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FIXED POINTS OF MULTIVALUED ALMOST CONTRACTIONS IN METRIC SPACES WITH w-DISTANCE

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Abstract. We introduce the notion of multivalued almost $w-$contractions and obtain extension of fixed point results for weak Ciric-contractive operators and multivalued almost contractions (multivalued $(\delta;L)-$weak contractions). We have also shown that the class of weak Ciric-contractive operators are included in the class of multivalued almost $w-$contractions introduced here.

Keywords: multivalued almost contractions; fixed points; metric space; $w-$distance; weak Ciric-contractions.

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1. Introduction

In 2007 Berinde and Berinde [4] obtained interesting and inspiring fixed point results for multi-valued $(\delta;k)-$weak contractions (here-after called almost contractions [5]) as an extension of equally interesting results for single-valued weak contractions introduced in [1] by Berinde. It is the aim of this paper to obtain these results in the context of $w-$distance in metric spaces. Let (X,d) be a metric space, a functional $w : X \times X \to [0,\infty)$ is called a $w-$distance on X if the following axioms are satisfied:

(1) $w(x,z) \leq w(x;y) + w(y,z)$;

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(2) for any $x \in X$ the mapping $w(x,.) : X \to [0.\infty)$ is lower semicontinuous;

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $w(z;x) \leq \delta$, $w(z,y) \leq \delta$ implies $d(x;y) \leq \varepsilon$.

The concept of $w-$distance (and its properties) was introduced and studied in 1996 by Kada, Suzuki and Takahashi [13] and examples of $w-$distance functions are given in [11]. Later, the same year, Suzuki and Takahashi [14] obtained some fixed point results for a new class of operators called weakly contractive operators based on the $w-$distance function. In [15] Ume obtained improvements of Kannan and Ciric fixed point theorems using the concept of $w-$distance. We recall that Ciric [10] obtained a splendid generalization of Banach contraction principle by unifying the weak contractive formulations of Kannan [12], Chatterjea [8] and Zamfirescu [16] into one contractive condition called quasi-contraction for which he also obtained a fixed point result in the multivalued context. The class of quasicontractions of Ciric is partially (to a large extent) included in the class of almost contractions of Berinde yet there are examples of almost contractions, including identity operator, which are not quasicontractions. This means that the the two classes of operators are independent. Following [14] Guran and Petrusel [11] initiated studies of the class of multivalued weakly Ciric-contractive operators. They proved a fixed point theorem for the class of weak-Ciric contractive operators and presented a data dependence result for the fixed point set.

2. PRELIMINARIES

We shall adopt the following notations: 2^X for the power set of X; $\mathscr{P} = 2^X \backslash \emptyset$; Ω_X for collection of all bounded subsets of X; $CB(X)$ for the collection of all bounded and closed subsets of X; $C(X)$ for the collection of all closed subsets of X. Likewise, the following definitions, for a metric space X:

Definition 1. *I . The gap functional $D : 2^X \times 2^X \to [0,\infty)$ is defined by;*

$$D(A;B) = \inf\{d(a,b) : a \in A \text{ and } b \in B\} : \tag{1}$$

II . The diameter functional $\delta : 2^X \times 2^X \to [0;\infty)$ is defined by;

$$\delta(A;B) = \sup\{d(a,b) : a \in A \text{ and } b \in B\} : \tag{2}$$

III . The excess functional $\rho : 2^X \times 2^X \to [0,\infty)$ *is defined by;*

$$\rho(A,B) = \sup\{D(a,B) : a \in A\} : \tag{3}$$

IV . The Pompeiu-Hausdorff functional $H : 2^X \times 2^X \to [0;\infty)$ *is defined by;*

$$\begin{aligned} H(A;B) &= \max\{sup_{a\in A} d(a;B), \sup_{b\in B} d(b;A)\} && (4)\\ &= \max\{\rho(A,B), \rho(B,A)\} && (5)\\ &= max\{\sup_{a\in A}\inf_{b\in B} d(a;b), \sup_{b\in B}\inf_{a\in A} d(a;b)\} && (6)\end{aligned}$$

Definition 2. [11] *Let* (X,d) *be a metric space and* $T : X \to P(X)$ *then* T *is called weakly Ciric-contractive operator if there exists a* $w-$*distance on* X *such that for every* $x,y \in X$ *and* $u \in Tx$ *there is* $v \in Ty$ *with:*

$$w(u,v) \le q\max\{w(x,y), D_w(x,Tx), D_w(y,Ty), \frac{1}{2}D_w(y,Tx)\}. \tag{7}$$

The fixed point theorem proved in [11] is as follows:

Theorem 3. [11] *Let* (X,d) *be a complete metric space,* $x_0 \in X$*, and* $T : B_w[x_0,r] \to C(X)$; $r > 0$, *a multivalued operator such that:*

(i) T *is weakly Ciric-contractive operator*

(ii) $D_w(x_0,Tx_0) \le (1-q)r$,

Then there exists $x^* \in X$ *such that* $x^* \in Tx^*$.

A crucial result for $w-$distance functions is the following Lemma:

Lemma 4. *Let* (X,d) *be a metric space, and let* w *be a* $w-$*distance on* X. *Let* α_n *and* β_n *be two sequences in* $[0;\infty)$ *converging to zero and let* $x,y,z \in X$. *Then the following hold:*

(a) If $w(x_n,y) \le \alpha_n$ *and* $w(x_n,z) \le \beta_n, n \in \mathbb{N}$, *then* $y = z$.

(b) If $w(x_n,y_n) \le \alpha_n$ *and* $w(x_n;z) \le \alpha_n, n \in \mathbb{N}$, *then* $\{y_n\}$ *converges to* z.

(c) If $w(x_n,x_m) \le \alpha_n$; $n,m \in \mathbb{N}$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence.*

(c) If $w(y,x_n) \le \alpha_n$; $m \in \mathbb{N}$ *then* $\{x_n\}$ *is a Cauchy sequence.*

The multivalued almost contractions of Berinde and Berinde [4] is defined as follows:

Definition 5. [4] *Let (X,d) be a metric space and $T : X \to \mathscr{P}(X)$ a multivalued operator. T is called multivalued almost contraction or multivalued $(\delta, L)-$weak contraction) if and only if there exist $\delta \in (0,1)$ and $L \geq 0$ such that*

$$H(Tx;Ty) \leq \delta d(x;y) + LD(y;Tx) : \tag{8}$$

3. Main Results

Following (7) and (8) we introduce the notion of almost $w-$contractions as follows:

Definition 6. *Let (X,d) be a metric space and $T : X \to P(X)$ a multivalued operator. T is called multivalued almost $w-$contraction (or multivalued $(\delta, L)-$weak $w-$contraction) if and only if there exist $\delta \in (0,1)$, $L \geq 0$ and a distance function w on X such that for every $x, y \in X$ and $u \in Tx$ there exists $v \in Ty$ with*

$$w(u,v) \leq \delta w(x,y) + LD_w(y,Tx). \tag{9}$$

We now present the main results of this paper:

Proposition 7. *The class of weak Ciric-contractive operators is a subclass of almost $w-$contractions.*

PROOF: The proof of Proposition 7 consists in showing that if T is a weak Ciric-contractive operator then for certain $u \in Tx$ and $v \in Ty$ the following hold:

$$w(u;v) = \begin{cases} qw(x;y) + Ld(y;Tx); & if\ M_{weak}(x,y) = w(x,y), foranyL \geq 0 \\ qw(x;y) + Ld(y;Tx); & if\ M_{weak}(x,y) = D_w(x,Tx) foranyL \geq q \\ qw(x;y) + Ld(y;Tx); & if\ M_{weak}(x,y) = D_w(x,Ty) foranyL \geq q \end{cases} \tag{10}$$

where $M_{weak}(x;y)$ denotes $\max\{w(x,y), Dw(x,Tx), Dw(y,Ty), \frac{1}{2}D_w(y,Tx)\}$.

The case when $M_{weak}(x,y) = w(x,y)$ is trivial, so we verify the cases when $M_{weak}(x,y) = D_w(x,Tx)$ and $M_{weak}(x,y) = D_w(x,Ty)$ knowing that verification of the the symmetric counterparts; $M_{weak}(x,y) = D_w(y,Ty)$ and $M_{weak}(x,y) = D_w(y,Tx)$ are respective replicas of the first two.

CASE (I): When $M_{weak}(x,y) = D_w(x,Tx)$, then

$$w(u,v) \leq qD_w(x,Tx) \leq qw(x;y) + qD_w(y,Tx). \tag{11}$$

Due to symmetric nature of (10) we are left to prove $w(u;v) \leq qw(x;y) + LD_w(x,Ty),\ L \geq q$. But $qD_w(x,Tx) \leq qw(x,Ty) + qD_w(Tx,Ty) \leq qw(x,Ty) + qw(u,v)$ gives $D_w(x,Tx) \leq \frac{1}{1-q} qD_w(x,Ty)$ which yields

$$w(u,v) \leq qw(xy) + LD_w(x,Ty),\ L \geq \frac{1}{1-q} \geq q. \tag{12}$$

From (11) and (12), $w(u,v) \leq qw(x,y) + qD_w(y,Tx)$ when $M_{weak}(x,y) = D_w(x,Tx)$.

CASE (II): When $M_{weak}(x,y) = D_w(x,Ty)$ we are to verify that the almost $w-$contraction condition, $w(u,v) \leq qw(x,y) + qD_w(y,Tx)$, holds knowing that its symmetric counterpart holds trivially. The argument is based on the fact that:

$$\begin{aligned}
qDw(x;Ty) &\leq qD_w(x;Tx) + qD_w(Tx;Ty) \\
&\leq qD_w(x,Tx) + qw(u,v) \\
&\leq qD_w(x,Tx) + \frac{1}{2}q^2 D_w(x,Ty). \\
\Longrightarrow \left(1 - \frac{1}{2}q\right) qD_w(x,Ty) &\leq qD_w(x,Tx), \\
\Longrightarrow \frac{1}{2} qD_w(x,Ty) &\leq qw(x,y) + qD_w(x,Tx) \text{ since } 1 - \frac{1}{2}q > \frac{1}{2}
\end{aligned}$$

End of proof. □

The last result is very significant in that, apart from extending and unifying many improvements on fixed point results concerning weak contractive operators, it also imply that our next result (for almost $w-$contractions) constitutes a natural generalization of the existence aspect of Theorem 3 proved in [11] and a theorem proved in [4].

Theorem 8. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ a multi-valued almost $w-$contraction. Then:*

(1) $Fix(T) \neq \emptyset$, where $Fix(T) = \{p \in X : p \in Tp\}$

(2) For any $x_0 \in X$ there exists an orbit $\{x_n\}_{n=0}^{\infty} = \{T^n x_0\}_{n=0}^{\infty}$ of T at the point x_0 that converges to a fixed point p of T for which the following estimates hold:

(2i) $w(x_n,p) \leq \frac{\delta^n}{1-\delta} w(x_0;x_1)$,

(2ii) $w(x_n,p) \leq \frac{\delta}{1-\delta} w(x_{n-1},x_n)$.

PROOF

It suffices to prove there exists an orbit $\{x_n\}_{n=0}^{\infty} = \{T^n x_0\}_{n=0}^{\infty}$ of T at the point $x_0 \in X$ which is a Cauchy sequence. To this end, we chose any $x_0 \in X$ and $x_1 \in Tx_0$ so that by definition of multivalued almost $w-$contraction (i.e (9)) we obtain a point $x_2 \in Tx_1$ such that $w(x_1,x_2) \leq \delta w(x_0,x_1) + LD_w(x_2,Tx_1)$ which yields $w(x_1,x_2) \leq \delta w(x_0,x_1)$. Following from the same definition, there exists $x_3 \in Tx_2$ such that $w(x_2,x_3) \leq \delta^2 w(x_0,x_1)$.

Continuing in this way we have:

$$w(x_n,x_{n+1}) \leq \delta^n w(x_0,x_1). \tag{13}$$

$$w(x_n,x_{n+k}) \leq [1+\delta+\delta^2+ ::: +\delta^{k-1}]\delta^n w(x_0,x_1).$$

$$w(x_n,x_{n+k}) \leq \frac{[1-\delta^{k-1}]\delta^n}{1-\delta} w(x_0,x_1) < \frac{\delta^n}{1-\delta} w(x_0,x_1). \tag{14}$$

Application of Lemma 4 to (14) guarantees that the sequence generated above is a Cauchy sequence in (X,d). Since X is a complete metric space we infer that the orbit $\{x_n\}_{n=0}^{\infty} = \{T^n x_0\}_{n=0}^{\infty}$ of T at the point $x_0 \in X$ converges to a limit $p \in X$. We are left to show that $p \in Tp$. In this direction, since $p, x_n \in X$, for some $u \in Tp$ there exists $x_{n+1} \in Tx_n$ such that $w(u,x_{n+1}) \leq \delta w(p,x_n)$. By hypothesis of lower semicontinuity of $w(p,.)$, we obtain $w(u,p) \leq \liminf w(u,x_{n+1}) \leq \delta \liminf w(p,x_n) = 0$. It follows that $w(u,x_{n+1}) \leq \alpha_n$ and $w(p,xn) \leq \beta_n$ for some null sequences $\{\alpha_n\}$ and $\{\beta_n\}$ so that by Lemma 4, $u = p$. Therefore $p \in Tp$, hence $Fix(T) \neq \emptyset$, establishing 1. The proof of $(2i)$ follows from from (13) and (14) using $p \in Tp$ while $(2ii)$ follows from semicontinuity of w. End of proof. □

4. Conclusion

In conclusion, we like to draw attention to effort of Berinde at unification of the fixed point theory of weak contractive operators following Ciric's generalization of Banach contraction principle. It would greatly foster a unifying theory to introduce appropriate class of operators which can contain the Ciric-type mappings and the class of almost contractions of Berinde as subclasses. A very close attempt at this was initiated in 2008 by Berinde [6, 7] which should motivate and stimulate researches in this direction.

Conflict of Interests

The authors declare that there is no conflict of interests.

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