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AN APPROACH TO THE APPROXIMATION OF COMMON COUPLED FIXED POINTS OF CONTRACTIVE MAPS IN METRIC-TYPE SPACES

HALLOWED OLAOLUWA*, JOHNSON O. OLALERU

Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria

Abstract. In this research work, some results on the existence and approximation of common coupled fixed points of contractive maps in cone metric spaces are unified and generalized based on a new method.

Keywords: common coupled fixed point; w^*-compatible; cone metric space.

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1. Introduction

Ever since the re-introduction of cone metric spaces by Huang and Zhang [1], Banach-valued metric spaces have been extensively used in the study of the existence and convergence of fixed points of different contractive-like mappings. From the contraction condition to more general conditions of two, three and four maps with the weak compatibility, researchers have established various fixed point theorems on contractive mappings in normal cones (e.g. [2-6]) and non necessarily normal cones (e.g. [7-12]). The following two classes of self-maps, studied

*Corresponding author

E-mail addresses: olu20_05@hotmail.com (H. Olaoliuwa), olaleru1@yahoo.co.ukl (J.O. Olaleru)

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by Abbas *et al.* [13], generalize the contractive conditions in [2-11] :

$$d(fx,gy) \leq pd(Sx,Ty)+qd(fx,Sx)+rd(gy,Ty)+t[d(fx,Ty)+d(gy,Sx)], \tag{1}$$

where $(0 < p+q+r+2t < 1)$ and

$$d(fx,gy) \leq ku_{x,y},\ u_{x,y} \in \left\{ d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(fx,Ty)+d(gy,Sx)}{2} \right\}, \tag{2}$$

where $0 < k < 1$. One observes that both classes are unified and generalized in the following mappings of Ciric-type,

$$d(fx,gy) \leq p(x,y)d(Sx,Ty)+q(x,y)d(fx,Sx)+r(x,y)d(gy,Ty)+t(x,y)[d(fx,Ty)+d(gy,Sx)] \tag{3}$$

where $p,q,r,t : X\times X \to [0,1)$ satisfy $0 < \sup\{p(x,y)+q(x,y)+r(x,y)+t(x,y)\} = \lambda < 1$. The case of one map ($f = g$ and $S = T = Id$) was introduced and studied by Ciric [14] in the usual metric space context.

Following the results on coupled fixed points in partially ordered metric spaces by Bhashkar and Lakshmikantham [15] and Lakshmikantham and Ciric [16], Sabetghadam *et al.* [17] and Abbas et al. [18] extended some special cases (cases of one and two maps) of the conditions (1.1) and (1.2) to the study of coupled fixed points in cone metric spaces by considering the following inequalities on maps $F : X\times X \to X$ and $g : X \to X$:

$$\begin{aligned} d(F(x,y),F(u,v)) &\leq a_1d(gx,gu)+a_2d(F(x,y),gx)+a_3d(gy,gv)+a_4d(F(u,v),gu) \\ &\quad +a_6d(F(x,y),gu)+a_6(F(u,v),gx), \end{aligned} \tag{4}$$

where $\sum_{i=1}^{6} a_i < 1$ and

$$\begin{aligned} &d(F(x,y),F(u,v)) \leq kU+mV, \\ &U,V \in S^{x,y}_{u,v} = \{d(gx,gu), d(gy,gv), d(F(x,y),gx), d(F(x,y),gu), d(f(u,v),gu)\}, \end{aligned} \tag{5}$$

where $k+m < 1$. The similarity between fixed points and coupled fixed points suggests that some coupled fixed points results are obtainable from existing fixed point results.

Recently, Samet *et al.* [21] noticed that a coupled fixed point of a map $F : X\times X \to X$ is exactly the fixed point of the associate map $\tilde{F}(x,y) = (F(x,y),F(y,x))$ while Olaleru and Olaoluwa [19] observed that a common coupled fixed point of two maps $F : X\times X \to X$ and $g : X \to X$ is a common fixed point of $\tilde{F}$ and $\tilde{g}$ defined by $\tilde{g}(x,y) := (gx,gy)$. Their methodology stands when a transfer of the topological structure of X to $X \times X$ is possible by inducing a metric on $X \times X$, mostly $D((x,y),(u,v)) = d(x,u) + d(y,v)$ and $D((x,y),(u,v)) = \max\{d(x,u), d(y,v)\}$, and when the coupled fixed point inequality in X, say $d(F(x,y),F(u,v)) \leq \mathscr{F}_d(F,g,x,y,u,v)$, can be transformed to a known fixed point inequality in $X \times X$, say $D(\tilde{F}(x,y),\tilde{F}(u,v)) \leq \mathscr{F}'_D(\tilde{F},\tilde{g},(x,y),(u,v))$. This methodology works with some coupled fixed point inequalities such as (1.4). However, it does not work for inequalities of the type (1.5) since they cannot be transformed to the product space $X \times X$.

In this work, we provide a novel technique of tackling coupled fixed point inequalities of type (1.5), which also works for inequalities of type (1.4), without the use of the classical concept of Cauchy sequences. It should be noted that this novel technique deals with the problems of existence and approximation. For this purpose, we study the fixed points of Ciric mappings (1.3) which is a simple way of unifying and generalizing inequalities of type (1.1) and (1.2). First, let us recall the following basic definitions.

Definition 1.1. [1] Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, non-empty and $P \neq \{0\}$;

(b) $a,b \in R, a,b \geq 0, x,y \in P$ imply that $ax+by \in P$;

(c) $P \cap (-P) = \{0\}$.

Given a cone P, define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y-x \in P$. We shall write $x \ll y$ for $y-x \in intP$, where $intP$ stands for interior of P. Also we will use $x < y$ to indicate that $x \leq y$ and $x \neq y$.

The cone P in a normed space E is called normal whenever there is a real number $k > 0$, such that for all $x,y \in E, 0 \leq x \leq y$ implies $\|x\| \leq k\|y\|$. The least positive number satisfying this norm inequality is called the normal constant of P.

Definition 1.2. [1] Let X be a non-empty set and let E be a real Banach space equipped with the partial ordering $\leq$ with respect to the cone $P \subset E$. Suppose that the mapping $d : X \times X \longrightarrow E$ satisfies:

(d_1) $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and ony if $x = y$;

(d_2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(d_3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 1.3. [1] Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. We say that $\{x_n\}$ is

(c_1) a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is some $k \in \mathbb{N}$ such that, for all $n,m \geq k$, $d(x_n,x_m) \ll c$;

(c_2) a convergent sequence if for every $c \in E$ with $0 \ll c$, there is some $k \in \mathbb{N}$ such that, for all $n \geq k$, $d(x_n,x) \ll c$. Such x is called limit of the sequence $\{x_n\}$.

Note that every convergent sequence in a cone metric space X is a Cauchy sequence. A cone metric space X is said to be complete if every Cauchy sequence in X is convergent.

Definition 1.4. [22] Let P be a cone and let $\{\omega_n\}$ be a sequence in P. One says that $\omega_n \stackrel{\ll}{\longrightarrow} 0$ if for every $\varepsilon \in P$ with $0 \ll \varepsilon$ there exists $N > 0$ such that $\omega_n \ll \varepsilon$ for all $n \geq N$.

Note that a sequence $\{x_n\}$ converges to x if and only if $d(x_n,x) \stackrel{\ll}{\longrightarrow} 0$.

Definition 1.5. [15] An element $(x,y) \in X \times X$ is called a coupled fixed point of mapping $F : X \times X \to X$ if $x = F(x,y)$ and $y = F(y,x)$.

Definition 1.6. [18] An element $(x,y) \in X \times X$ is called:

(g_1) a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x,y)$ and $g(y) = F(y,x)$, and (gx,gy) is called coupled point of coincidence;

(g_2) a common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x,y)$ and $y = g(y) = F(y,x)$.

Definition 1.7. ([18], [23]). The mappings $F : X \times X \to X$ and $g : X \to X$ are called:

(w_1) w-compatible if $g(F(x,y)) = F(gx,gy)$ whenever $g(x) = F(x,y)$ and $g(y) = F(y,x)$;

(w_2) w*-compatible if $g(F(x,x)) = F(gx,gx)$ whenever $g(x) = F(x,x)$.

The w*-compatibility condition is less restrictive than the w-compatibility condition.

We prove the following lemma, crucial in the establishing of some of the results in this paper.

Lemma 1.8. *Let X be a cone metric space with respect to a cone P in a Banach space E, with $int(P) \neq \emptyset$. Let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ be sequences in P and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$,*

$\{f_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{d'_n\}$, $\{e'_n\}$, $\{f'_n\}$ *be real positive sequences satisfying for all* $n \in \mathbb{N}$,

$$\begin{cases} u_{n+1} \leq a_n u_n + b_n v_n + c_n w_n + d_n z_n \\ v_{n+1} \leq a'_n u_n + b'_n v_n + c'_n w_n + d'_n z_n \\ w_n \leq e_n w_{n-1} + f_n z_{n-1} \\ z_n \leq e'_n w_{n-1} + f'_n z_{n-1}, \end{cases}$$

where $a_n + b_n + c_n + d_n \leq \lambda,\ a'_n + b'_n + c'_n + d'_n \leq \lambda,\ e_n + f_n \leq \lambda,\ e'_n + f'_n \leq \lambda$, *with* $\lambda < 1$. *Then* $u_n \xrightarrow{\ll} 0$ *and* $v_n \xrightarrow{\ll} 0$.

Proof.

$$\begin{aligned} u_{n+1} &\leq a_n[a_{n-1}u_{n-1} + b_{n-1}v_{n-1} + c_{n-1}w_{n-1} + d_{n-1}z_{n-1}] \\ &\quad + b_n[a'_{n-1}u_{n-1} + b'_{n-1}v_{n-1} + c'_{n-1}w_{n-1} + d'_{n-1}z_{n-1}] \\ &\quad + c_n[e_n w_{n-1} + f_n z_{n-1}] + d_n[e'_n w_{n-1} + f'_n z_{n-1}] \\ &= [a_n a_{n-1} + b_n a'_{n-1}]u_{n-1} + [a_n b_{n-1} + b_n b'_{n-1}]v_{n-1} \\ &\quad + [a_n c_{n-1} + b_n c'_{n-1} + c_n e_n + d_n e'_n]w_{n-1} \\ &\quad + [a_n d_{n-1} + b_n d'_{n-1} + c_n f_n + d_n f'_n]z_{n-1} \\ &= a^1_n u_{n-1} + b^1_n v_{n-1} + c^1_n w_{n-1} + d^1_n z_{n-1}, \end{aligned} \tag{6}$$

where

$$\begin{cases} a^1_n = a_n a_{n-1} + b_n a'_{n-1}, \\ b^1_n = a_n b_{n-1} + b_n b'_{n-1}, \\ c^1_n = a_n c_{n-1} + b_n c'_{n-1} + c_n e_n + d_n e'_n, \\ d^1_n = a_n d_{n-1} + b_n d'_{n-1} + c_n f_n + d_n f'_n, \end{cases}$$

$$\begin{aligned} a^1_n + b^1_n + c^1_n + d^1_n &= a_n[a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}] + b_n[a'_{n-1} + b'_{n-1} + c'_{n-1} + d'_{n-1}], \\ &\quad + c_n[e_n + f_n] + d_n[e'_n + f'_n], \\ &\leq \lambda[a_n + b_n + c_n + d_n] \leq \lambda^2 \end{aligned}$$

and

$$
\begin{aligned}
u_{n+1} &\leq a_n^1[a_{n-2}u_{n-2}+b_{n-2}v_{n-2}+c_{n-2}w_{n-2}+d_{n-2}z_{n-2}], \\
&\quad +b_n^1[a'_{n-2}u_{n-2}+b'_{n-2}v_{n-2}+c'_{n-2}w_{n-2}+d'_{n-2}z_{n-2}], \\
&\quad +c_n^1[e_{n-1}w_{n-2}+f_{n-1}z_{n-2}]+d_n^1[e'_{n-1}w_{n-2}+f'_{n-1}z_{n-2}], \\
&= [a_n^1a_{n-2}+b_n^1a'_{n-2}]u_{n-2}+[a_n^1b_{n-2}+b_n^1b'_{n-2}]v_{n-2}, \\
&\quad +[a_n^1c_{n-2}+b_n^1c'_{n-2}+c_n^1e_{n-1}+d_n^1e'_{n-1}]w_{n-2}, \\
&\quad +[a_n^1d_{n-2}+b_n^1d'_{n-2}+c_n^1f_{n-1}+d_n^1f'_{n-1}]z_{n-2}, \\
&= a_n^2u_{n-2}+b_n^2v_{n-2}+c_n^2w_{n-2}+d_n^2z_{n-2},
\end{aligned}
\tag{7}
$$

where

$$
\begin{cases}
a_n^2=a_n^1a_{n-2}+b_n^1a'_{n-2}, \\
b_n^2=a_n^1b_{n-2}+b_n^1b'_{n-2}, \\
c_n^2=a_n^1c_{n-2}+b_n^1c'_{n-2}+c_n^1e_{n-1}+d_n^1e'_{n-1}, \\
d_n^2=a_n^1d_{n-2}+b_n^1d'_{n-2}+c_n^1f_{n-1}+d_n^1f'_{n-1},
\end{cases}
$$

$$
\begin{aligned}
a_n^2+b_n^2+c_n^2+d_n^2 &= a_n^1[a_{n-2}+b_{n-2}+c_{n-2}+d_{n-2}]+b_n^1[a'_{n-2}+b'_{n-2}+c'_{n-2}+d'_{n-2}], \\
&\quad +c_n^1[e_{n-1}+f_{n-1}]+d_n^1[e'_{n-1}+f'_{n-1}], \\
&\leq \lambda[a_n^1+b_n^1+c_n^1+d_n^1] \leq \lambda^3.
\end{aligned}
$$

Repeating the process, we have

$$u_{n+1} \leq a_n^n u_0+b_n^n v_0+c_n^n w_0+d_n^n z_0,$$

where

$$
\begin{cases}
a_n^n=a_n^{n-1}a_0+b_n^{n-1}a'_0, \\
b_n^n=a_n^{n-1}b_0+b_n^{n-1}b'_0, \\
c_n^n=a_n^{n-1}c_0+b_n^{n-1}c'_0+c_n^{n-1}e_1+d_n^{n-1}e'_1, \\
d_n^n=a_n^{n-1}d_0+b_n^{n-1}d'_0+c_n^{n-1}f_1+d_n^{n-1}f'_1,
\end{cases}
$$

$$
\begin{aligned}
a_n^n+b_n^n+c_n^n+d_n^n &= a_n^{n-1}[a_0+b_0+c_0+d_0]+b_n^{n-1}[a'_0+b'_0+c'_0+d'_0], \\
&\quad +c_n^{n-1}[e_1+f_1]+d_n^{n-1}[e'_1+f'_1], \\
&\leq \lambda[a_n^{n-1}+b_n^{n-1}+c_n^{n-1}+d_n^{n-1}] \leq \lambda^{n+1}.
\end{aligned}
$$

Thus $u_{n+1} \leq (u_0+v_0+w_0+z_0)\lambda^{n+1}$. Given $0 \ll c$, choose $\tau > 0$ such that $c+\{y \in P : y < \tau\} \subset P$. Since $\lambda^{n+1} \to 0$, there is $N \in \mathbb{N}$ such that $\lambda^{n+1}(u_0+v_0+w_0+z_0) \in \{y \in P : y < \tau\}$

for all $n > N$. It follows that $\lambda^{n+1}(u_0 + v_0 + w_0 + z_0) \ll c$ for all $n > N$. Thus, for all $n > N$, $u_{n+1} \ll c$ and so $u_n \xrightarrow{\ll} 0$. Similarly, $v_n \xrightarrow{\ll} 0$.

As explained earlier, the Ciric classes of mappings are one of the widest generalizations of contraction mappings in literature. In this section, we prove the existence of common fixed points for four maps useful in the proof of our result on coupled fixed points even though it is of independent interest in that it generalizes most common fixed point theorems in cone metric spaces.

Theorem 1.9. *Let f, g, S and T be self-mappings of a cone metric space X with cone P having non-empty interior, satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$ and*

$$\begin{aligned} d(fx, gy) &\leq p(x,y)d(Sx, Ty) + q(x,y)d(fx, Sx) + r(x,y)d(gy, Ty) \\ &\quad + t(x,y)[d(fx, Ty) + d(gy, Sx)] \end{aligned} \tag{8}$$

for all $x, y \in X$, where $p, q, r, t : X \times X \to [0, 1)$ satisfy

$$\sup_{x,y\in X} \{p(x,y) + q(x,y) + r(x,y) + 2t(x,y)\} = \lambda < 1. \tag{9}$$

If one of $f(X)$, $g(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence in X. Moreover if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. Given that $f(X) \subset T(X)$ and $g(X) \subset S(X)$, one can define sequences $\{x_n\}$ and $\{y_n\}$ such that $y_{2n-1} := fx_{2n-2} = Tx_{2n-1}$ and $y_{2n} := gx_{2n-1} = Sx_{2n}$.

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n-1}) \\ &\leq p(x_{2n}, x_{2n-1})d(Sx_{2n}, Tx_{2n-1}) + q(x_{2n}, x_{2n-1})d(fx_{2n}, Sx_{2n}) \\ &\quad + r(x_{2n}, x_{2n-1})d(gx_{2n-1}, Tx_{2n-1}) + t(x_{2n}, x_{2n-1})[d(fx_{2n}, Tx_{2n-1}) \\ &\quad + d(gx_{2n-1}, Sx_{2n})] \\ &\leq p(x_{2n}, x_{2n-1})d(y_{2n-1}, y_{2n}) + q(x_{2n}, x_{2n-1})d(y_{2n+1}, y_{2n}) \\ &\quad + r(x_{2n}, x_{2n-1})d(y_{2n}, y_{2n-1}) + t(x_{2n}, x_{2n-1})d(y_{2n+1}, y_{2n-1}) \\ &\leq [p(x_{2n}, x_{2n-1}) + r(x_{2n}, x_{2n-1}) + t(x_{2n}, x_{2n-1})]d(y_{2n-1}, y_{2n}) \\ &\quad + [q(x_{2n}, x_{2n-1}) + t(x_{2n}, x_{2n-1})]d(y_{2n}, y_{2n+1}). \end{aligned}$$

Hence $d(y_{2n},y_{2n+1}) \leq \delta(x_{2n},x_{2n-1})d(y_{2n-1},y_{2n})$, where $\delta(x,y) = \frac{p(x,y)+r(x,y)+t(x,y)}{1-q(x,y)-t(x,y)}$. Note that $\lambda < 1$. From $p(x,y)+\lambda q(x,y)+r(x,y)+\lambda t(x,y)+t(x,y) \leq \lambda$, we have $\delta(x,y) \leq \lambda$. Hence

$$d(y_{2n},y_{2n+1}) \leq \lambda d(y_{2n-1},y_{2n}). \tag{10}$$

$$\begin{aligned}
d(y_{2n+1},y_{2n+2}) &= d(fx_{2n},gx_{2n+1}) \\
&\leq p(x_{2n},x_{2n+1})d(Sx_{2n},Tx_{2n+1})+q(x_{2n},x_{2n+1})d(fx_{2n},Sx_{2n}) \\
&\quad +r(x_{2n},x_{2n+1})d(gx_{2n+1},Tx_{2n+1})+t(x_{2n},x_{2n+1})[d(fx_{2n},Tx_{2n+1}) \\
&\quad +d(gx_{2n+1},Sx_{2n})] \\
&\leq [p(x_{2n},x_{2n+1})+q(x_{2n},x_{2n+1})+r(x_{2n},x_{2n+1}) \\
&\quad +t(x_{2n},x_{2n+1})]d(y_{2n+1},y_{2n+2})+t(x_{2n},x_{2n+1})d(y_{2n},y_{2n+1}),
\end{aligned}$$

i.e. $d(y_{2n+1},y_{2n+2}) \leq \delta'(x_{2n},x_{2n+1})d(y_{2n},y_{2n+1})$, where $\delta'(x,y) = \frac{t(x,y)}{1-p(x,y)-q(x,y)-r(x,y)-t(x,y)}$. Note that $\lambda < 1$. From $\lambda[p(x,y)+q(x,y)+r(x,y)+t(x,y)]+t(x,y) \leq \lambda$, we have $\delta'(x,y) \leq \lambda$. It follows that

$$d(y_{2n+1},y_{2n+2}) \leq \lambda d(y_{2n},y_{2n+1}). \tag{11}$$

From (10) and (11), we have $d(y_n,y_{n+1}) \leq \lambda d(y_{n-1},y_n)$ for all $n \geq 1$. Repeating this argument n-times, we have $d(y_n,y_{n+1}) \leq \lambda^n d(y_0,y_1)$. For any $n,m \in \mathbb{N}$ with $m > n$, it follows that $d(y_n,y_m) \leq \sum_{i=n}^{m-1} d(y_i,y_{i+1}) \leq \sum_{i=n}^{m-1} \lambda^i d(y_0,y_1) = \lambda^n d(y_0,y_1)\sum_{j=0}^{m-n-1} \lambda^j \leq \frac{\lambda^n}{1-\lambda}d(y_0,y_1)$. Given $0 \ll c$, choose $\tau > 0$ such that $c+\{y \in P : y < \tau\} \subset P$. Since $\frac{\lambda^n}{1-\lambda} \to 0$, there is $N \in \mathbb{N}$ such that $\frac{\lambda^n}{1-\lambda}d(y_0,y_1) \in \{y \in P : y < \tau\}$ for all $n > N$. It follows that $\frac{\lambda^n}{1-\lambda}d(y_0,y_1) \ll c$ for all $n > N$. Thus, for all $m > n > N$, $d(y_n,y_m) \ll c$ and $\{y_n\}$ is Cauchy. Suppose that $S(X)$ is complete. Then there exists $u \in S(X)$, say $u = Sv$, such that $Sx_{2n} = y_{2n} \to u$ as $n \to \infty$. In fact, $y_n \to u$ as

$n \to \infty$. Let us prove that $fv = u$.

$$\begin{aligned} d(fv,u) &\leq d(fv, gx_{2n-1}) + d(gx_{2n-1}, u) \\ &\leq p(v,x_{2n-1})d(Sv, Tx_{2n-1}) + q(v,x_{2n-1})d(fv,Sv) \\ &\quad + r(v,x_{2n-1})d(gx_{2n-1}, Tx_{2n-1}) + t(v,x_{2n-1})[d(fv, Tx_{2n-1}) + d(gx_{2n-1}, Sv)] \\ &\quad + d(gx_{2n-1}, u) \\ &\leq p(v,x_{2n-1})d(u,y_{2n-1}) + q(v,x_{2n-1})d(fv,u) \\ &\quad + r(v,x_{2n-1})d(y_{2n}, y_{2n-1}) + t(v,x_{2n-1})[d(fv,u) + d(u,y_{2n-1}) + d(y_{2n},u)] \\ &\quad + d(y_{2n},u), \end{aligned}$$

$$\begin{aligned} d(fv,u) &\leq [p(v,x_{2n-1}) + t(v,x_{2n-1})]d(u,y_{2n-1}) + [q(v,x_{2n-1}) + t(v,x_{2n-1})]d(fv,u) \\ &\quad + r(v,x_{2n-1})d(y_{2n}, y_{2n-1}) + [1 + t(v,x_{2n-1})]d(y_{2n},u) \\ &\leq [p(v,x_{2n-1}) + t(v,x_{2n-1})]d(u,y_{2n-1}) + \lambda d(fv,u) + r(v,x_{2n-1})d(y_{2n}, y_{2n-1}) \\ &\quad + [1 + t(v,x_{2n-1})]d(y_{2n},u), \end{aligned}$$

which, on taking $n \to \infty$, yields $d(fv,u) \leq \lambda d(fv,u)$. Since $\lambda < 1$, we have that $d(fv,u) = 0$, i.e. $fu = v$. Since $u \in f(X) \subset T(X)$, there exists $w \in X$ such that $Tw = u$. Now we shall show that $gw = u$.

$$\begin{aligned} d(gw,u) &\leq d(fx_{2n}, gw) + d(fx_{2n}, u) \\ &\leq p(x_{2n},w)d(Sx_{2n}, Tw) + q(x_{2n},w)d(fx_{2n}, Sx_{2n}) + r(x_{2n},w)d(gw,Tw) \\ &\quad + t(x_{2n},w)[d(fx_{2n}, Tw) + d(gw, y_{2n})] + d(fx_{2n}, u) \\ &\leq p(x_{2n},w)d(y_{2n},u) + q(x_{2n},w)d(y_{2n+1}, y_{2n}) + r(x_{2n},w)d(gw,Tw) \\ &\quad + t(x_{2n},w)[d(y_{2n+1}, Tw) + d(gw,Tw) + d(Tw, y_{2n})] + d(y_{2n+1}, u) \\ &\leq p(x_{2n},w)d(y_{2n},u) + q(x_{2n},w)d(y_{2n+1}, y_{2n}) + [r(x_{2n},w) + t(x_{2n},w]d(gw,Tw) \\ &\quad + t(x_{2n},w)[d(y_{2n+1},u) + d(u,y_{2n})] + d(y_{2n+1}, u) \\ &\leq p(x_{2n},w)d(y_{2n},u) + q(x_{2n},w)d(y_{2n+1}, y_{2n}) + \lambda d(gw,u) \\ &\quad + t(x_{2n},w)[d(y_{2n+1},u) + d(u,y_{2n})] + d(y_{2n+1}, u), \end{aligned}$$

which, on taking $n \to \infty$, yields $d(gw,u) \leq \lambda d(gw,u)$, i.e., $gw = u$. If the pairs $\{f,S\}$ and $\{g,T\}$ are weakly compatible, then $fu = fSv = Sfv = Su = w_1$ (say) and $gu = gTw = Tgw = Tu = w_2$

(say).

$$\begin{aligned} d(w_1,w_2)=d(fu,gu) &\leq p(u,u)d(Su,Tu)+q(u,u)d(fu,Su) \\ &\quad +r(u,u)d(gu,Tu) \\ &\quad +t(u,u)[d(fu,Tu)+d(gu,Su)] \\ &= [p(u,u)+2t(u,u)]d(w_1,w_2) \leq \lambda d(w_1,w_2), \end{aligned}$$

which implies that $w_1 = w_2$. Therefore $fu = gu = Su = Tu$. Next, we show that $u = gu$.

$$\begin{aligned} d(u,gu)=d(fv,gu) &\leq p(v,u)d(Sv,Tu)+q(v,u)d(fv,Sv)+r(v,u)d(gu,Tu) \\ &\quad +t(v,u)[d(fv,Tu)+d(gu,Sv)] \\ &= [p(v,u)+2t(v,u)]d(gu,u) \leq \lambda d(gu,u) \end{aligned}$$

and $gu = u$. Thus u is a common fixed point of f, g, S and T. The uniqueness of the common fixed point is an immediate consequence of the generalized contractive condition.

If the functions p,q,r,t in the previous theorem are constants, we have the first main result in [13]. The second main result in [13], which is a generalization of many results in literature, is retrieved as follows:

Corollary 1.10. [13] *Let f,g,S and T be self-maps of a cone metric space X with cone P having non-empty interior, satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$ and*

$$d(fx,gy) \leq hu_{x,y}(f,g,S,T), \tag{12}$$

where $h \in (0,1)$ and

$$u_{x,y}(f,g,S,T) \in \left\{ d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(fx,Ty)+d(gy,Sx)}{2} \right\} \tag{13}$$

for all $x,y \in X$. If one of $f(X), g(X), S(X)$ or $T(X)$ is a complete subspace of X, then $\{f,S\}$ and $\{g,T\}$ have a unique common fixed point of coincidence. Moreover, if $\{f,S\}$ and $\{g,T\}$ are weakly compatible, then f,g,S and T have a unique common fixed point.

Proof. If

$$u_{x,y}(f,g,S,T) \in \left\{ d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(fx,Ty)+d(gy,Sx)}{2} \right\},$$

then

$$\begin{aligned} hu_{x,y}(f,g,S,T) &= p(x,y)d(Sx,Ty)+q(x,y)d(fx,SX)+r(x,y)d(gy,Ty) \\ &\quad +2t(x,y)[d(fx,Ty)+d(gy,Sx)] \end{aligned}$$

where $p,q,r : X\times X \to \{0,h\}$ and $t : X\times X \to \{0,\frac{h}{2}\}$ are chosen such that for each $(x,y) \in X\times X$, one and only one of $p(x,y),q(x,y),r(x,y),t(x,y)$ is non null. With such choice, $p(x,y)+q(x,y)+r(x,y)+2t(x,y) = h < 1$. It follows that f,g,S and T satisfy the contractive condition of Theorem 2.1. Hence, they have a unique common fixed point.

Example 1.11. Let $E = C^1([0,1],\mathbb{R})$, $P = \{\varphi \in E : \varphi(t) \geq 0, t \in [0,1]\}$, $X = [0,\infty)$ and $d : X\times X \to E$ be defined by $d(x,y) = |x-y|\varphi$. The space X together with d is a non-normal cone metric space. The self maps of $f,g,S,T : X \to X$ defined by $fx = \begin{cases} \frac{x}{3},\ x\in[0,1] \\ \frac{x}{4},\ x\in(1,\infty) \end{cases}$, $gx = 0$, $Sx = \begin{cases} 3x,\ x\in[0,1] \\ 2x,\ x\in(1,\infty) \end{cases}$ and $Tx = x$ satisfy the conditions of Theorem 1.9. with $q(x,y) = \frac{1}{8}$ and $p(x,y) = r(x,y) = t(x,y) = 0$. The four maps have 0 as common fixed point.

2. Coupled fixed points in cone metric spaces

In this section, we study the existence and approximation of coupled fixed points of four maps of Ciric-type in cone metric spaces. The method used is unique and differs from the existing ones in literature.

Theorem 2.1. *Let (X,d) be a cone metric space, $f : X\times X \to X$, $g : X\times X \to X$, $S : X \to X$ and $T : X \to X$ be four mappings such that $f(X\times X) \subset T(X)$, $g(X\times X) \subset S(X)$ and*

$$\begin{aligned} d(f(x,y),g(u,v)) &\leq p_1(x,y,u,v)d(Sx,Tu)+p_2(x,y,u,v)d(Sy,Tv) \\ &\quad +q(x,y,u,v)d(f(x,y),Sx)+r(x,y,u,v)d(g(u,v),Tu) \\ &\quad +t(x,y,u,v)[d(f(x,y),Tu)+d(g(u,v),Sx)] \end{aligned} \tag{14}$$

for all $x,y,u,v \in X$, where $p_1,p_2,q,r,t : X^4 \to [0,1)$ and
$\sup_{(x,y,u,v)\in X}[(p_1+p_2+q+r+2t)(x,y,u,v)] = \lambda < 1.$
(1) If one of $f(X\times X)$, $g(X\times X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then $\{f,S\}$ and

$\{g,T\}$ have a unique coupled coincidence point in X. Moreover, if $\{f,S\}$ and $\{g,T\}$ are w-compatible, then f, g, S and T have a unique coupled common fixed point $(x^*,x^*)\in X\times X$.*
(2) For every $(x_0,y_0)\in X\times X$, there exist a pair of sequences $\{(u_n,v_n)\}$ not necessarily lying on the main diagonal, such that $\{u_n\}$ and $\{v_n\}$ converge each to $x^*\in X$.

Proof. (1) Existence: Let $x=y$ and $u=v$ be in the contractive condition. From Theorem 1.9, the functions $x\mapsto f(x,x)$, $x\mapsto g(x,x)$, $x\mapsto Sx$ and $x\mapsto Tx$ have a unique common fixed point $u^*\in X$. The element $(u^*,u^*)\in X\times X$ is a common coupled fixed fixed point of f, g, S and T since $u=f(u,u)=g(u,u)=Su=Tu$. It is the unique common coupled fixed point of f, g, S and T that lies on the main diagonal $\{(x,x),\ x\in X\}$.

Uniqueness: Let (x,y) be a common coupled fixed point of f, g, S and T. We have

$$\begin{cases} x=Sx=Tx=f(x,y)=g(x,y)\\ y=Sy=Ty=f(y,x)=g(y,x).\end{cases}$$

From the contractive condition,

$$\begin{aligned} d(f(x,y),g(y,x)) &\le p_1(x,y,y,x)d(Sx,Ty)+p_2(x,y,u,v)d(Sy,Tx)\\ &\quad +q(x,y,y,x)d(f(x,y),Sy)+r(x,y,y,x)d(g(y,x),Ty)\\ &\quad +t(x,y,y,x)[d(f(x,y),Ty)+d(g(y,x),Sx)],\end{aligned}$$

i.e. $d(x,y)\le[p_1(x,y,y,x)+p_2(x,y,y,x)+2t(x,y,y,x)]d(x,y)\le\lambda d(x,y)$. Since $\lambda<1$, $d(x,y)=0$, i.e. $x=y$. This shows that every common coupled fixed point of f, g, S and T belongs to the main diagonal and so, the common coupled fixed point of $f.g,S$ and T is unique.

(2) Convergence: Let (x^*,x^*) be the unique common coupled fixed point of f,g,S and T. When no confusion is possible, we will simply denote it (x,x). For any $(x_0,y_0)\in X\times X$, the conditions $f(X\times X)\subset T(X)$ and $g(X\times X)\subset S(X)$ guarantee the existence of sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ such that

$$\begin{cases} f(x_{2n-2},y_{2n-2})=Tx_{2n-1}:=u_{2n-1},\\ f(y_{2n-2},x_{2n-2})=Ty_{2n-1}:=v_{2n-1},\\ g(x_{2n-1},y_{2n-1})=Sx_{2n}:=u_{2n},\\ g(y_{2n-1},x_{2n-1})=Sy_{2n}:=v_{2n}.\end{cases}\tag{15}$$

From the contractive condition, we have

$$\begin{aligned} d(f(x_{2n-2},y_{2n-1}),g(x,x)) &\leq p_1(\alpha_n)d(Sx_{2n-2},Tx)+p_2(\alpha_n)d(Sy_{2n-2},Tx) \\ &\quad +q(\alpha_n)d(f(x_{2n-2},y_{2n-2}),Sx_{2n-2})+r(\alpha_n)d(g(x,x),Sx) \\ &\quad +t(\alpha_n)[d(f(x_{2n-2},y_{2n-2}),Tx)+d(g(x,x),Sx_{2n-2})], \end{aligned}$$

where $\alpha_n=(x_{2n-2},y_{2n-2},x,x)$,

$$\begin{aligned} d(u_{2n-1},x) &\leq [p_1(\alpha_n)+t(\alpha_n)]d(u_{2n-2},x)+p_2(\alpha_n)d(v_{2n-2},x) \\ &\quad +q(\alpha_n)d(u_{2n-1},u_{2n-2})+t(\alpha_n)d(u_{2n-1},x) \end{aligned}$$

(16)
$$d(u_{2n-1},x)\leq \frac{(p_1+t)(\alpha_n)}{1-t(\alpha_n)}d(u_{2n-2},x)+\frac{p_2(\alpha_n)}{1-t(\alpha_n)}d(v_{2n-2},x)+\frac{q(\alpha_n)}{1-t(\alpha_n)}d(u_{2n-1},u_{2n-2})$$

and

$$\frac{(p_1+t)(\alpha_n)}{1-t(\alpha_n)}+\frac{p_2(\alpha_n)}{1-t(\alpha_n)}+\frac{q(\alpha_n)}{1-t(\alpha_n)}\leq\lambda. \tag{17}$$

$$\begin{aligned} d(f(x,x),g(x_{2n-1},y_{2n-1})) &\leq p_1(\beta_n)d(Sx,Tx_{2n-1})+p_2(\beta_n)d(Sx,Ty_{2n-1}) \\ &\quad +q(\beta_n)d(f(x,x),Sx)+r(\beta_n)d(g(x_{2n-1},y_{2n-1}),Tx_{2n-1}) \\ &\quad +t(\beta_n)[d(f(x,x),Tx_{2n-1})+d(g(x_{2n-1},y_{2n-1}),Sx)], \end{aligned}$$

where $\beta_n=(x,x,x_{2n-1},y_{2n-1})$,

$$d(u_{2n},x)\leq \frac{(p_1+t)(\beta_n)}{1-t(\beta_n)}d(u_{2n-1},x)+\frac{p_2(\beta_n)}{1-t(\beta_n)}d(v_{2n-1},x)+\frac{q(\beta_n)}{1-t(\beta_n)}d(u_{2n-1},u_{2n}) \tag{18}$$

and

$$\frac{(p_1+t)(\beta_n)}{1-t(\beta_n)}+\frac{p_2(\beta_n)}{1-t(\beta_n)}+\frac{q(\beta_n)}{1-t(\beta_n)}\leq\lambda. \tag{19}$$

Hence, from (16)-(19), $\forall n\in\mathbb{N}$, $\exists a_n,b_n,c_n\geq 0$, such that

$$d(u_n,x)\leq a_nd(u_{n-1},x)+b_nd(v_{n-1},x)+c_nd(u_{n-1},u_n) \tag{20}$$

with $a_n+b_n+c_n\leq\lambda<1$. Similarly $\forall n\in\mathbb{N}$, $\exists a'_n,b'_n,c'_n\geq 0$, such that

$$d(v_n,x)\leq a'_nd(u_{n-1},x)+b'_nd(v_{n-1},x)+c'_nd(v_{n-1},v_n) \tag{21}$$

with $a'_n + b'_n + c'_n \leq \lambda < 1$. From the contractive condition, we have that

$$\begin{aligned}
& d(f(x_{2n-2}, y_{2n-2}), g(x_{2n-1}, y_{2n-1})) \\
& \leq p_1(\gamma_n) d(Sx_{2n-2}, Tx_{2n-1}) + p_2(\gamma_n) d(Sy_{2n-2}, Ty_{2n-1}) \\
& + q(\gamma_n) d(f(x_{2n-2}, y_{2n-2}), Sx_{2n-2}) + r(\gamma_n) d(g(x_{2n-1}, y_{2n-1}), Tx_{2n-1}) \\
& + t(\gamma_n)[d(f(x_{2n-2}, y_{2n-2}), Tx_{2n-1}) + d(g(x_{2n-1}, y_{2n-1}), Sx_{2n-2})],
\end{aligned}$$

where $\gamma_n = (x_{2n-2}, y_{2n-2}, x_{2n-1}, y_{2n-1})$,

$$\begin{aligned}
d(u_{2n-1}, u_{2n}) & \leq p_1(\gamma_n) d(u_{2n-2}, u_{2n-1}) + p_2(\gamma_n) d(v_{2n-2}, v_{2n-1}) + q(\gamma_n) d(u_{2n-1}, u_{2n-2}) \\
& \quad + r(\gamma_n) d(u_{2n}, u_{2n-1}) + t(\gamma_n) d(u_{2n}, u_{2n-2}) \\
& \leq p_1(\gamma_n) d(u_{2n-2}, u_{2n-1}) + p_2(\gamma_n) d(v_{2n-2}, v_{2n-1}) + q(\gamma_n) d(u_{2n-1}, u_{2n-2}) \\
& \quad + r(\gamma_n) d(u_{2n}, u_{2n-1}) + t(\gamma_n)[d(u_{2n}, u_{2n-1}) + d(u_{2n-1}, u_{2n-2})]
\end{aligned}$$

$$d(u_{2n-1}, u_{2n}) \leq \frac{(p_1 + q + t)(\gamma_n)}{1 - (r+t)(\gamma_n)} d(u_{2n-1}, u_{2n-2}) + \frac{p_2(\gamma_n)}{1 - (r+t)(\gamma_n)} d(v_{2n-2}, v_{2n-1}) \tag{22}$$

and

$$\frac{(p_1 + q + t)(\gamma_n)}{1 - (r+t)(\gamma_n)} + \frac{p_2(\gamma_n)}{1 - (r+t)(\gamma_n)} \leq \lambda. \tag{23}$$

From the contractive condition

$$\begin{aligned}
& d(f(x_{2n}, y_{2n}), g(x_{2n-1}, y_{2n-1})) \\
& \leq p_1(\delta_n) d(Sx_{2n}, Tx_{2n-1}) + p_2(\delta_n) d(Sy_{2n}, Ty_{2n-1}) \\
& + q(\delta_n) d(f(x_{2n}, y_{2n}), Sx_{2n}) + r(\delta_n) d(g(x_{2n-1}, y_{2n-1}), Tx_{2n-1}) \\
& + t(\delta_n)[d(f(x_{2n}, y_{2n}), Tx_{2n-1}) + d(g(x_{2n-1}, y_{2n-1}), Sx_{2n})]
\end{aligned}$$

where $\delta_n = (x_{2n}, y_{2n}, x_{2n-1}, y_{2n-1})$.

$$\begin{aligned}
d(u_{2n}, u_{2n+1}) & \leq p_1(\delta_n) d(u_{2n}, u_{2n-1}) + p_2(\delta_n) d(v_{2n}, v_{2n-1}) + q(\delta_n) d(u_{2n+1}, u_{2n}) \\
& \quad + r(\delta_n) d(u_{2n}, u_{2n-1}) + t(\delta_n) d(u_{2n+1}, u_{2n-1}) \\
& \leq p_1(\delta_n) d(u_{2n}, u_{2n-1}) + p_2(\delta_n) d(v_{2n}, v_{2n-1}) + q(\delta_n) d(u_{2n+1}, u_{2n}) \\
& \quad + r(\delta_n) d(u_{2n}, u_{2n-1}) + t(\delta_n)[d(u_{2n+1}, u_{2n}) + d(u_{2n}, u_{2n-1})],
\end{aligned}$$

$$d(u_{2n}, u_{2n+1}) \leq \frac{(p_1 + r + t)(\delta_n)}{1 - (q+t)(\delta_n)} d(u_{2n-1}, u_{2n}) + \frac{p_2(\delta_n)}{1 - (q+t)(\delta_n)} d(v_{2n}, v_{2n-1}) \tag{24}$$

and

$$\frac{(p_1+r+t)(\delta_n)}{1-(q+t)(\delta_n)}+\frac{p_2(\delta_n)}{1-(q+t)(\delta_n)}\leq\lambda. \tag{25}$$

Hence, from (22)-(25), $\forall n\in\mathbb{N}$, $\exists e_n, f_n\geq 0$, such that

$$d(u_{n-1},u_n)\leq e_n d(u_{n-2},u_{n-1})+f_n d(v_{n-2},v_{n-1}),\ e_n+f_n\leq\lambda<1. \tag{26}$$

Similarly, we have

$$d(v_{n-1},v_n)\leq e'_n d(u_{n-2},u_{n-1})+f'_n d(v_{n-2},v_{n-1}),\ e'_n+f'_n\leq\lambda<1. \tag{27}$$

The equations (20), (21), (26) and (27) yield the system

$$\begin{cases}\bar{u}_{n+1}\leq a_n\bar{u}_n+b_n\bar{v}_n+c_n\bar{w}_n+d_n\bar{z}_n,\\ \bar{v}_{n+1}\leq a'_n\bar{u}_n+b'_n\bar{v}_n+c'_n\bar{w}_n+d'_n\bar{z}_n,\\ \bar{w}_n\leq e_n\bar{w}_{n-1}+f_n\bar{z}_{n-1},\\ \bar{z}_n\leq e'_n\bar{w}_{n-1}+f'_n\bar{z}_{n-1},\end{cases}$$

with $c'_n=d_n=0$, $a_n+b_n+c_n+d_n\leq\lambda$, $a'_n+b'_n+c'_n+d'_n\leq\lambda$, $e_n+f_n\leq\lambda$, $e'_n+f'_n\leq\lambda$, $\lambda<1$, where $\bar{u}_n=d(u_{n-1},x)$, $\bar{v}_n=d(\bar{v}_{n-1},x)$, $\bar{w}_n=d(u_{n-1},u_n)$ and $\bar{z}_n=d(v_{n-1},v_n)$. From Lemma 1.8, we have that $\bar{u}_n,\bar{v}_n\overset{\ll}{\to}0$ and so, by implication $\{u_n\}$ and $\{v_n\}$ converge both to x.

We have the following corollaries which extend the results of Abbas et al. [18] to four maps with w-compatibility replaced with w*-compatibility thus improving their results.

Corollary 2.2. *Let (X,d) be a cone metric space, $f:X\times X\to X$, $g:X\times X\to X$, $S:X\to X$ and $T:X\to X$ be four mappings such that $f(X\times X)\subset T(X)$, $g(X\times X)\subset S(X)$ and*

$$d(f(x,y),g(u,v))\leq h_1U+h_2V,\ h_1+h_2<1, \tag{28}$$

where

$$\begin{aligned}U,V\ \ \in\ \ &\{d(Sx,Tu),d(Sy,Tv),d(f(x,y),Sx),d(g(u,v),Tu),\\ &\tfrac{1}{2}[d(f(x,y),Tu)+d(g(u,v),Sx)]\}\end{aligned} \tag{29}$$

for all $x,y,u,v\in X$. If one of $f(X\times X)$, $g(X\times X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then $\{f,S\}$ and $\{g,T\}$ have a unique coupled coincidence point in X. Moreover, if $\{f,S\}$ and $\{g,T\}$ are w-compatible, then f, g, S and T have a unique coupled common fixed point*

$(u,u)\in X\times X$ and for every $(x_0,y_0)\in X\times X$, the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ defined by

$$\begin{cases} f(x_{2n-2},y_{2n-2}) = Tx_{2n-1} := u_{2n-1}, \\ f(y_{2n-2},x_{2n-2}) = Ty_{2n-1} := v_{2n-1}, \\ g(x_{2n-1},y_{2n-1}) = Sx_{2n} := u_{2n}, \\ g(y_{2n-1},x_{2n-1}) = Sy_{2n} := v_{2n}, \end{cases} \tag{30}$$

converge each to $u\in X$.

Proof.

$$\begin{aligned} U,V &\in \{d(Sx,Tu),d(Sy,Tv),d(f(x,y),Sx),d(g(u,v),Tu),\\ &\quad \tfrac{1}{2}[d(f(x,y),Tu)+d(g(u,v),Sx)]\} \end{aligned}$$

implies that

$$\begin{aligned} h_1U+h_2V &= p_1(x,y,u,v)d(Sx,Tu)+p_2(x,y,u,v)d(Sy,Tv)\\ &\quad +q(x,y,u,v)d(f(x,y),Sx)+r(x,y,u,v)d(g(u,v),Tu)\\ &\quad +t(x,y,u,v)[d(f(x,y),Tu)+d(g(u,v),Sx)] \end{aligned}$$

for all $x,y,u,v\in X$, where $p_1,p_2,q,r : X^4\to\{0,h_1,h_2\}$ and $t : X^4\to\{0,\frac{h_1}{2},\frac{h_2}{2}\}$ are chosen such that $[(p_1+p_2+q+r+2t)(x,y,u,v)] = h_1+h_2 < 1$.
It follows that f,g,S and T satisfy the contractive condition of Theorem 3.1 , hence, they have a unique common fixed point.

Corollary 2.3 *Let (X,d) be a cone metric space, $f : X\times X\to X$, $g : X\times X\to X$, $S : X\to X$ and $T : X\to X$ be four mappings such that $f(X\times X)\subset T(X)$, $g(X\times X)\subset S(X)$ and*

$$\begin{aligned} d(f(x,y),g(u,v)) &\leq p_1d(Sx,Tu)+p_2d(Sy,Tv)+qd(f(x,y),Sx)+rd(g(u,v),Tu)\\ &\quad +t[d(f(x,y),Tu)+d(g(u,v),Sx)] \end{aligned} \tag{31}$$

for all $x,y,u,v\in X$, where $p_1,p_2,q,r,t\in[0,1)$ and $p_1+p_2+q+r+2t<1$.
If one of $f(X\times X)$, $g(X\times X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then $\{f,S\}$ and $\{g,T\}$ have a unique coupled coincidence point in X.
Moreover, if $\{f,S\}$ and $\{g,T\}$ are w-compatible, then f, g, S and T have a unique coupled common fixed point $(u,u)\in X\times X$ and for every $(x_0,y_0)\in X\times X$, the sequences $\{u_n\}$ and $\{v_n\}$*

defined by

$$\begin{cases} f(x_{2n-2},y_{2n-2}) = Tx_{2n-1} := u_{2n-1} \\ f(y_{2n-2},x_{2n-2}) = Ty_{2n-1} := v_{2n-1} \\ g(x_{2n-1},y_{2n-1}) = Sx_{2n} := u_{2n} \\ g(y_{2n-1},x_{2n-1}) = Sy_{2n} := v_{2n} \end{cases} \tag{32}$$

converge each to $u \in X$.

A weaker version of this corollary was proved in [19] via product cone metric spaces.

Remark 2.4. (a) The substitutions made to obtain the existence results were also made by the authors in [20] and [21] for two maps satisfying a less general contractive condition.
(b) The importance of the use of Lemma 1.8 should be stressed. The existence of a common coupled fixed point of the four maps in Theorem 2.1 is guaranteed when the case $x = y$ and $u = v$ is linked to Theorem 1.9.. However, such substitution restricts the iterative schemes considered in (15) to lie in the main diagonal, from a starting point $x_0 \in X$. Indeed, the sequences $\{u_n\}$ and $\{x_n\}$ defined by

$$\begin{cases} x_0 \in X, \\ f(x_{2n-2},x_{2n-2}) = Tx_{2n-1} := u_{2n-1}, \\ g(x_{2n-1},x_{2n-1}) = Sx_{2n} := u_{2n}, \end{cases}$$

would then converge each to $x^* \in X$, where (x^*,x^*) is the common coupled fixed point. The use of the lemma of convergence gives a higher degree of freedom to the iterative scheme and the initial point (x_0,y_0). By the lemma, it is shown in the proof of Theorem 2.1 that the sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$ and $\{y_n\}$ defined by

$$\begin{cases} f(x_{2n-2},y_{2n-2}) = Tx_{2n-1} := u_{2n-1}, \\ f(y_{2n-2},x_{2n-2}) = Ty_{2n-1} := v_{2n-1}, \\ g(x_{2n-1},y_{2n-1}) = Sx_{2n} := u_{2n}, \\ g(y_{2n-1},x_{2n-1}) = Sy_{2n} := v_{2n}, \end{cases}$$

where (x_0,y_0) is any element in $X \times X$, converge each to $x^* \in X$, despite the fact that the sequences $\{(u_n,v_n)\}$ do not belong to the main diagonal.

The following example is given to stress the theorem and the remark.

Example 2.5. Let $E=\mathbb{R}^2$, $P=\{(x,y)\in E : x,y\geq 0\}$, $X=\mathbb{R}$ and $d: X\times X\to E$ defined by $d(x,y)=(|x-y|,|x-y|)$. Consider the maps $f,g: X\times X\to X$ and $S,T: X\to X$ defined by $f(x,y)=g(x,y)=\frac{x+2y}{3}$ and $S(x)=T(x)=2x$. These maps satisfy the conditions of Theorem 3.1 with $p_1(x,y,u,v)=p_2(x,y,u,v)=\frac{1}{3}$ and $q(x,y,u,v)=r(x,y,u,v)=t(x,y,u,v)=0$. Their common coupled fixed point is $(0,0)$. The sequences (15) which can be written

$$\begin{cases} \frac{1}{2}u_{2n-1} & = & x_{2n-1} & = & \frac{1}{6}[x_{2n-2}+2y_{2n-2}], \\ \frac{1}{2}v_{2n-1} & = & y_{2n-1} & = & \frac{1}{6}[2x_{2n-2}+y_{2n-2}], \\ \frac{1}{2}u_{2n} & = & x_{2n} & = & \frac{1}{6}[x_{2n-1}+2y_{2n-1}], \\ \frac{1}{2}v_{2n} & = & y_{2n} & = & \frac{1}{6}[2x_{2n-1}+y_{2n-1}], \end{cases}$$ converges to $(0,0)$ for any $x_0,y_0\in X$.

A MATLAB script yields $u_{26}=0.0000$ and $v_{26}=0.0001$ for $x_0=41$ and $y_0=7$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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