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APPROXIMATING FIXED POINTS OF GENERALIZED NONEXPANSIVE MAPPINGS VIA FASTER ITERATION SCHEMES

ANUPAM SHARMA*, MOHAMMAD IMDAD

Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

Abstract. In this paper, we approximate fixed points of generalized nonexpansive mappings in Banach spaces under relatively faster iteration schemes and also prove some weak and strong convergence theorems. Our results generalize and improve several previously known results of the existing literature.

Keywords: generalized nonexpansive mapping; weak convergence; strong convergence; condition (A').

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1. Introduction and preliminaries

Generally, a Banach space E is said to have the fixed point property for nonexpansive mappings provided every nonexpansive self-mapping of every nonempty, closed, convex, bounded subset K of E has a fixed point. The fixed point property heavily depends upon geometric characteristics of the classes of Banach spaces (under consideration) e.g., uniformly convex Banach spaces, strictly convex Banach spaces and similar others. Nevertheless, if the norm of a Banach space E has suitable geometric properties such as: uniform convexity and strict convexity, then

*Corresponding author
E-mail address: annusharma241@gmail.com
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every nonexpansive self-mapping defined on every weakly compact and convex subset of E has a fixed point. In such instances, E is said to have the weak fixed point property. Here, it can be pointed out that a nonexpansive self-mapping defined on a weakly compact and convex subset K of E need not have a fixed point. A self-mapping T defined on a subset K of a Banach space E is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in K.$$

Although nonexpansive mappings are the most important class of mappings studied in metric fixed point theory, yet one can find considerable research literature dealing with fixed points of more general classes of mappings; see, e.g., [1, 2]. With similar quest, Suzuki [3] introduced another class of self-mappings which falls in between the classes of nonexpansive and quasi-nonexpansive mappings and referred such maps as maps satisfying condition (C) and utilized the same to prove some fixed point theorems.

A mapping $T : K \to K$ is said to satisfy condition (C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|, \ \forall\, x, y \in K.$$

It is easy to see that every nonexpansive mapping satisfies condition (C) on K, but one can find some examples of noncontinuous mappings satisfying condition (C) in [3].

Recently, García-Falsat *et al.* [4] defined two new generalizations of condition (C) and term their new conditions as condition (E) and condition (C_λ) and also studied their asymptotic behavior as well as existence of fixed points.

Definition 1.1. [4] A mapping $T : K \to K$ satisfies condition (C_λ) on K if for all $x, y \in K$ and $\lambda \in (0, 1)$,

$$\lambda\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|.$$

For $\lambda = \frac{1}{2}$, we recapture the class of mappings satisfying condition (C). Notice that if $0 < \lambda_1 < \lambda_2 < 1$, then condition (C_{λ_1}) implies condition (C_{λ_2}) but converse fails (e.g. Example 5 in [4]).

Now, we recall another generalization of nonexpansive map under the name of condition (E).

Definition 1.2. [4] A mapping $T : K \to K$ is said to satisfy condition (E_μ) if for some $\mu \geq 1$ and for all $x, y \in K$,

$$\|x - Ty\| \leq \mu \ \|x - Tx\| + \|x - y\|.$$

We say that T satisfies condition (E) on K whenever T satisfies condition (E_μ) for some $\mu \geq 1$.

In view of the foregoing definitions, one can make the following remarks:

1. If $T : K \to K$ is a nonexpansive mapping, then T satisfies condition (E_1). But the converse is not true in general.
2. From Lemma 1 in [3], one can see that if $T : K \to K$ satisfies condition (C), then T satisfies condition (E_3) but the converse is not true in general.

The following example supports the two preceding facts.

Example 1.1. [4] In the Banach space $X = C[0,1]$ under supremum norm, consider a nonempty subset K of X defined as follows:

$$K = \{f \in C[0,1] \ : \ 0 = f(0) \leq f(x) \leq f(1) = 1\}.$$

To any $g \in K$, associate a function $F_g : K \to K$ defined by $F_g h(t) = (goh)(t) = g(h(t))$.
It is easy to verify that F_g satisfies condition (E_1) but does not satisfy condition (C).

For approximating fixed points of nonlinear mappings, Picard [5], Mann [6] and Ishikawa [7] introduced iteration schemes, which are respectively described in the following lines:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = Tx_n, \ n \in \mathbb{N}, \end{cases}$$

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1 - a_n)x_n + a_n T x_n, \ n \in \mathbb{N}, \end{cases}$$

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - a_n)x_n + a_n T y_n, \\ y_n = (1 - b_n)x_n + b_n T x_n, \ n \in \mathbb{N}, \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$.

It is well-known that Picard iteration scheme converges for contractions but may not converge for nonexpansive mappings whereas Mann iteration scheme converges for nonexpansive mappings as well. Agarwal *et al.* [8] posed the following question:

Is there any scheme for contraction mappings which converges at a rate similar to Picard iteration scheme?

In an attempt to answer the same, they introduced the following iteration scheme known as S-iteration scheme:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-a_n)Tx_n + a_n T y_n, \\ y_n = (1-b_n)x_n + b_n T x_n, \ n \in \mathbb{N}, \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$.

On the other hand, Yao and Chen [9] introduced an iteration scheme for approximating the common fixed points of two mappings $T, S : K \to K$, which runs as follows:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n S x_n, \ n \in \mathbb{N}, \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $[0,1]$ and $a_n + b_n + c_n = 1$. Notice that this scheme reduces to Mann iteration scheme when $T = I$ or $S = I$.

Agarwal *et al.* [8] also posed the following question.

Is there any scheme to compute the common fixed points for two contraction mappings which converges at a rate similar to Picard scheme and faster than its counter parts?

In an attempt to answer the preceding question, Khan *et al.* [10] introduced the following iteration scheme to compute the common fixed points of mappings S and T.

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-a_n)Tx_n + a_n S y_n, \\ y_n = (1-b_n)x_n + b_n T x_n, \ n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$.

Further, Khan *et al.* [10] introduced yet another iteration scheme for nonexpansive non-self mappings, wherein the idea of retraction map is utilized:

A subset K of E is called a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if P is a retraction, then $Py = y$ for all y in the range of P.

Let $T, S : K \to E$ be two nonexpansive non-self mappings and $P : E \to K$ a nonexpansive retraction. Define $\{x_n\}$ by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1-a_n)Tx_n + a_n Sy_n), \\ y_n = P((1-b_n)x_n + b_n Tx_n),\ n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$.

Motivated by S-iteration process, Sahu [11] introduced the normal S-iteration process as follows:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = T((1-a_n)x_n + a_n Tx_n),\ n \in \mathbb{N}, \end{cases}$$

where $\{a_n\}$ is a sequence in $(0,1)$.

Agarwal *et al.* [8] showed that S-iteration process involving contractions converges at the rate of convergence of Picard iteration and even faster than Mann iteration. Sahu [11] demonstrated that S-iteration process converges at a rate faster than both Picard and Mann iterations (involving contractions) with the help of a numerical example.

Very recently, Kadioglu and Yildirim [12] introduced the following iteration scheme for one mapping, whose rate of convergence is faster than both the S-iteration process and the normal

S-iteration process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = Ty_n, \\ y_n = (1-a_n)z_n + a_nTz_n, \\ z_n = (1-b_n)x_n + b_nTx_n, \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$.

The process (1.3) is independent of all Picard, Mann, Ishikawa and S-iteration processes as $\{a_n\}$ and $\{b_n\}$ are in $(0,1)$. Even if it is allowed to take $a_n = 0$ and $a_n = b_n = 0$ in the process (1.3), then this process reduces to normal S-iteration process and Picard iteration process respectively. To appreciate the rate of convergence of scheme (1.3), one can see [12].

The main purpose of this paper is to prove some weak and strong convergence theorems for approximating fixed points of generalized nonexpansive mappings.

With a view to make, our presentation self contained, we collect some basic definitions, needed results and some iterative methods which will be used frequently in the text later.

Let $S = \{x \in E : \|x\| = 1\}$ and E^* be the dual of E. Then the space E has :

(i) Gâteaux differentiable norm if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t},$$

exists for each x and y in S;

(ii) Fréchet differentiable norm ([13]) if for each $x \in S$, the above limit exists and is attained uniformly for $y \in S$ and in this case, it is also well-known that

$$\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 \le \frac{1}{2}\|x+h\|^2 \le \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|h\|)$$

for all $x, h \in E$, where J is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x \in X$, $\langle ., . \rangle$ is the dual pairing between E and E^*, and b is an increasing function defined on $[0,\infty)$ such that $\lim\limits_{t\to 0} \frac{b(t)}{t} = 0$;

(iii) Opial's condition ([14]) if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\lim_{n\to\infty} \sup \|x_n - x\| < \lim_{n\to\infty} \sup \|x_n - y\|, \ \forall\, y \in E, \ y \neq x.$$

Examples of Banach spaces satisfying this condition are Hilbert spaces and all l_p spaces $(1 < p < \infty)$. On the other hand, $L_p[0,2]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.

Definition 1.3. [15] Let K be a non-empty subset of a Banach space E. Two mappings $T,S : K \to K$ are said to satisfy the condition (A') if there exists a nondecreasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0,\ g(r) > 0$ for all $r \in (0,\infty)$ such that either $\|x - Tx\| \geq g(D(x,F))$ or $\|x - Sx\| \geq g(D(x,F))$ for all $x \in K$, where $D(x,F) = \inf\{\|x - z\|\ :\ z \in F\}$ and $F = F(T) \cap F(S)$.

Remark 1.1. For $S = T$ in Definition 1.3, condition (A') reduces to condition (A), (see [16]).

An important property of uniformly convex Banach space is the following lemma:

Lemma 1.1. [17] *Let E be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\lim_{n\to\infty} \sup \|x_n\| \leq r$, $\lim_{n\to\infty} \sup \|y_n\| \leq r$, and*

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

We use the following lemmas in order to prove our main results:

Lemma 1.2. [18] *Let T be a mapping on a bounded and convex subset K of a uniformly convex Banach space E. Assume that T satisfies condition (C). Then $(I - T)$ is demiclosed at 0. That is, if $\{x_n\}$ in K converges weakly to $w \in K$ and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tw = w$.*

Lemma 1.3. [3] *Let T be a mapping on a subset K of a Banach space E. Assume that T satisfies condition (C). Then for each $x, y \in K$,*

(i) $\|x - Ty\| \leq 3\|Tx - x\| + \|x - y\|$,

(ii) $\|y - Ty\| \leq 3\|Tx - x\| + 2\|x - y\|$.

2. Main results

Now we prove some weak and strong convergence theorems of generalized nonexpansive mappings. We prove our results in three sections. In first two sections we prove some weak and strong convergence theorems for approximating common fixed points of two generalized nonexpansive self mappings and non-self mappings through iteration schemes (1.1) and (1.2) respectively while in the last section we prove some convergence theorems with the help of iteration scheme (1.3) for the class of generalized nonexpansive mapping in Banach spaces.

2.1. Convergence theorems for generalized nonexpansive mappings

We start this section with the following existence theorem through an iteration scheme (1.1). In the sequel, $F = F(T) \cap F(S)$ denotes the set of common fixed points of the mappings T and S.

Lemma 2.1.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Let T and S be two self mappings of K satisfying condition (C). Let $\{x_n\}$ be defined by the iteration scheme (1.1) where $\{a_n\}$ and $\{b_n\}$ are in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some $\varepsilon \in (0,1)$. If $F \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - z\|$ exists and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0 = \lim_{n\to\infty} \|x_n - Sx_n\|$.*

Proof. Let $z \in F$. By use of condition (C), we get

$$\frac{1}{2}\|z - Tz\| = 0 \leq \|x_n - z\| \;\Rightarrow\; \|Tx_n - Tz\| \leq \|x_n - z\|, \tag{2.1}$$

$$\frac{1}{2}\|z - Sz\| = 0 \leq \|y_n - z\| \;\Rightarrow\; \|Sy_n - Sz\| \leq \|y_n - z\|. \tag{2.2}$$

Using inequalities (2.1) and (2.2) along with (1.1), we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|(1-a_n)(Tx_n - z) + a_n(Sy_n - z)\| \\
&\leq (1-a_n)\|Tx_n - z\| + a_n\|Sy_n - z\| \\
&\leq (1-a_n)\|x_n - z\| + a_n\|y_n - z\| \\
&= (1-a_n)\|x_n - z\| + a_n\|(1-b_n)x_n + b_nTx_n - z\| \\
&\leq (1-a_n)\|x_n - z\| + a_n(1-b_n)\|x_n - z\| + a_nb_n\|Tx_n - z\| \\
&\leq (1-a_n)\|x_n - z\| + a_n(1-b_n)\|x_n - z\| + a_nb_n\|x_n - z\| \\
&= \|x_n - z\|.
\end{aligned}$$

Therefore $\lim\limits_{n\to\infty} \|x_n - z\|$ exists for any $z \in F$. Let $\lim\limits_{n\to\infty} \|x_n - z\| = a$. Consider

$$\begin{aligned}
\|y_n - z\| &= \|b_nTx_n + (1-b_n)x_n - z\| \\
&\leq b_n\|Tx_n - z\| + (1-b_n)\|x_n - z\| \\
&\leq b_n\|x_n - z\| + (1-b_n)\|x_n - z\| \\
&= \|x_n - z\|,
\end{aligned}$$

which implies that

$$\limsup_{n\to\infty} \|y_n - z\| \le a. \tag{2.3}$$

Using (2.1) and (2.2), we have

$$\limsup_{n\to\infty} \|Tx_n - z\| \le a \text{ and } \limsup_{n\to\infty} \|Sy_n - z\| \le a. \tag{2.4}$$

Moreover, we have

$$a = \lim_{n\to\infty} \|x_{n+1} - z\| = \lim_{n\to\infty} \|(1-a_n)(Tx_n - z) + a_n(Sy_n - z)\|. \tag{2.5}$$

Therefore by using (2.4), (2.5) and Lemma 1.1, we have

$$\limsup_{n\to\infty} \|Tx_n - Sy_n\| = 0. \tag{2.6}$$

On the other hand, we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|(1-a_n)Tx_n + a_nSy_n - z\| \\ &= \|(Tx_n - z) + a_n(Sy_n - Tx_n)\| \\ &\le \|Tx_n - z\| + a_n\|Tx_n - Sy_n\|. \end{aligned}$$

Taking lim inf on both the sides, we get $a \le \liminf_{n\to\infty} \|Tx_n - z\|$, which implies from (2.4) that

$$\lim_{n\to\infty} \|Tx_n - z\| = a. \tag{2.7}$$

Using (2.7), we have

$$\begin{aligned} \|Tx_n - z\| &\le \|Tx_n - Sy_n\| + \|Sy_n - z\| \\ &\le \|Tx_n - Sy_n\| + \|y_n - z\|. \end{aligned}$$

Taking lim inf on both the sides and using (2.7), we find that

$$a \le \liminf_{n\to\infty} \|y_n - z\|. \tag{2.8}$$

Hence by (2.3) and (2.8), we have

$$\lim_{n\to\infty} \|y_n - z\| = a. \tag{2.9}$$

Since

$$a = \lim_{n\to\infty} \|y_n - z\| = \lim_{n\to\infty} \|(1-b_n)(x_n - z) + b_n(Tx_n - z)\|,$$

we find from Lemma 1.1 that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{2.10}$$

Since $\|y_n - x_n\| = b_n\|Tx_n - x_n\|$, making use of (2.10), we get

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.11}$$

Using (2.6), (2.10), (2.11) and Lemma 1.3 (ii), we have

$$\begin{aligned}\|x_n - Sx_n\| &\leq 3\|y_n - Sy_n\| + 2\|x_n - y_n\| \\ &\leq 3\|y_n - Tx_n\| + 3\|Tx_n - Sy_n\| + 2\|x_n - y_n\| \\ &= 3\|(1-b_n)x_n + b_nTx_n - Tx_n\| + 3\|Tx_n - Sy_n\| + 2\|x_n - y_n\| \\ &= 3(1-b_n)\|x_n - Tx_n\| + 3\|Tx_n - Sy_n\| + 2\|x_n - y_n\|,\end{aligned}$$

yielding thereby $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$. This concludes the proof.

Lemma 2.1.2. *In addition to the hypotheses of Lemma 2.1.1, suppose that $z_1, z_2 \in F$. Then, $\lim_{n\to\infty} \langle x_n, J(z_1 - z_2)\rangle$ exists. In particular, $\langle p - q, J(z_1 - z_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$, the set of all weak limits of sequence $\{x_n\}$.*

Proof. The proof of this lemma is the same as that of Lemma 2.3 of Khan *et al.* [10], hence proof is omitted.

By using Lemma 2.1.2, we prove the following weak convergence theorem:

Theorem 2.1.1. *Let E be a uniformlye convex Banach space and let K, T, S and $\{x_n\}$ be the same as in Lemma 2.1.1. Assume that*

(a) E satisfies Opial's condition,

(b) E has a Fréchet differentiable norm.

If $F \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T and S.

Proof. Let $z \in F$. Then by Lemma 2.1.1, $\lim_{n\to\infty} \|x_n - z\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in F. Let w_1 and w_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. From Lemma 2.1.1, we see that $\lim_{n\to\infty} \sup \|x_n - Tx_n\| = 0$ and $(I - T)$ is demiclosed with respect to zero from Lemma 1.2. Therefore $Tw_1 = w_1$. Similarly, $Sw_1 = w_1$. In the same way, we can prove that $w_2 \in F$. Next, we prove the uniqueness. To this end, first we assume that (a) holds. Let us suppose that $w_1 \neq w_2$. Since $x_{n_i} \rightharpoonup w_1$ and $w_1 \neq w_2$, by Opial's condition, we have

$$\lim_{n\to\infty}\|x_n - w_1\| = \lim_{i\to\infty}\|x_{n_i} - w_1\| < \lim_{i\to\infty}\|x_{n_i} - w_2\| = \lim_{n\to\infty}\|x_n - w_2\|.$$

Again as $x_{n_j} \rightharpoonup w_2$ and $w_2 \neq w_1$, by Opial's condition, we have

$$\lim_{n\to\infty}\|x_n - w_2\| = \lim_{j\to\infty}\|x_{n_j} - w_2\| < \lim_{j\to\infty}\|x_{n_j} - w_1\| = \lim_{n\to\infty}\|x_n - w_1\|.$$

Thus, we get a contradiction. Hence $w_1 = w_2$.

Next, we assume that (b) holds. From Lemma 2.1.2, $\langle p-q, J(z_1-z_2)\rangle = 0$ for all $p,q \in \omega_w(x_n)$. Therefore $\|w_1-w_2\|^2 = \langle w_1-w_2, J(w_1-w_2)\rangle = 0$ implies $w_1 = w_2$.

Now, we prove two strong convergence theorems in a real Banach space under iterative scheme (1.1).

Theorem 2.1.2. *Let E be a real Banach space and $K, T, S, \{x_n\}$ and F be the same as in Lemma 2.1.1. Then $\{x_n\}$ converges strongly to a point of F if and only if $\lim_{n\to\infty}\inf D(x_n,F) = 0$, where $D(x,F) = inf\ \{\|x-z\|\ :\ z \in F\}$.*

Proof. Necessity is obvious. Suppose that $\lim_{n\to\infty}\inf D(x_n,F) = 0$. By Lemma 2.1.1, $\lim_{n\to\infty}\|x_n - z\|$ exists for all $z \in F$. Therefore $\lim_{n\to\infty} D(x_n,F)$ exists. In view of the hypothesis, $\liminf_{n\to\infty} D(x_n,F) = 0$. Therefore we have $\lim_{n\to\infty} D(x_n,F) = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence in K. Since $\lim_{n\to\infty}\inf D(x_n,F) = 0$, for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $D(x_n,F) < \frac{\varepsilon}{2}$. In particular, inf $\{\|x_{n_0} - z\|\ :\ z \in F\} < \frac{\varepsilon}{2}$. Hence, there exists $z^* \in F$ such that $\|x_{n_0} - z^*\| < \frac{\varepsilon}{2}$. Hence for $m,n \geq n_0$, we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - z^*\| + \|x_n - z^*\| \leq 2\|x_{n_0} - z^*\| < \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence and therefore converges. Let $\{x_n\}$ converges to z. Since $\lim_{n\to\infty} D(x_n,F) = 0$, we get $D(z,F) = 0$. Consequently, $z \in F$ which amounts to say that $\{x_n\}$ converges weakly to a point of F. This completes the proof.

By using Theorem 2.1.2 under condition (A'), we prove another convergence theorem as follows:

Theorem 2.1.3. *Let E be a real Banach space and $K,S,T,\{x_n\}$ and F be the same as in Lemma 2.1.1. Let T,S satisfy the condition (A') and $F \neq \emptyset$. Then $\{x_n\}$ converges strongly to a point of F.*

Proof. In view of Lemma 2.1.1, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0 = \lim_{n\to\infty} \|x_n - Sx_n\|$. Hence, owing to condition (A'), we have

$$\lim_{n\to\infty} g(D(x_n,F)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

$$\text{or, } \lim_{n\to\infty} g(D(x_n,F)) \leq \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Thus in both the cases, we have $\lim_{n\to\infty} g(D(x_n,F)) = 0$. Since $g : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $g(0) = 0$, $g(r) > 0$ for all $r \in (0,\infty)$, therefore we have $\lim_{n\to\infty} D(x_n,F) = 0$. Hence by Theorem 2.1.2, $\{x_n\}$ converges strongly to a point of F.

Corollary 2.1.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T : K \to K$ be a mapping satisfying condition (C) and $\{x_n\}$ a S-iteration scheme where $\{a_n\}$ and $\{b_n\}$ are in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some $\varepsilon \in (0,1)$. If $F(T)$, (the set of fixed points of T) is nonempty, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Putting $S = T$ in Theorem 2.1.3, we conclude the desired result.

Corollary 2.1.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T : K \to K$ be a mapping satisfying condition (C) and $\{x_n\}$ be the Mann iteration scheme where $\{a_n\}$ is in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some $\varepsilon \in (0,1)$. If $F(T)$, (the set of fixed points of T) is nonempty, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Putting $T = I$ in Theorem 2.1.3, we conclude the desired result.

2.2. Convergence theorems for generalized nonexpansive non-self mappings

In this section, we outline the proofs of the theorems proved in the preceding section for generalized nonexpansive non-self mappings. We begin with the following existence theorem under the iteration scheme (1.2). As earlier, $F = F(T) \cap F(S)$ denotes the set of common fixed points of the mappings T and S.

Lemma 2.2.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T,S : K \to E$ be two non-self mappings satisfying condition (C) and $P : E \to K$ be a retraction satisfying condition (C). Let $\{x_n\}$ be defined by the iteration scheme (1.2), where $\{a_n\}$ and $\{b_n\}$ are in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ with some ε in $(0,1)$. If $F \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - z\|$ exists and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0 = \lim_{n\to\infty} \|x_n - Sx_n\|$.*

Proof. Let $z \in F$. Then by using condition (C), we get

$$\frac{1}{2}\|z - Tz\| = 0 \leq \|x_n - z\| \;\Rightarrow\; \|Tx_n - Tz\| \leq \|x_n - z\|, \tag{2.12}$$

$$\frac{1}{2}\|z - Sz\| = 0 \leq \|y_n - z\| \;\Rightarrow\; \|Sy_n - Sz\| \leq \|y_n - z\|. \tag{2.13}$$

In the same way, we have

$$\frac{1}{2}\|z - Pz\| = 0 \leq \|(1-a_n)Tx_n + a_nSy_n - z\|,$$

$$\Rightarrow\; \|P((1-a_n)Tx_n + a_nSy_n) - Pz\| \leq \|(1-a_n)Tx_n + a_nSy_n - z\|. \tag{2.14}$$

Similarly, we have

$$\frac{1}{2}\|z - Pz\| = 0 \leq \|(1-b_n)x_n + b_nTx_n - z\|,$$

$$\Rightarrow\; \|P((1-b_n)x_n + b_nTx_n) - Pz\| \leq \|(1-b_n)x_n + b_nTx_n - z\|. \tag{2.15}$$

Employing the iterative scheme (1.2) and using (2.12)-(2.15), we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|P((1-a_n)Tx_n + a_nSy_n) - z\| \\
&\leq \|(1-a_n)Tx_n + a_nSy_n - z\| \\
&\leq (1-a_n)\|Tx_n - z\| + a_n\|Sy_n - z\| \\
&\leq (1-a_n)\|x_n - z\| + a_n\|y_n - z\| \\
&= (1-a_n)\|x_n - z\| + a_n\|P((1-b_n)x_n + b_nTx_n) - z\| \\
&\leq (1-a_n)\|x_n - z\| + a_n\|(1-b_n)x_n + b_nTx_n - z\| \\
&\leq (1-a_n)\|x_n - z\| + a_n(1-b_n)\|x_n - z\| + a_nb_n\|x_n - z\| \\
&= \|x_n - z\|.
\end{aligned}$$

Therefore $\lim_{n\to\infty}\|x_n - z\|$ exists for any $z \in F$. Let $\lim_{n\to\infty}\|x_n - z\| = a$. Consider

$$\begin{aligned} \|y_n - z\| &= \|P((1-b_n)x_n + b_nTx_n) - z\| \\ &\le \|(1-b_n)x_n + b_nTx_n - z\| \\ &\le (1-b_n)\|x_n - z\| + b_n\|Tx_n - z\| \\ &\le \|x_n - z\|, \end{aligned}$$

which implies that $\limsup_{n\to\infty}\|y_n - z\| \le a$. By using (2.12) and (2.13), we have

$$\limsup_{n\to\infty}\|Tx_n - z\| \le a \text{ and } \limsup_{n\to\infty}\|Sy_n - z\| \le a.$$

It follows that

$$\lim_{n\to\infty}\|(1-a_n)(Tx_n - z) + a_n(Sy_n - z)\| = a. \tag{2.16}$$

Therefore by using Lemma 1.1, we have $\limsup_{n\to\infty}\|Tx_n - Sy_n\| = 0$. Again we consider

$$\begin{aligned} \|x_{n+1} - z\| &= \|P((1-a_n)Tx_n + a_nSy_n) - z\| \\ &\le \|(1-a_n)Tx_n + a_nSy_n - z\| \\ &\le \|Tx_n - z\| + a_n\|Tx_n - Sy_n\|. \end{aligned}$$

Taking lim inf of both the sides, we get

$$a \le \liminf_{n\to\infty}\|Tx_n - z\| \text{ so that } \lim_{n\to\infty}\|Tx_n - z\| = a.$$

On the lines similar to (2.16), we obtain

$$\lim_{n\to\infty}\|(1-b_n)(x_n - z) + b_n(Tx_n - z)\| = a.$$

Using Lemma 1.1, we have

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0.$$

As $x_n \in K$, the range of P, therefore $Px_n = x_n$ for all $n \in \mathbb{N}$. Owing to Lemma 1.3 (i), we have

$$\begin{aligned} \|y_n - x_n\| &= \|P((1-b_n)x_n + b_nTx_n) - x_n\| \\ &\le 3\|Px_n - x_n\| + \|(1-b_n)x_n + b_nTx_n - x_n\| \\ &= b_n\|x_n - Tx_n\|. \end{aligned}$$

Therefore $\lim_{n\to\infty}\|y_n - x_n\| = 0$. Using Lemma 1.3 (ii) and the foregoing inequalities, we have

$$\begin{aligned}\|x_n - Sx_n\| &\leq 3\|y_n - Sy_n\| + 2\|x_n - y_n\| \\ &\leq 3\|y_n - Tx_n\| + 3\|Tx_n - Sy_n\| + 2\|x_n - y_n\| \\ &= 3\|(1-b_n)x_n + b_nTx_n - Tx_n\| + 3\|Tx_n - Sy_n\| + 2\|x_n - y_n\| \\ &= 3(1-b_n)\|x_n - Tx_n\| + 3\|Tx_n - Sy_n\| + 2\|x_n - y_n\|,\end{aligned}$$

so that $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$.

The following theorems for generalized nonexpansive non-self mappings can now be proved with appropriate modifications in the proofs of Theorems 2.1.1, 2.1.2, and 2.1.3.

Theorem 2.2.1. *Let E be a uniformly convex Banach space and K,T,S and $\{x_n\}$ be the same as in Lemma 2.2.1. Assume that*
(a) E satisfies Opial's condition,
(b) E has a Fréchet differentiable norm.
If $F \neq \emptyset$, then $\{x_n\}$ converges weakly to a point of F.

Theorem 2.2.2. *Let E be a real Banach space and K,T,S and $\{x_n\}$ be the same as in Lemma 2.2.1. Then $\{x_n\}$ converges to a point of F if and only if $\lim_{n\to\infty}\inf D(x_n,F) = 0$.*

Theorem 2.2.3. *Let E be a real Banach space and K,T,S and $\{x_n\}$ be the same as in Lemma 2.2.1. Let T,S satisfy condition (A') and $F \neq \emptyset$. Then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Remark 2.2.1. The above theorems can also be proved by using the iteration scheme (1.2) with error terms.

2.3. Convergence theorems via a faster iteration scheme

In this section, we give some convergence theorems for a generalized nonexpansive mapping using iteration process (1.3), whose rate of convergence is faster than the S-iteration process as well a normal S-iteration process.

Lemma 2.3.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E and T be a self mapping of K satisfying condition (C). Let $\{x_n\}$ be defined by the iteration* *process (1.3) where $\{a_n\}$ and $\{b_n\}$ are in (0,1) for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in F(T)$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. Let $z \in F(T)$. Then using condition (C) on T in iterative scheme (1.3), we have

$$\begin{aligned} \|z_n - z\| &= \|(1-b_n)(x_n - z) + b_n(Tx_n - z)\| \\ &\leq (1-b_n)\|x_n - z\| + b_n\|Tx_n - z\| \\ &\leq (1-b_n)\|x_n - z\| + b_n\|x_n - z\| \\ &= \|x_n - z\|. \end{aligned} \tag{2.17}$$

As earlier, using condition (C) on T in iterative scheme (1.3), we get

$$\begin{aligned} \|x_{n+1} - z\| = \|Ty_n - z\| &\leq \|y_n - z\| \\ &= \|(1-a_n)z_n + a_nTz_n - z\| \\ &\leq (1-a_n)\|z_n - z\| + a_n\|Tz_n - z\| \\ &\leq \|z_n - z\| \leq \|x_n - z\|. \end{aligned}$$

This shows that $\{\|x_n - z\|\}$ is decreasing, and hence $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in F(T)$. Let

$$\lim_{n\to\infty} \|x_n - z\| = a. \tag{2.18}$$

Owing to condition (C) on T, we have

$$\limsup_{n\to\infty} \|Tx_n - z\| \leq a. \tag{2.19}$$

As $\|x_{n+1} - z\| \leq \|z_n - z\| \Rightarrow \liminf_{n\to\infty} \|x_{n+1} - z\| \leq \liminf_{n\to\infty} \|z_n - z\|$, and therefore,

$$a \leq \liminf_{n\to\infty} \|z_n - z\|. \tag{2.20}$$

Moreover, (2.17) implies that

$$\limsup_{n\to\infty} \|z_n - z\| \leq a. \tag{2.21}$$

In view of (2.20) and (2.21), we have $\lim_{n\to\infty} \|z_n - z\| = a$, which implies that

$$a = \lim_{n\to\infty} \|z_n - z\| = \lim_{n\to\infty} \|(1-b_n)(x_n - z) + b_n(Tx_n - z)\|. \tag{2.22}$$

Using (2.18), (2.19), (2.22) and Lemma 1.1, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Lemma 2.3.2. *Assume that all the conditions of Lemma 2.3.1 are satisfied. Then for* $z_1, z_2 \in F(T)$, $\lim_{n\to\infty} \langle x_n, J(z_1 - z_2)\rangle$ *exists. In particular,* $\langle p - q, J(z_1 - z_2)\rangle = 0$ *for all* $p, q \in \omega_w(x_n)$, *the set of all weak limits of sequence* $\{x_n\}$.

Proof. The proof of this lemma is the same as that of Lemma 2.3 of Khan *et al.* [10].

Now, we have our convergence theorems. The proofs of Theorems 2.3.1-2.3.3 run on the lines of the proofs of Theorems 2.1.1-2.1.3 by setting $S = T$, so we omit the proof.

Theorem 2.3.1. *Let* E *be a uniformly convex Banach space and let* K, T *and* $\{x_n\}$ *be as in Lemma 2.3.1. Assume that*

(a) E *satisfies Opial's condition or*

(b) E *has a Fréchet differentiable norm.*

If $F(T) \neq \emptyset$, *then* $\{x_n\}$ *converges weakly to a fixed point of* T.

Theorem 2.3.2. *Let* E *be a uniformly convex Banach space and let* $K, T, \{x_n\}$ *and* $F(T)$ *be as in Lemma 2.3.1. Then* $\{x_n\}$ *converges strongly to a point of* $F(T)$ *if and only if* $\lim_{n\to\infty} \inf D(x_n, F(T)) = 0$.

Notice that this condition is weaker than the requirement "T is demicompact or K is compact" (see [16]). Applying Theorem 2.3.2, we obtain a strong convergence of the iterative scheme (1.3) under condition (A) as follows:

Theorem 2.3.3. *Let* E *be a uniformly convex Banach space and let* K, T *and* $\{x_n\}$ *be the same as in Lemma 2.3.1. If* T *satisfies condition* (A), *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Now, we prove Lemma 2.3.1 using condition (E).

Proposition 2.3.1. *Let* K *be a nonempty closed convex subset of a uniformly convex Banach space* E *and* T *be a self mapping of* K *satisfying condition* (E). *Let* $\{x_n\}$ *be defined by the iteration process (1.3) where* $\{a_n\}$ *and* $\{b_n\}$ *are in (0, 1) for all* $n \in \mathbb{N}$. *Then* $\lim_{n\to\infty} \|x_n - z\|$ *exists for all* $z \in F(T)$ *and* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. Let $z \in F(T)$. Since T satisfies condition (E), then for $\mu \geq 1$, we have

$$\begin{cases} \|z - Tz_n\| \leq \mu\|z - Tz\| + \|z_n - z\| \ \Rightarrow \ \|Tz_n - z\| \leq \|z_n - z\|, \\ \|z - Ty_n\| \leq \mu\|z - Tz\| + \|y_n - z\| \ \Rightarrow \ \|Ty_n - z\| \leq \|y_n - z\|, \\ \|z - Tx_n\| \leq \mu\|z - Tz\| + \|x_n - z\| \ \Rightarrow \ \|Tx_n - z\| \leq \|x_n - z\|. \end{cases} \tag{2.23}$$

Hence, using iterative scheme (1.3) and (2.23), we have

$$\|z_n - z\| \leq (1 - b_n)\|x_n - z\| + b_n\|Tx_n - z\| \leq \|x_n - z\|, \tag{2.24}$$

while using (1.3) and inequalities (2.23) and (2.24), we have

$$\|x_{n+1} - z\| = \|Ty_n - z\| \leq \|y_n - z\| \leq \|z_n - z\| \leq \|x_n - z\|,$$

which shows that $\{\|x_n - z\|\}$ is decreasing so that $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in$ F(T). Hence proceeding on the lines of the proof of Lemma 2.3.1, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Remark 2.3.1. Theorems 2.3.1-2.3.3 remain valid if condition (C) is replaced by condition (E).

Conflict of Interests

The authors declare that there is no conflict of interests.

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