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HYBRID CONTRACTIONS WITH IMPLICIT RELATIONS

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Abstract. In this paper, we prove the existence of fixed points for two set-valued mappings and two single-valued mappings satisfying generalized contractive conditions by using the concept of weakly compatible mappings with control functions and implicit relations in complete metric spaces. Our results extend and generalize the corresponding result in Mehta and Joshi [21].

Keywords: fixed point; weakly compatible mapping; control function; Implicit relation; weak contraction.

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1. Introduction and preliminaries

Let (X,d) be a complete metric space and $B(X)$ be the set of all non-empty, bounded subsets of X. The function $\delta(A,B)$ with A and B in $B(X)$ is defined by

$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}.$$

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If A consists of a single point $\{a\}$, we write $\delta(A,B) = \delta(a,B)$ and if B consists also of a single point $\{b\}$, we write $\delta(A,B) = \delta(a,b) = d(a,b)$. It follows immediately from the definition that

$$\delta(A,B) = \delta(B,A) \geq 0, \delta(A,A) = \text{ diam}A,$$

$$\delta(A,B) \leq \delta(A,C) + \delta(C,B),$$

for all A,B and C in $B(X)$ and if $\delta(A,B) = 0$, then $A = B = \{a\}$.

The following lemma was proved in [10].

Lemma 1.1. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded, nonempty subsets of a complete metric space (X,d), which converge to the bounded subsets A and B respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.*

Hybrid fixed point theory is a recent development is the ambit of fixed point theorems for contracting single-valued and multi-valued mappings in metric spaces. Indeed, the study of such mappings was initiated during 1980-1983 by Bhaskaran-Subrahmanyam [7], Hadžić [13], Singh- Kulshrestha [27], Kaneko-Sessa [19], Naimpally *et al.* [22], Fisher-Sessa [11]. For a history of fundamental work on this line refer to Singh and Mishra [28], Sessa- Fisher [26] and for more recent work on this line Beg-Azam [5], Jungck-Rhoades [17], Kaneko [18].

Hybrid fixed point theory has potential applications in functional inclusions, optimization theory, fractal graphics and discrete dynamics for set-valued operator. By extending the definition of compatible mappings of Jungck [14], Jungck-Rhoades [16] gave the following definition for a pair multi-valued and single-valued mappings.

Definition 1.1. Let (X,d) be a metric space. Let $f : X \to X$ and $S : X \to B(X)$. S and f are δ-compatible iff $fSx \in B(X)$ for all $x \in X$ and $\delta(fSx_n, Sfx_n) \to 0$, whenever $\{x_n\}$ is a sequence in X such that $fx_n \to t$ and $Sx_n \to \{t\}$ for some $t \in X$.

Following Jungck [15] and Jungck-Rhoades [17], we have the following definition of weakly compatible mappings.

Definition 1.2. The mappings $f : X \to X$ and $S : X \to B(X)$ are weakly compatible iff they commute at their coincidence points, that is $Sfx = fSx$, whenever $fx \in Sx$.

The example 5.1 in [17] shows that weak compatibility is more general than compatibility. In 1984, Khan *et al.* [20] generalized the notion of altering distance and used it solving for fixed point problems in metric spaces.

Definition 1.3. A function $\varphi : [0,\infty) \to [0,\infty)$ is called an altering distance function (control function) if the following properties are satisfied:
I. φ is monotone increasing and continuous,
II. $\varphi(t) = 0$ if and only if $t = 0$.

Later on, the concept of weak contractions was introduced in 1997 by Alber *et al.* [2] in Hilbert spaces and subsequently it was extended to metric spaces by Rhoades [25]. Following Rhoades [25], we have the following definition.

Definition 1.4. A mapping $f : X \to X$, where (X,d) is a metric space, is said to be weakly contractive if for all $x,y \in X$ $d(fx,fy) \leq d(x,y) - \phi(d(x,y))$, where $\phi : [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$. If one takes $\phi(t) = (1-k)t$, where $0 < k < 1$, a weak contraction reduces to a Banach contraction.

Subsequently, a number of fixed point theorems in metric spaces have been proved by extending and generalizing the weak contractive conditions based on the concept of weaker forms of commutativity of mappings (see, for instance [1], [8], [9], [12], [29] and references therein). Afterward, several classical fixed point theorems and common fixed point theorems were unified by considering general contractive conditions expressed by an implicit condition. This approach has been initiated in the seminal papers Popa [23], [24]. Following Popa's approach, a consistent literature on fixed points, common fixed points and coincidence point theorems for both single-valued and multi-valued mappings, in various ambient spaces has been developed (see, for instances [3], [4], [6], [21] and references therein).

Definition 1.5. Let F^* be the set of continuous functions $F(t_1,t_2,t_3,t_4) : [0,\infty)^4 \to [0,\infty)$ satisfying the following conditions,
(F_1) F is non-decreasing in variable t_1.
(F_2) For $u \geq 0$, $v \geq 0$, $F(u+v,0,u,v) \leq u$.
(F_3) $F(u,v,0,0) \leq u$, $F(0,u,0,u) \leq u \; \forall \; u > 0$ and $F(u,0,u,0) \leq u \;\; \forall \; u > 0$.

The purpose of this paper is to prove a common fixed point theorem for hybrid contractions with generalized contractive conditions by using the concept of weakly compatible mappings with control functions and implicit relations in complete metric spaces.

2. Main results

Now we state our main result.

Theorem 2.1. *Let (X,d) be a complete metric space. Let $S,T : X \to B(X)$ be two set-valued mappings and $f,g : X \to X$ be two single-valued mappings such that for all $x,y \in X$*

$$S(X) \subseteq f(X), \;\; T(X) \subseteq g(X), \tag{1}$$

$$\varphi(\delta(Sx,Ty)) \leq \varphi(M(x,y)) - \phi(M(x,y)), \tag{2}$$

where

$$M(x,y) = F\{\delta(Ty,gx), \delta(Sx,fy), \delta(Ty,fy), \delta(Sx,gx)\}$$

and $\varphi : [0,\infty) \to [0,\infty)$ is continuous and monotonic increasing function and $\phi : [0,\infty) \to [0,\infty)$ is continuous and monotonic decreasing function with $\varphi(x) = 0 = \phi(x)$ if and only if $x = 0$ and $F \in F^*$. Further if $\{S,g\}$ and $\{T,f\}$ are weakly compatible pairs, then S,T,f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X and define the sequence $\{x_n\}$ inductively. Since $S(X) \subseteq f(X)$ and $T(X) \subseteq g(X)$, we can choose points $x_1, x_2 \in X$ such that $fx_1 \in Sx_0 = Z_0$ and $gx_2 \in Tx_1 = Z_1$. We continue this process to obtain a sequence $fx_{2n+1} \in Sx_{2n} = Z_{2n}$ and

$$gx_{2n+2} \in Tx_{2n+1} = Z_{2n+1}, \quad \text{for all } n = 0,1,2,3.... \, .$$

Applying contractive condition (2), we obtain

$$\begin{aligned} \varphi(\delta(Z_{2n}, Z_{2n+1})) &= \varphi(\delta(Sx_{2n}, Tx_{2n+1})) \\ &\leq \varphi(M(x_{2n}, x_{2n+1})) - \phi(M(x_{2n}, x_{2n+1})), \end{aligned} \tag{3}$$

where

$$
\begin{aligned}
M(x_{2n}, x_{2n+1}) &= F\left\{\begin{array}{c} \delta(Tx_{2n+1}, gx_{2n}), \delta(Sx_{2n}, fx_{2n+1}), \\ \delta(Tx_{2n+1}, fx_{2n+1}), \delta(Sx_{2n}, gx_{2n}) \end{array}\right\} \\
&\leq F\left\{\begin{array}{c} \delta(Z_{2n+1}, Z_{2n-1}), \delta(Z_{2n}, Z_{2n}), \\ \delta(Z_{2n+1}, Z_{2n}), \delta(Z_{2n}, Z_{2n-1}) \end{array}\right\} \\
&\leq F\left\{\begin{array}{c} (\delta(Z_{2n+1}, Z_{2n}) + \delta(Z_{2n}, Z_{2n-1})), 0, \\ \delta(Z_{2n+1}, Z_{2n}), \delta(Z_{2n}, Z_{2n-1}) \end{array}\right\}.
\end{aligned}
$$

From (F_2), we have $M(x_{2n}, x_{2n+1}) \leq \delta(Z_{2n}, Z_{2n-1})$. From (3), we obtain

$$
\begin{aligned}
\varphi(\delta(Z_{2n}, Z_{2n+1})) &\leq \varphi(\delta(Z_{2n}, Z_{2n-1})) - \phi(\delta(Z_{2n}, Z_{2n-1})) \\
\varphi(\delta(Z_{2n}, Z_{2n+1})) &\leq \varphi(\delta(Z_{2n}, Z_{2n-1})).
\end{aligned}
$$

Now φ is monotonic increasing function. This implies that the sequence $\{\delta(Z_n, Z_{n+1})\}$ is monotonic decreasing and bounded sequence of real numbers. Hence, there exists an $r \geq 0$ such that

$$
\lim_{n\to\infty} \{\delta(Z_n, Z_{n+1})\} = r. \tag{4}
$$

Taking limit as $n \to \infty$ in (3) and using the continuity of φ and ϕ, we have $\varphi(r) \leq \varphi(r) - \phi(r)$, which is a contradiction unless $r = 0$. Let $\{z_n\}$ be an arbitrary point in Z_n for $n = 0, 1, 2....$. Hence

$$
\lim_{n\to\infty} d(z_n, z_{n+1}) \leq \lim_{n\to\infty} \delta(Z_n, Z_{n+1}) = 0. \tag{5}
$$

Next, we show that $\{z_n\}$ is a Cauchy sequence. It is sufficient to show that $\{z_{2n}\}$ is a Cauchy sequence. Suppose that $\{z_{2n}\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ such that each positive integer there exist sequences $\{2n(k)\}$ and $\{2m(k)\}$ such that for all positive integer k,

$$
2m(k) > 2n(k) > k
$$

such that

$$
\varepsilon < d(z_{2n(k)}, z_{2m(k)}) \leq \delta(Z_{2n(k)}, Z_{2m(k)}). \tag{6}
$$

Now we may suppose that $2m(k)$ is the smallest integer exceeding $2n(k)$ for which (6) holds, that is, $\delta(Z_{2n(k)}, Z_{2m(k)-2}) < \varepsilon$. Using the triangle inequality, we have

$$\begin{aligned}\varepsilon \leq \delta(Z_{2n(k)}, Z_{2m(k)}) &\leq \delta(Z_{2n(k)}, Z_{2m(k)-2}) + \delta(Z_{2m(k)-2}, Z_{2m(k)-1}) \\ &\quad + \delta(Z_{2m(k)-1}, Z_{2m(k)}),\end{aligned}$$

that is,

$$\varepsilon \leq \delta(Z_{2n(k)}, Z_{2m(k)}) < \varepsilon + \delta(Z_{2m(k)-2}, Z_{2m(k)-1}) + \delta(Z_{2m(k)-1}, Z_{2m(k)}).$$

Letting $k \to \infty$ in the above inequality and using (5), we have

$$\lim_{k\to\infty} \delta(Z_{2n(k)}, Z_{2m(k)}) = \varepsilon. \tag{7}$$

On the other hand, we have

$$\begin{aligned}\delta(Z_{2n(k)-1}, Z_{2m(k)+1}) &\leq \delta(Z_{2n(k)-1}, Z_{2n(k)}) + \delta(Z_{2n(k)}, Z_{2m(k)}) \\ &\quad + \delta(Z_{2m(k)}, Z_{2m(k)+1})\end{aligned}$$

and

$$\begin{aligned}\delta(Z_{2n(k)}, Z_{2m(k)}) &\leq \delta(Z_{2n(k)}, Z_{2n(k)-1}) + \delta(Z_{2n(k)-1}, Z_{2m(k)+1}) \\ &\quad + \delta(Z_{2m(k)+1}, Z_{2m(k)}).\end{aligned}$$

It follows that

$$\begin{aligned}&|\delta(Z_{2n(k)-1}, Z_{2m(k)+1}) - \delta(Z_{2n(k)}, Z_{2m(k)})| \\ &\leq \delta(Z_{2n(k)-1}, Z_{2n(k)}) + \delta(Z_{2m(k)}, Z_{2m(k)+1}).\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (5) and (7), we have

$$\lim_{k\to\infty} \delta(Z_{2n(k)-1}, Z_{2m(k)+1}) = \varepsilon. \tag{8}$$

Similarly, we have

$$\begin{aligned}\delta(Z_{2n(k)}, Z_{2m(k)}) &\leq \delta(Z_{2n(k)}, Z_{2n(k)+1}) + \delta(Z_{2n(k)+1}, Z_{2m(k)+1}) \\ &\quad + \delta(Z_{2m(k)+1}, Z_{2m(k)})\end{aligned}$$

and

$$\begin{aligned}\delta(Z_{2n(k)+1},Z_{2m(k)+1}) &\leq \delta(Z_{2n(k)+1},Z_{2n(k)})+\delta(Z_{2n(k)},Z_{2m(k)})\\ &\quad +\delta(Z_{2m(k)},Z_{2m(k)+1}).\end{aligned}$$

It follows that

$$\begin{aligned}&|\delta(Z_{2n(k)},Z_{2m(k)})-\delta(Z_{2n(k)+1},Z_{2m(k)+1})|\\ &\leq \delta(Z_{2n(k)},Z_{2n(k)+1})+\delta(Z_{2m(k)+1},Z_{2m(k)}).\end{aligned}$$

Letting $k\to\infty$ in the above inequality and using (5) and (7), we have

$$\lim_{k\to\infty}\delta(Z_{2n(k)+1},Z_{2m(k)+1})=\varepsilon. \tag{9}$$

Further

$$\delta(Z_{2n(k)},Z_{2m(k)+1}) \leq \delta(Z_{2n(k)},Z_{2n(k)+1})+\delta(Z_{2n(k)+1},Z_{2m(k)+1})$$

and

$$\delta(Z_{2n(k)+1},Z_{2m(k)+1}) \leq \delta(Z_{2n(k)+1},Z_{2n(k)})+\delta(Z_{2n(k)},Z_{2m(k)+1}).$$

Hence, we have

$$|\delta(Z_{2n(k)},Z_{2m(k)+1})-\delta(Z_{2n(k)+1},Z_{2m(k)+1})|\leq \delta(Z_{2n(k)},Z_{2n(k)+1}).$$

Letting $k\to\infty$ in the above inequality and using (5) and (9), we have

$$\lim_{k\to\infty}\delta(Z_{2n(k)},Z_{2m(k)+1})=\varepsilon. \tag{10}$$

Putting $x=x_{2n(k)}$ and $y=x_{2m(k)+1}$ in (2), we have

$$\begin{aligned}\varphi(\delta(Z_{2n(k)},Z_{2m(k)+1})) &= \varphi(\delta(Sx_{2n(k)},Tx_{2m(k)+1}))\\ &\leq \varphi(M(x_{2n(k)},x_{2m(k)+1}))-\phi(M(x_{2n(k)},x_{2m(k)+1})),\end{aligned} \tag{11}$$

where

$$\begin{aligned}M(x_{2n(k)},x_{2m(k)+1}) &= F\left\{\begin{array}{l}\delta(Tx_{2m(k)+1},gx_{2n(k)}),\delta(Sx_{2n(k)},fx_{2m(k)+1}),\\ \delta(Tx_{2m(k)+1},fx_{2m(k)+1}),\delta(Sx_{2n(k)},gx_{2n(k)})\end{array}\right\}\\ &\leq F\left\{\begin{array}{l}\delta(Z_{2m(k)+1},Z_{2n(k)-1}),\delta(Z_{2n(k)},Z_{2m(k)}),\\ \delta(Z_{2m(k)+1},Z_{2m(k)}),\delta(Z_{2n(k)},Z_{2n(k)-1})\end{array}\right\}.\end{aligned}$$

By taking limit $k \to \infty$ in the above inequality and using (5), (7) and (8), we get

$$\lim_{k\to\infty} M(x_{2n(k)}, x_{2m(k)+1}) \leq F(\varepsilon, \varepsilon, 0, 0).$$

From (F_3), we have $\lim_{k\to\infty} M(x_{2n(k)}, x_{2m(k)+1}) \leq \varepsilon$. Therefore from equation (11), we have

$$\varphi(\varepsilon) \leq \varphi(\varepsilon) - \phi(\varepsilon),$$

which is a contradiction by virtue of a property of ϕ. Therefore the sequence $\{z_n\}$ is a Cauchy sequence in complete metric space X and so has a limit z in X. So the sequences $\{fx_{2n+1}\}$ and $\{gx_{2n+2}\}$ converges to z and further, the sequences of sets $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ converges to set $\{z\}$. By (1), there exist points u and v in X such that $gu = z$ and $fv = z$.

Next, we show that $z = fv \in Tv$. By (2), we have

$$\begin{aligned} \varphi(\delta(fx_{2n+1}, Tv)) &\leq \varphi(\delta(Sx_{2n}, Tv)) \\ &\leq \varphi(M(x_{2n}, v)) - \phi(M(x_{2n}, v)), \end{aligned}$$

where

$$M(x_{2n}, v) = F\left\{ \begin{array}{c} \delta(Tv, gx_{2n}), \delta(Sx_{2n}, fv), \\ \delta(Tv, fv), \delta(Sx_{2n}, gx_{2n}) \end{array} \right\}.$$

By taking limit $n \to \infty$ in the above inequality and using Lemma 1.1, we have

$$\begin{aligned} \lim_{n\to\infty} M(x_{2n}, v) &\leq F\{\delta(Tv, z), d(z, fv), \delta(Tv, z), d(z, z)\} \\ &= F\{\delta(Tv, z), 0, \delta(Tv, z), 0\}. \end{aligned}$$

From (F_3), we have $\lim_{n\to\infty} M(x_{2n}, u) \leq \delta(Tv, z)$. It follows that $\varphi(\delta(z, Tv)) \leq \varphi(\delta(z, Tv)) - \phi(\delta(z, Tv))$, which in turn implies that $\phi(\delta(z, Tv)) = 0$. Hence $\delta(z, Tv) = 0$, that is $z \in Tv$. Thus $Tv = \{z\} = \{fv\}$. But $T(X) \subseteq g(X)$, so there exists $u \in X$ such that $Tv = \{z\} = \{fv\} = \{gu\}$.

Now, we are in a position to show that $z = gu \in Su$. By (2.2), we have

$$\begin{aligned} \varphi(\delta(Su, gx_{2n+2})) &\leq \varphi(\delta(Su, Tx_{2n+1})) \\ &\leq \varphi(M(u, x_{2n+1})) - \phi(M(u, x_{2n+1})), \end{aligned}$$

where

$$M(u,x_{2n+1}) = F\left\{\begin{array}{l}\delta(Tx_{2n+1},gu),\delta(Su,fx_{2n+1}),\\ \delta(Tx_{2n+1},fx_{2n+1}),\delta(Su,gu)\end{array}\right\}.$$

By taking limit $n\to\infty$ in the above inequality and using Lemma 1.1, we have

$$\begin{aligned}\lim_{n\to\infty}M(u,x_{2n+1}) &\leq F\{d(z,gu),\delta(Su,z),d(z,z),\delta(Su,z)\}\\ &= F\{0,\delta(Su,z),0,\delta(Su,z)\}.\end{aligned}$$

In view of (F_3), we have $\lim_{n\to\infty}M(u,x_{2n+1})\leq\delta(Su,z)$. It follows that $\varphi(\delta(Su,z))\leq\varphi(\delta(Su,z))-\phi(\delta(Su,z))$, which implies that $\phi(\delta(Su,z))=0$. Hence $\delta(Su,z)=0$, that is, $z\in Su$. Then $Su=Tv=\{z\}=\{fv\}=\{gu\}$. Since $Su=\{gu\}$ and $\{S,g\}$ is weakly compatible $Sgu=gSu$ gives $Sz=\{gz\}$. Next, we show that z is a fixed point of g. By (2), we have

$$\begin{aligned}\varphi(\delta(Sz,fv)) &\leq \varphi(\delta(Sz,Tv))\\ &\leq \varphi(M(z,v))-\phi(M(z,v)),\end{aligned}$$

where

$$\begin{aligned}M(z,v) &= F\{\delta(Tv,gz),\delta(Sz,fv),\delta(Tv,fv),\delta(Sz,gz)\}\\ &\leq F\{d(z,gz),d(gz,z),d(z,z),d(gz,gz)\}\\ &= F\{d(z,gz),d(gz,z),0,0\}.\end{aligned}$$

From (F_3), we have $M(z,v)\leq d(z,gz)$. Therefore

$$\varphi(d(gz,z))\leq\varphi(d(gz,z))-\phi(d(gz,z)),$$

which implies that $\phi(d(gz,z))=0$. Hence $d(gz,z)=0$, that is, $z=gz$. Then $Sz=\{gz\}=\{z\}$. Similarly $Tv=\{fv\}$ and $\{T,f\}$ is weakly compatible $Tfv=fTv$ gives $Tz=\{fz\}$. Now we show that z is a fixed point of f. By (2), we have

$$\begin{aligned}\varphi(\delta(gu,Tz)) &\leq \varphi(\delta(Su,Tz))\\ &\leq \varphi(M(u,z))-\phi(M(u,z)),\end{aligned}$$

where

$$\begin{aligned} M(u,z) &= F\{\delta(Tz,gu),\delta(Su,fz),\delta(Tz,fz),\delta(Su,gu)\} \\ &\leq F\{d(fz,z),d(z,fz),d(fz,fz),d(z,z)\} \\ &= F\{d(fz,z),d(z,fz),0,0\}. \end{aligned}$$

From (F_3), we have $M(u,z) \leq d(z,fz)$. Therefore $\varphi(d(z,fz)) \leq \varphi(d(z,fz)) - \phi(d(z,fz))$, which implies that $\phi(d(z,fz)) = 0$. Hence $d(z,fz) = 0$, that is, $z = fz$. Then $Tz = \{z\} = \{fz\}$. Thus $Sz = Tz = \{z\} = \{fz\} = \{gz\}$ and z is a common fixed point of S, T, f and g.

Next, we show that z is unique. Suppose $p \neq z$ such that $Sp = Tp = \{p\} = \{fp\} = \{gp\}$. From (2), we have

$$\begin{aligned} \varphi(d(z,p)) &= \varphi(\delta(Sz,Tp)) \\ &\leq \varphi(M(z,p)) - \phi(M(z,p)), \end{aligned}$$

where

$$\begin{aligned} M(z,p) &= F\{\delta(Tp,gz),\delta(Sz,fp),\delta(Tp,fp),\delta(Sz,gz)\} \\ &\leq F\{d(p,z),d(z,p),d(p,p),d(z,z)\} \\ &= F\{d(p,z),d(z,p),0,0\}. \end{aligned}$$

By use of (F_3), we have $M(z,p) \leq d(z,p)$. Therefore, we obtain $\varphi(d(z,p)) \leq \varphi(d(z,p)) - \phi(d(z,p))$, which implies that $\phi(d(z,p)) = 0$. Hence $d(z,p) = 0$, that is, $z = p$. This completes the proof.

Corollary 2.2. *Let (X,d) be a complete metric space. Let $S,T : X \to B(X)$ be two set-valued mappings and $f,g : X \to X$ be two single-valued mappings such that for all $x,y \in X$*

$$S(X) \subseteq f(X), \ \ T(X) \subseteq g(X) \tag{12}$$

$$\delta(Sx,Ty) \leq M(x,y) - \phi(M(x,y)), \tag{13}$$

where

$$M(x,y) = F\{\delta(Ty,gx),\delta(Sx,fy),\delta(Ty,fy),\delta(Sx,gx)\}$$

and $\phi : [0,\infty) \to [0,\infty)$ *is continuous and monotonic decreasing function with* $\phi(x) = 0$ *if and only if* $x = 0$ *and* $F \in F^*$. *Further if* $\{S, g\}$ *and* $\{T, f\}$ *are weakly compatible pairs, then* S, T, f *and* g *have a unique common fixed point.*

Proof. By taking φ as an identity function in the proof of Theorem 2.1, we find the desired conclusion immediately.

Corollary 2.3. *Let* (X,d) *be a complete metric space. Let* $S : X \to B(X)$ *be a set-valued mapping and* $f : X \to X$ *be a single-valued mapping such that for all* $x, y \in X$

$$S(X) \subseteq f(X), \tag{14}$$

$$\varphi(\delta(Sx,Sy)) \leq \varphi(M(x,y)) - \phi(M(x,y)), \tag{15}$$

where

$$M(x,y) = F\{\delta(Sy, fx), \delta(Sx, fy), \delta(Sy, fy), \delta(Sx, fx)\}$$

and $\varphi : [0,\infty) \to [0,\infty)$ *is continuous and monotonic increasing function and* $\phi : [0,\infty) \to [0,\infty)$ *is continuous and monotonic decreasing function with* $\varphi(x) = 0 = \phi(x)$ *if and only if* $x = 0$ *and* $F \in F^*$. *Further if* S *and* f *are weakly compatible mappings, then* S *and* f *have a unique common fixed point.*

Proof. Let $S = T$ and $f = g$. From Theorem 2.1, we find the desired conclusion immediately.

Remark 2.4. If we take S as a single-valued mapping in Corollary 2.3, then we get Theorem 3.1 of [21].

Conflict of Interests

The authors declare that there is no conflict of interests.

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