorff metric induced by p.

Remark 2.2. Let (X,p) be a partial metric space and A be any nonempty set in (X,p), then $a \in \overline{A}$ if and only if $p(a,A) = p(a,a)$, where $\overline{A}$ denotes the closure of A with respect to the partial metric p. Thus, A is closed in (X,p) if and only if $A = \overline{A}$.

Theorem 2.1 *Let (X,p) be a partial metric space. For all $A,B,C \in CB^p(X)$, we have*

(i). $H_p(A,A) \leq H_p(A,B)$

(ii). $H_p(A,B) = H_p(B,A)$

(iii). $H_p(A,B) \leq H_p(A,C) + H_p(C,B) - \inf_{c\in C} p(c,c)$.

Corollary 2.1 *Let (X,p) be a partial metric space. For $A,B \in CB^p(X)$ then, $H_p(A,B) = 0$ implies that $A = B$.*

Remark 2.3 The converse of Corollary 2.1 is not true in general. Also, note that any Hausdorff metric is a partial Hausdorff metric, but the converse is not true.

Lemma 2.2 *Let (X,p) be a partial metric space, $A,B \in CB^p(X)$ and $h > 1$. For any $a \in A$, there exists $b = b(a) \in B$ such that $p(a,b) \leq hH_p(A,B)$.*

For our results, we use the following relations between nonempty subsets of a partially ordered partial metric space which give us a notion of monotone multivalued mappings. The first relation appeared in [13].

Definition 2.5 Let $(X,\preceq)$ be a partially ordered set, and suppose that there exists a partial metric p on X such that (X,p) is a complete partial metric space. Let $F:X\to CB^p(X)$ be a multivalued mapping. Define the following relations:

(i). $A <^{(I)} B$ if for each $a\in A$, there exists $b\in B$ such that $a\preceq b$.

(ii). $A <^{(II)} B$ if for each $a\in A$, there exists $b\in B$ such that $a\preceq b$ and $p(a,b)\leq H_p(A,B)$.

Moreover, let $x,y\in X$ such that $x\preceq y$, then F is said to be:

(i). monotone nondecreasing of type (I) if $Fx <^{(I)} Fy$.

(ii). monotone nondecreasing of type (II) if $Fx <^{(II)} Fy$.

3. Main results

Theorem 3.1 *Let $(X,\preceq)$ be a partially ordered set. Suppose that there exists a partial metric p on X such that (X,p) is a complete partial metric space. Let $F:X\to CB^p(X)$ be a multivalued mapping such that the following conditions are satisfied:*

(i). There exists $x_0\in X$ such that $\{x_0\} <^{(I)} Fx_0$.

(ii). F is monotone nondecreasing of type (II).

(iii). If $x_n\to x$ is a nondecreasing sequence in X then $x_n\preceq x$ for all n.

(iv). For all $x,y\in X$ with $x\preceq y$,

$$H_p(Fx,Fy)\leq \varphi(p(x,y))p(x,y), \tag{1}$$

where φ is an $\mathscr{M}\mathscr{T}-$function.

Then F has a fixed point.

Proof. By assumption (i), there exists $x_0\in X$ and $x_1\in Fx_0$ such that $x_0\preceq x_1$. If $x_0=x_1$, then x_0 is a fixed point of F, and we are done.

Suppose that $x_0 \neq x_1$, then by assumptions (ii) and (iv), there exists $x_2 \in Fx_1$ such that $x_1 \preceq x_2$ and

$$\begin{aligned} p(x_1,x_2) &\leq H_p(Fx_0,Fx_1) \\ &\leq \varphi(p(x_0,x_1))p(x_0,x_1) \\ &< p(x_0,x_1), \end{aligned}$$

where φ is an $\mathscr{M}\mathscr{T}-$function. By induction, we obtain a sequence $\{x_n\} \in X$ with the property that $x_{n+1} \in Fx_n$ and $x_n \preceq x_{n+1}$ such that

$$\begin{aligned} p(x_n,x_{n+1}) &\leq H_p(Fx_{n-1},Fx_n) \\ &\leq \varphi(p(x_{n-1},x_n))p(x_{n-1},x_n) \\ &< p(x_{n-1},x_n) \quad \forall n \in \mathbb{N}. \end{aligned}$$

Note that $\{p(x_n,x_{n+1})\}$ is strictly decreasing and bounded below. It follows that

$$\lim_{n\to\infty} p(x_n,x_{n+1}) = r \geq 0.$$

Suppose that $r > 0$. By assumptions (ii) and (iv), we find that

$$\begin{aligned} p(x_n,x_{n+1}) &\leq H_p(Fx_{n-1},Fx_n) \\ &\leq \varphi(p(x_{n-1},x_n))p(x_{n-1},x_n). \end{aligned}$$

Therefore, we have

$$\begin{aligned} r &= \limsup_n p(x_n,x_{n+1}) \\ &\leq \limsup_n \varphi(p(x_{n-1},x_n)) \limsup_n p(x_{n-1},x_n) \\ &< \limsup_n p(x_{n-1},x_n) = r, \end{aligned}$$

which is a contradiction. Thus, $r = 0$, that is,

$$\lim_{n\to\infty} p(x_n,x_{n+1}) = 0$$

and it follows from (P2) that

$$\lim_{n\to\infty} p(x_n,x_n) = 0. \tag{2}$$

By definition, we also have

$$\lim_{n\to\infty} p^s(x_n, x_{n+1}) = 0. \tag{3}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in (X, p^s). By sense of contradiction, suppose that $\{x_n\}$ is not a Cauchy sequence in (X, p^s). Then there exists $\varepsilon > 0$ and $m(k), n(k) \in \mathbb{N}$ such that for all $k \geq 0$

$$n(k) > m(k) > k,$$

$$p^s(x_{m(k)}, x_{n(k)}) \geq \varepsilon$$

and

$$p^s(x_{m(k)}, x_{n(k)-1}) < \varepsilon.$$

Hence, for $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon &\leq p^s(x_{m(k)}, x_{n(k)}) \\ &\leq p^s(x_{m(k)}, x_{n(k)-1}) + p^s(x_{n(k)-1}, x_{n(k)}) \\ &< p^s(x_{n(k)-1}, x_{n(k)}) + \varepsilon. \end{aligned}$$

Taking $k \to \infty$ in the above inequality and using (3), we obtain

$$\lim_{k\to\infty} p^s(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{4}$$

By definition, we have

$$p^s(x_{m(k)}, x_{n(k)}) = 2p(x_{m(k)}, x_{n(k)}) - p(x_{m(k)}, x_{m(k)}) - p(x_{n(k)}, x_{n(k)}).$$

Taking $k \to \infty$ and using (2) and (4), we obtain

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}) = \frac{\varepsilon}{2}. \tag{5}$$

Since $m(k) < n(k)$ and $\{x_n\}$ is a nondecreasing sequence, we find that $x_{m(k)} \preceq x_{n(k)}$, and by condition ii, for $x_{m(k)+1} \in Fx_{m(k)}$, there exists $x_{n(k)+1} \in Fx_{n(k)}$ such that $x_{m(k)+1} \preceq x_{n(k)+1}$ and

$$\begin{aligned} p(x_{m(k)+1}, x_{n(k)+1}) &\leq H_p(Fx_{m(k)}, Fx_{n(k)}) \\ &\leq \varphi(p(x_{m(k)}, x_{n(k)}))p(x_{m(k)}, x_{n(k)}) \\ &< p(x_{m(k)}, x_{n(k)}). \end{aligned}$$

With this, we have

$$\begin{aligned} p^s(x_{m(k)+1},x_{n(k)+1}) &= 2p(x_{m(k)+1},x_{n(k)+1}) - p(x_{m(k)+1},x_{m(k)+1}) - p(x_{n(k)+1},x_{n(k)+1}) \\ &< 2p(x_{m(k)},x_{n(k)}) - p(x_{m(k)+1},x_{m(k)+1}) - p(x_{n(k)+1},x_{n(k)+1}). \end{aligned}$$

Taking $k \to \infty$ and using (2) and (5), we have

$$\lim_{n\to\infty} p^s(x_{m(k)+1},x_{n(k)+1}) < \varepsilon. \tag{6}$$

By using triangle inequality, we have

$$p^s(x_{m(k)},x_{n(k)}) \leq p^s(x_{m(k)},x_{m(k)+1}) + p^s(x_{m(k)+1},x_{m(k)+1}) + p^s(x_{n(k)+1},x_{n(k)}).$$

Taking $k \to \infty$ and using (3), (4), and (6), we obtain

$$\begin{aligned} \varepsilon &\leq 0 + \lim_{k\to\infty} p^s(x_{m(k)+1},x_{n(k)+1}) + 0 \\ &< \varepsilon, \end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence in (X,p^s). Since (X,p) is complete. Then (X,p^s) is a complete metric space, and $\{x_n\}$ converges to some $x \in X$ with respect to the metric p^s, that is, $\lim_{n\to\infty} p^s(x_n,x) = 0$.

Now, we need to show that $x \in X$ is a fixed point of F. By definition, $\{x_n\} \to x$ in (X,p^s) implies that

$$p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(x_n,x_n) = 0. \tag{7}$$

By conditions iii and iv, since $\{x_n\}$ is a nondecreasing sequence that converges to $x \in X$, then $x_n \preceq x$ and $H_p(Fx_n,Fx) \leq \varphi(p(x_n,x))p(x_n,x)$ for all n. Taking (7) into consideration, we obtain $\lim_{n\to\infty} H_p(Fx_n,Fx) = 0$. Since $x_{n+1} \in Fx_n$, we see that

$$p(x_{n+1},Fx) \leq \delta_p(Fx_n,Fx) \leq H_p(Fx_n,Fx)$$

and hence

$$\lim_{n\to\infty} p(x_{n+1},Fx) = 0. \tag{8}$$

By Property 4 (P4) of a partial metric, we have

$$
\begin{aligned}
p(x,Fx) &\leq p(x,x_{n+1})+p(x_{n+1},Fx)-p(x_{n+1},x_{n+1}) \\
&\leq p(x,x_{n+1})+p(x_{n+1},Fx).
\end{aligned}
$$

Taking $n\to\infty$ and using (7) and (8), we obtain $p(x,Fx)=0$ Therefore, $p(x,x)=p(x,Fx)$, and this implies that $x\in\overline{Fx}=Fx$, because $Fx\in CB^p(X)$. Thus x is a fixed point of F.

The following corollary considers the case when the $\mathscr{M}\mathscr{T}-$function is constant.

Corollary 3.1 *Let $(X,\preceq)$ be a partially ordered set, and suppose that there exists a partial metric p on X such that (X,p) is a complete partial metric space. Let $F: X\to CB^p(X)$ be a multivalued mapping such that the following conditions are satisfied:*

(i). There exists $x_0\in X$ such that $\{x_0\}<^{(I)} Fx_0$.

(ii). F is monotone nondecreasing of type (II).

(iii). If $x_n\to x$ is a nondecreasing sequence in X then $x_n\preceq x$ for all n.

(iv). For all $x,y\in X$ with $x\preceq y$, there exists $\alpha\in(0,1)$ such that

$$H_p(Fx,Fy)\leq\alpha p(x,y). \tag{9}$$

Then F has a fixed point.

Remark 3.1 Single valued fixed point theorems in partial metric spaces analogous to our multivalued results can also be established, and are stated as follows.

Corollary 3.2 *Let $(X,\preceq)$ be a partially ordered set, and suppose that there exists a partial metric p on X such that (X,p) is a complete partial metric space. Let $f: X\to X$ be a single valued nondecreasing mapping such that the following conditions are satisfied:*

(i). There exists $x_0\in X$ such that $x_0\preceq fx_0$.

(ii). If $x_n\to x$ is a nondecreasing sequence in X then $x_n\preceq x$ for all n.

(iii). For all $x,y\in X$ with $x\preceq y$,

$$p(fx,fy)\leq\varphi(p(x,y))p(x,y), \tag{10}$$

where φ is an $\mathscr{M}\mathscr{T}-$function.

Then f has a fixed point.

Corollary 3.3 *Let $(X,\preceq)$ be a partially ordered set, and suppose that there exists a partial metric p on X such that (X,p) is a complete partial metric space. Let $f:X\to X$ be a single valued nondecreasing mapping such that the following conditions are satisfied:*

(i). There exists $x_0\in X$ such that $x_0\preceq fx_0$.

(ii). If $x_n\to x$ is a nondecreasing sequence in X then $x_n\preceq x$ for all n.

(iii). For all $x,y\in X$ with $x\preceq y$, there exists $\alpha\in(0,1)$ such that

$$p(fx,fy)\leq \alpha p(x,y), \tag{11}$$

where φ is an $\mathscr{M}\mathscr{T}-$function.

Then f has a fixed point.

Next theorem gives additional condition to ensure uniqueness of fixed point of single valued mapping.

Theorem 3.2 *Suppose that all assumptions of Corollary 3.2 are satisfied, and in addition, for arbitrary elements $x,y\in X$, there exists $z\in X$ which is comparable with both x and y. Then the fixed point of f is unique.*

Proof. Let $u,v\in X$ be two fixed points of f, *i.e.*, $fu=u$ and $fv=v$. We note the following two cases:

Case 1. u and v are comparable. WLOG, let $u\preceq v$. By assumption (iv) of Corollary 3.2,

$$p(u,v)=p(fu,fv)\leq \varphi(p(u,v))p(u,v)$$

and this implies that $p(u,v)=0$, thus $u=v$.

Case 2. u and v are not comparable. By the virtue of the additional assumption, there exists $w\in X$ that is comparable to both of u and v. Note also that $u=f^nu$ is comparable with f^nw for each n because of transitivity and nondecreasing property of f. Let $n\in\mathbb{N}$, then by assumption (iv) of Corollary 3.2,

$$\begin{aligned} p(u,f^nw) &= p(ff^{n-1}u,ff^{n-1}w)\leq \varphi(p(f^{n-1}u,f^{n-1}w))p(f^{n-1}u,f^{n-1}w) \\ &< p(f^{n-1}u,f^{n-1}w)=p(u,f^{n-1}w). \end{aligned}$$

Hence, $\{p(u, f^{n-1}w)\}$ is strictly decreasing and bounded, thus

$$\lim_{n\to\infty} p(u, f^n w) = l \geq 0.$$

Suppose that $l > 0$. Getting the limit of $p(u, f^n w) \leq \varphi(p(u, f^{n-1}w))p(u, f^{n-1}w)$ as $n \to \infty$, one arrives at a contradiction, and thus $\lim\limits_{n\to\infty} p(u, f^n w) = 0$. Similarly, it can be shown that $\lim\limits_{n\to\infty} p(v, f^n w) = 0$.

Consider,

$$p(u,v) \leq p(u, f^n w) + p(f^n w, v). \tag{12}$$

Taking $n \to \infty$ in (12), we have $p(u,v) = 0$, and so, $u = v$.

Therefore, in any case, the fixed point of f is unique.

To illustrate the usability of our results, we provide the following example where the result in [10] is not applicable.

Example 3.1. Let $X = \left\{(0,0), \left(0, \frac{1}{5}\right), \left(\frac{1}{8}, \frac{1}{6}\right), \left(\frac{1}{7}, \frac{1}{7}\right)\right\}$, where $\preceq$ is defined as: for $x, y \in X$ such that $x = (x_1, y_1)$, and $y = (x_2, y_2)$, $x \preceq y$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$, and let X be endowed with the partial metric $p : X \times X \to \mathbb{R}^+$ defined as

$$p(x,y) = \frac{1}{4}|x_1 - x_2| + \frac{1}{2}\max\{x_1, x_2\} + \frac{1}{4}|y_1 - y_2| + \frac{1}{2}\max\{y_1, y_2\}. \tag{13}$$

Note that $p\left(\left(0, \frac{1}{5}\right), \left(0, \frac{1}{5}\right)\right) = \frac{1}{10} \neq 0$, $p\left(\left(\frac{1}{6}, \frac{1}{8}\right), \left(\frac{1}{8}, \frac{1}{6}\right)\right) = \frac{7}{48} \neq 0$, and $p\left(\left(\frac{1}{7}, \frac{1}{7}\right), \left(\frac{1}{7}, \frac{1}{7}\right)\right) = \frac{1}{7} \neq 0$, so p is not a metric on X. Moreover $p^s = |x_1 - x_2| + |y_1 - y_2|$ is a complete metric space, and so (X, p) is a complete partial metric space. Let $F : X \to CB^p(X)$ be defined as

$$Fx = \begin{cases} \{(0,0)\} & \text{if } x = (0,0), \\ \{(0,0)\} & \text{if } x = (0, \frac{1}{5}), \\ \{(0,0), (\frac{1}{7}, \frac{1}{7})\} & \text{if } x = (\frac{1}{8}, \frac{1}{6}), \\ \{(0,0), (0, \frac{1}{5})\} & \text{if } x = (\frac{1}{7}, \frac{1}{7}). \end{cases}$$

Note that $\{(0,0)\}$, $\{(0,0),(\frac{1}{7},\frac{1}{7})\}$, and $\{(0,0),(0,\frac{1}{5})\}$ are bounded sets in (X,p). If $(x,y)\in X$, then

$$\begin{aligned}
(x,y)\in\overline{\{(0,0)\}} &\iff p((x,y),\{(0,0)\})=p((x,y),(x,y)\\
&\iff \frac{3}{4}x+\frac{3}{4}y=\frac{1}{2}x+\frac{1}{2}y\\
&\iff x=0,y=0\\
&\iff (x,y)\in\{(0,0)\}.
\end{aligned}$$

Hence, $\{(0,0)\}$ is closed with respect to the partial metric p. Moreover,

$$\begin{aligned}
(x,y)\in\overline{\{(0,0),(\frac{1}{7},\frac{1}{7})\}} &\iff p((x,y),\{(0,0),(\frac{1}{7},\frac{1}{7})\})=p((x,y),(x,y)\\
&\iff \min\{\frac{3}{4}x+\frac{3}{4}y,\frac{1}{4}|x-\frac{1}{7}|+\frac{1}{2}\max\{x,\frac{1}{7}\}+\frac{1}{4}|y-\frac{1}{7}|+\\
&\qquad \frac{1}{2}\max\{y,\frac{1}{7}\}=\frac{1}{2}x+\frac{1}{2}y\\
&\iff (x,y)\in\{(0,0),(\frac{1}{7},\frac{1}{7})\}.
\end{aligned}$$

Hence, $\{(0,0),(\frac{1}{7},\frac{1}{7})\}$ is closed with respect to the partial metric p. Also,

$$\begin{aligned}
(x,y)\in\overline{\{(0,0),(0,\frac{1}{5})\}} &\iff p((x,y),\{(0,0),(0,\frac{1}{5})\})=p((x,y),(x,y)\\
&\iff \min\{\frac{3}{4}x+\frac{3}{4}y,\frac{1}{4}x+\frac{1}{2}\max\{x,0\}+\frac{1}{4}|y-\frac{1}{5}|+\\
&\qquad \frac{1}{2}\max\{y,\frac{1}{5}\}=\frac{1}{2}x+\frac{1}{2}y\\
&\iff (x,y)\in\{(0,0),(0,\frac{1}{5})\}.
\end{aligned}$$

Hence, $\{(0,0),(0,\frac{1}{5})\}$ is closed with respect to the partial metric p.

Let us consider $(0,\frac{1}{5})$, $(\frac{1}{8},\frac{1}{6})\in X$. For these elements of X, $F(0,\frac{1}{5})=\{(0,0)\}$ and $F(\frac{1}{8},\frac{1}{6})=\{(0,0),(\frac{1}{7},\frac{1}{7})\}$, and $p\left((0,\frac{1}{5}),(\frac{1}{8},\frac{1}{6})\right)=\frac{97}{480}$ while $H_p(F(0,\frac{1}{5}),F(\frac{1}{8},\frac{1}{6}))=\max\{0,\frac{3}{14}\}=\frac{3}{14}$. Note that these two specific elements of X do not satisfy the contraction condition established by Aydi et al. [10], that is, there is no $\alpha\in(0,1)$ such that $H_p(Fx,Fy)\le\alpha p(x,y)$. Therefore, we cannot use the fixed point theorem established by Aydi et al. [10] to show that F has a fixed point.

Now, we use our result to show that F has fixed point. Let us first demonstrate that the contraction condition is satisfied for comparable elements of X. For this, we consider cases summarized in Table 1.

Table 1. $p(x,y)$ and $H_p(Fx,Fy)$ of comparable elements $x,y \in X$

$x \preceq y$	$p(x,y)$	$H_p(Fx,Fy)$
$(0,0) \preceq (0,\frac{1}{5})$	$\frac{3}{20}$	0
$(0,0) \preceq (\frac{1}{8},\frac{1}{6})$	$\frac{7}{32}$	$\frac{3}{14}$
$(0,0) \preceq (\frac{1}{7},\frac{1}{7})$	$\frac{3}{14}$	$\frac{3}{20}$
$(0,0) \preceq (0,0)$	0	0
$(0,\frac{1}{5}) \preceq (0,\frac{1}{5})$	$\frac{1}{10}$	0
$(\frac{1}{8},\frac{1}{6}) \preceq (\frac{1}{8},\frac{1}{6})$	$\frac{7}{48}$	$\frac{1}{7}$
$(\frac{1}{7},\frac{1}{7}) \preceq (\frac{1}{7},\frac{1}{7})$	$\frac{1}{7}$	$\frac{1}{10}$

Thus we can choose $\alpha = \frac{859}{875} \in (0,1)$ such that $H_p(Fx,Fy) \leq \alpha p(x,y)$ whenever $x \preceq y$. It can also be shown that the remaining assumptions of our result (specifically Theorem 3.1 and Corollary 3.1) are satisfied. Therefore, it can be invoked F has a fixed point.

Conflict of Interests

The author declares that there is no conflict of interests.

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