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CONVERGENCE THEOREMS FOR TOTAL ASYMPTOTICALLY QUASI-NONEXPANSIVE NONSELF MAPPINGS IN UNIFORMLY CONVEX METRIC SPACES

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Abstract. In this paper, we study and prove some fixed point theorems for fixed points of total asymptotically quasi-nonexpansive nonself mappings in uniformly convex metric spaces.

Keywords: total asymptotically quasi-nonexpansive nonself; uniformly convex metric spaces; fixed points.

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1. Introduction

Takahashi [1] introduced the notion of convex metric spaces and studied the fixed point theory for nonexpansive mappings. A convex structure in a metric space (X,d) is a mapping $W : X \times X \times [0,1] \to X$ satisfying, for all $x,y,u \in X$ and all $\lambda \in [0,1]$

$$d(u,W(x,y,\lambda)) \leq (1-\lambda)d(u,x)+\lambda d(u,y). \tag{1.1}$$

A metric space with convex structure is called a convex metric space. Examples for convex metric are convex Banach space, $CAT(0)$ spaces and $CAT(1)$ spaces with small diameters (see

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[7]). A subset C of convex Metric space X is said to be convex if $W(x,y,\lambda) \in C$ for all $x,y \in C$ and $\lambda \in [0,1]$

A uinformly convex metric space have various authors(see [3, 5, 6, 7, 8]) study fixed point theory for nonexpansive mappings by using the Ishikawa iteration method (see e,g.,[7]).

Extend a convex structure as follows:

Definition 1.1. *Let (X,d) be a convex metric space with a convex structure and $I \in [0,1]$, $W : X^3 \times I^3 \to X, T : X \to X$ be an asymptotically quasi-nonexpansive mapping of X $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ be six sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n = 1$, $\alpha'_n+\beta'_n+\gamma'_n = 1$, $n = 0, 1, 2, ...,$and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\gamma'_n < \infty$, for any given $x_0 \in X$ define a sequence x_n as follows,*

$$\begin{aligned} x_{n+1} &= W(Ty_n, x_n, u_n, \alpha_n, \beta_n, \gamma_n), \\ y_n &= W(Tx_n, x_n, u_n, \alpha'_n, \beta'_n, \gamma'_n), \end{aligned} \tag{1.2}$$

where $\{u_n\}, \{v_n\}$ are two sequences in X satisfying the following conditions. For any nonnegative integers n, m, $0 \le n < M$ if $\delta(A_{n,m}) > 0$, then

$$\max_{n \le i,j \le m}\left\{d(x,y) : x \in \{u_i, v_i\}, y \in \{x_j, y_j, u_j, v_j\}\right\} < \delta(A_{n,m}) \tag{1.3}$$

where $A_{n,m} = \{x_i, y_i, Tx_i, Ty_i, u_i, v_i : n \le i \le m\}$,

$$\delta(A_{n,m}) = \sup_{x,y \in A_{n,m}} d(x,y), \tag{1.4}$$

then $\{x_n\}$ is called the Ishikawa type iterative sequence with errors of asymptotically quasi-nonexpansive mapping T.

Obviously, the Ishikawa iterative sequence is a special case of with $\gamma_n = 0$, $\gamma'_n = 0$ and $\{u_n\} = \{v_n\} = 0$.

Lemma 1.1. *Let E be a nonempty closed convex subset of a complete convex metric space X, $T : E \to E$, an asymptotically quasi-nonexpansive mapping of E with $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T)$, nonempty. Suppose that $\{x_n\}$ is defined by*

$$\begin{aligned} x_0 \in E \quad x_{n+1} &= W(T^n y_n, x_n, \alpha_n), \\ y_n &= W(T^n x_n, x_n, \alpha_n), \quad n = 0, 1, 2, ..., \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *satisfy that* $0 \leq \alpha_n, \beta_n \leq 1$ *then, (a)* $d(x_{n+1}, p) \leq d(1+k_n)^2 d(x_n, p)$,*for all* $p \in F(T)$ *and for all* $n \geq 1$, *(b) there exists a constant* $M > 0$, *such that* $d(x_{n+m}, p) \leq Md(x_n, p)$ *for all* $p \in F(T)$ *and for all* $n, m \geq 1$.

Theorem 1.1. *Let E be a nonempty closed convex subset a complete convex metric space X,T* $: E \to E$, *an asymptotically quasi-nonexpansive mapping of E (T need not be continuous), and* $F(T)$, *nonempty. Suppose that* $\{x_n\}_{n=1}^{\infty}$ *is an Ishikawa type lterative sequence with errors defined by* (1.4). *Then,* $\{x_n\}_{n=1}^{\infty}$ *converges to a fixed point of T if and only if* $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$, *where* $d(y,X)$ *denotes the distance of y to set X, i.e.,* $d(y,X) = \inf_{x\in X} d(y,x)$.

2. Preliminaries

Let (X,d) be a metric space and $x, y \in X$ with $d(x,y) = l$. A geodesic path from x to y is a isometry $c : [0,l] \to X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic part is called a geodesic segment. A metric space X is a (uniquely) geodesic space, if every two point of X are joined by only one geodesic segment. We will use $[x,y]$ to denote a geodesic segment joining x and y. A subset C of a geodesic space is said to be convex if $[x,y] \in C$ for any $x, y \in C$.

Definition 2.1 *A geodesic metric space* (X,d) *is called uniformly convex if for any* $r > 0$ *and any* $\varepsilon \in (0,2]$ *there exists* $\delta \in (0,1]$ *such that for all* $a, x, y \in X$ *with* $d(x,a) < r, d(y,a)$ $< r$ *and* $d(x,y) \leq \varepsilon r$. *It is the case that*

$$d(m,a) \leq (1-\delta), \tag{2.1}$$

where m stands for any midpoint of any geodesic segment $[x,y]$. *A mapping* $\delta : (0,\infty) \times (0,2] \to (0,1]$ *providing such* $\delta = \delta(r,\varepsilon)$ *for a given* $r > 0$ *and* $\varepsilon \in (0,2]$ *is called a modulus of uniform convexity.*

From Definition 2.1, it is clear that uniformly convex metric spaces are uniquely geodesic. The mapping δ is monotone (resp. lower semi-continuous from the right) if for every fixed ε it decreases (resp. is lower semi-continuous from the right) with respect to r (see [5]).

In this paper, we assume that all uniformly convex metric spaces have monotone or lower semi-continuous from the right modulus of uniform convexity.

Definition 2.2 *A mapping $T : X \to X$ is called:*

(a) Nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$.

(b) Quasi-nonexpansive if $d(Tx,p) \leq d(x,p)$ for all $x \in X$ and for all $p \in F(T)$.

(c) Asymptotically nonexpansive if there exists $k_n \in [0,1)$ for all $n \geq 1$ with $\lim_{n\to\infty} k_n = 0$ such that $d(T^n x, T^n y) \leq (1+k_n)\, d(x,y)$ for all $x,y \in X$.

(d) Asymptotically quasi-nonexpansive if there exists $k_n \in [0,1)$ for all $n \geq 1$ with $\lim_{n\to\infty} k_n = 0$ such that $d(T^n x, p) \leq (1+k_n)\, d(x,p)$ for all $x \in X$, for all $p \in F(T)$.

(e) Total Asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists nonnegative real sequence $\{k_n\}$ and $\{u_n\}$ with $\lim_{n\to\infty} k_n = \lim_{n\to\infty} u_n = 0$, and strictly increasing and continuous functions $\xi : [0,\infty) \to [0,\infty)$ with $\xi(0) = 0$ such that $d(T^n x, p) \leq d(x,p) + k_n d(x,p) + u_n$ all $x,y \in X, n \geq 1$ and for all $p \in F(T)$.

Remark 2.1 *From Definition 2.2, the following implications are obvious:*

(a) Nonexpansiveness implies Quasi-nonexpansiveness.

(b) Nonexpansiveness implies Asymptotically nonexpansiveness.

(c) Quasi-nonexpansiveness implies Asymptotically quasi-nonexpansiveness.

(d) Asymptotically nonexpansiveness implies Asymptotically quasi-nonexpansiveness.

(e) Asymptotically quasi-nonexpansiveness implies Total asymptotically quasi-nonexpansiveness. The converses of these implications may not be true.

Let (X,d) be a metric space, and let C be a nonempty subset of X. Recall that C is said to be a retract of X if there exists a continuous map $P : X \to C$ such that $Px = x$, for all $x \in C$. Amap $P : X \to C$ is said to be a retraction if $P^2 = P$. If P is a retraction, then $Py = y$ for all y in the range of P.

Definition 2.3. *Let C be a bounded closed convex subset of a complete uniformly convex metric space X and P be the nonexpansive retraction of X onto C. Let $T : C \to X$ be said to be uniformly total quasi-asymptotically nonexpansive nonself mapping if $F(T) \neq \emptyset$ and there exist nonnegative real sequence $\{k_n\}, \{u_n\}$ with $\lim_{n\to\infty} k_n = 0$, $\lim_{n\to\infty} u_n = 0$ and a strictly* *increasing continuous function* $\xi : [0,\infty) \to [0,\infty)$ $\xi(0) = 0$ *with such that all* $x \in C,\ p \in F(T)$

$$d\left(p, T(PT)^{n-1}x\right) \leq d(p,x) + k_n\xi(d(p,x)) + u_n \qquad \forall n \geq 1 \tag{2.2}$$

where P *is a nonexpansive retraction of* X *onto* C.

Each quasi-asymptotically nonexpansive nonself mapping must be a total quasi-asymptotically nonexpansive nonself mapping, but the converse is not true.

Lemma 2.1[6] *Let* $\{a_n\},\{b_n\}$ *and* $\{c_n\}$ *be three nonnegative sequences satisfying*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad n \geq 1. \tag{2.3}$$

If $\sum_{n=1}^{\infty} c_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$*,then*
(a) $\lim_{n\to\infty} a_n$ *exists,*
(b) If $\{a_n\}$ *has a subsequence which converges strongly to zero, then* $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we start by proving the following important result.

Theorem 3.1. *Let* C *be a bounded closed convex subset of a complete uniformly convex metric space* X *and* P *be the nonexpansive retraction of* X *onto* C. *Let* $T : C \to X$ *be uniformly total quasi-asymptotically nonexpansive nonself mapping. Let* $T_i : C \to X,\ i = 1,2,$ *be uniformly total quasi-asymptotically nonexpansive nonself mappings with sequences* $\left\{k_n^{(i)}\right\}$ *and* $\left\{u_n^{(i)}\right\}$ *satisfying* $\lim_{n\to\infty} k_n^{(i)} = 0$ *and* $\lim_{n\to\infty} u_n^{(i)} = 0$ *and strictly increasing function* $\xi^{(i)} : [0,\infty) \to [0,\infty)$ *with* $\xi^{(i)}(0) = 0,\ i = 1,2$. *For arbitrarily chosen* $x_1 \in C,$ *the sequence* $\{x_n\}$ *is defined as follows:*

$$\begin{aligned} x_{n+1} &= P\left(W\left(x_n, T_1(PT_1)^{n-1}y_n, \alpha_n\right)\right), \\ y_n &= P\left(W\left(x_n, T_2(PT_2)^{n-1}x_n, \beta_n\right)\right), \end{aligned} \tag{3.1}$$

with $\left\{k_n^{(1)}\right\}, \left\{k_n^{(2)}\right\}, \left\{u_n^{(1)}\right\}, \left\{u_n^{(2)}\right\}, \xi^{(1)}, \xi^{(2)}, \{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following condition-s:*
(a) $\sum\limits_{n=1}^{\infty} k_n^{(i)} < \infty,\ \ \sum\limits_{n=1}^{\infty} u_n^{(i)} < \infty, i = 1,2,$

(b) there exist constants $a,b\in(0,1)$ *with* $0<b(1-a)\leq\frac{1}{2}$ *such that* $\{\alpha_n\}\subset[a,b]$ *and* $\{\beta_n\}\subset[a,b]$

(c) there exists a constant $M^*>0$ *such that* $\xi^{(i)}(r)\leq M^*r, r\geq 0$, $i=1,2$ *and* $F:=F(T_1)\cap F(T_2)=\{x\in C:T_1x=T_2x=x\}\neq\emptyset$. *Then, the sequence* $\{x_n\}$ *is bounded and* $\lim_{n\to\infty}d(x_n,q)$ *and* $\lim_{n\to\infty}d(x_n,F)$ *exists,* $q\in F$.

Proof. Let $q\in F$. Set $k_n=\max\left\{k_n^{(1)},k_n^{(2)}\right\}$ and $u_n=\max\left\{u_n^{(1)},u_n^{(2)}\right\}$, $n=1,2,...,\infty$ and condition (a), we have

$$\begin{aligned} d(x_{n+1},q) &= d\left(P\left(W\left(x_n,T_1(PT_1)^{n-1}y_n,\alpha_n\right)\right),q\right)\\ &\leq d\left(W\left(x_n,T_1(PT_1)^{n-1}y_n,\alpha_n\right),q\right)\\ &\leq (1-\alpha_n)d(x_n,q)+\alpha_n d\left(T_1(PT_1)^{n-1}y_n,q\right)\\ &\leq (1-\alpha_n)d(x_n,q)+\alpha_n\left[d(y_n,q)+k_n\xi^{(1)}(d(y_n,q))+u_n\right]. \end{aligned} \tag{3.2}$$

Since $\xi^{(1)}$ is an strictly increasing function, we see that there exists a constant $M^*>0$ such that $\xi^{(1)}(r)\leq M^*r, r\geq 0$. It follows that

$$\begin{aligned} d(x_{n+1},q) &\leq (1-\alpha_n)d(x_n,q)+\alpha_n\left[d(y_n,q)+k_n\xi^{(1)}(d(y_n,q))+u_n\right]\\ &\leq (1-\alpha_n)d(x_n,q)+\alpha_n[d(y_n,q)+k_nM^*d(y_n,q)+u_n]\\ &\leq (1-\alpha_n)d(x_n,q)+\alpha_n[(1+k_nM^*)d(y_n,q)+u_n] \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d(y_n,q) &= d\left(P\left(W\left(x_n,T_2(PT_2)^{n-1}x_n,\beta_n\right)\right),q\right)\\ &\leq d\left(W\left(x_n,T_2(PT_2)^{n-1}x_n,\beta_n\right),q\right)\\ &\leq (1-\beta_n)d(x_n,q)+\beta_n d\left(T_2(PT_2)^{n-1}x_n,q\right)\\ &\leq (1-\beta_n)d(x_n,q)+\beta_n[d(x_n,q)+k_n\xi^{(2)}(d(x_n,q))+u_n]. \end{aligned} \tag{3.4}$$

Since $\xi^{(2)}$ is an strictly increasing function, we find that there exists a constant $M^* > 0$ such that $\xi^{(2)}(r) \leq M^* r, r \geq 0$. Hence, we have

$$\begin{aligned} d(y_n,q) &\leq (1-\beta_n)\, d(x_n,q) + \beta_n [d(x_n,q) + k_n \xi^{(2)}(d(x_n,q)) + u_n] \\ &\leq (1-\beta_n)\, d(x_n,q) + \beta_n [d(x_n,q) + k_n M^* d(x_n,q) + u_n] \\ &\leq d(x_n,q) - \beta_n d(x_n,q) + \beta_n d(x_n,q) + \beta_n k_n M^* d(x_n,q) + \beta_n u_n \\ &\leq d(x_n,q) + \beta_n k_n M^* d(x_n,q) + \beta_n u_n \\ &\leq (1+\beta_n k_n M^*)\, d(x_n,q) + \beta_n u_n. \end{aligned} \tag{3.5}$$

Substituting (3.3) and (3.5) gives that

$$\begin{aligned} d(x_{n+1},q) &\leq (1-\alpha_n)\, d(x_n,q) + \alpha_n [(1+k_n M^*)\, ((1+\beta_n k_n M^*)\, d(x_n,q) + \beta_n u_n) + u_n] \\ &= [1-\alpha_n + \alpha_n (1+k_n M^*)(1+\beta_n k_n M^*)] d(x_n,q) + \alpha_n (1+k_n M^*)\, \beta_n u_n + \alpha_n u_n \\ &= [1-\alpha_n + \alpha_n + \alpha_n k_n M^* + \alpha_n \beta_n k_n M^* + (\alpha_n k_n M^*)(\beta_n k_n M^*)] d(x_n,q) \\ &+ \alpha_n (1+k_n M^*)\, \beta_n u_n + \alpha_n u_n \\ &= [1 + (1+\beta_n + \beta_n k_n M^*)\, \alpha_n k_n M^*] d(x_n,q) + [(1+k_n M^*)\, \beta_n + 1] \alpha_n u_n \\ &= [1 + (1+\beta_n + \beta_n k_n M^*)\, \alpha_n k_n M^*] d(x_n,q) + (1+\beta_n + \beta_n k_n M^*)\, \alpha_n u_n \end{aligned} \tag{3.6}$$

and

$$d(x_{n+1},F) \leq [1 + (1+\beta_n + \beta_n k_n M^*)\, \alpha_n k_n M^*] d(x_n,q) + (1+\beta_n + \beta_n k_n M^*)\, \alpha_n u_n. \tag{3.7}$$

Since $\sum_{n=1}^{\infty} k_n < \infty$, $\sum_{n=1}^{\infty} u_n < \infty$, we find from Lemma 2.1 that the sequence $\{x_n\}$ is bounded, $\lim_{n\to\infty} d(x_n,q)$ and $\lim_{n\to\infty} d(x_n,F)$ exists, q $\in F$. This completes the proof.

Theorem 3.2. *Let C be a bounded closed convex subset of a complete uniformly convex metric space X and P be the nonexpansive retraction of X onto C. Let $T : C \to X$ be uniformly total quasi-asymptotically nonexpansive nonself mapping. Let $T_i : C \to X$, $i = 1,2$, uniformly total quasi-asymptotically nonexpansive nonself mappings with sequences $\left\{k_n^{(i)}\right\}$ and $\left\{u_n^{(i)}\right\}$ satisfying $\lim_{n\to\infty} k_n^{(i)} = 0$ and $\lim_{n\to\infty} u_n^{(i)} = 0$ and strictly increasing function $\xi^{(i)} : [0,\infty) \to [0,\infty)$ with*

$\xi^{(i)}(0)=0,\ i=1,2$. *For arbitrarily chosen* $x_1\in C$, *the sequence* $\{x_n\}$ *is defined as follows:*

$$\begin{aligned} x_{n+1} &= P\left(W\left(x_n, T_1(PT_1)^{n-1}y_n, \alpha_n\right)\right), \\ y_n &= P\left(W\left(x_n, T_2(PT_2)^{n-1}x_n, \beta_n\right)\right), \end{aligned} \tag{3.8}$$

with $\left\{k_n^{(1)}\right\}, \left\{k_n^{(2)}\right\}, \left\{u_n^{(1)}\right\}, \left\{u_n^{(2)}\right\}, \xi^{(1)}, \xi^{(2)}, \{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions:*

(a) $\sum_{n=1}^{\infty} k_n^{(i)} < \infty,\ \sum_{n=1}^{\infty} u_n^{(i)} < \infty, i=1,2,$

(b) there exist constants $a,b\in(0,1)$ *with* $0<b(1-a)\leq\frac{1}{2}$ *such that* $\{\alpha_n\}\subset[a,b]$ *and* $\{\beta_n\}\subset[a,b]$

(c) there exists a constant $M^*>0$ *such that* $\xi^{(i)}(r)\leq M^* r, r\geq 0,\ i=1,2$ *and* $F:=F(T_1)\cap F(T_2)=\{x\in C: T_1x=T_2x=x\}\neq\emptyset$. *Then, the sequence* $\{x_n\}$ *converges strongly to a common fixed point of* $T_i, i=1,2,$ *if and only if* $\liminf_{n\to\infty} d(x_n,F)=0,$ *where* $d(x_n,F)=\inf_{q\in F} d(x_n,q), n\geq 1.$

Proof. It follows from Theorem 3.1 that $\lim_{n\to\infty} d(x_n,q)$ *exists*. Without loss of generality, we may assume that $\lim_{n\to\infty} d(x_n,q)=v>0$. Form (3.7), we have

$$\begin{aligned} d(x_{n+1},q) &\leq [1+(1+\beta_n+\beta_n k_n M^*)\alpha_n k_n M^*]d(x_n,q)+(1+\beta_n+\beta_n k_n M^*)\alpha_n u_n \\ &= d(x_n,q)+l_n, \end{aligned} \tag{3.9}$$

where $l_n=(1+\beta_n+\beta_n k_n M^*)\alpha_n k_n M^* d(x_n,q)+(1+\beta_n+\beta_n k_n M^*)\alpha_n u_n$. Since $\{d(x_n,q)\}$ is bounded and $\sum_{n=1}^{\infty} k_n<\infty,\ \sum_{n=1}^{\infty} u_n<\infty$, we have $\sum_{n=1}^{\infty} l_n<\infty$. Hance, (3.9) implies that

$$\inf_{q\in F} d(x_{n+1},q)\leq \inf_{q\in F} d(x_n,q)+l_n, \tag{3.10}$$

that is

$$d(x_{n+1},F)\leq d(x_n,F)+l_n. \tag{3.11}$$

It follows from lemma 2.1 and (3.10) that we have $\lim_{n\to\infty} d(x_n,F)=0$. Next, we show that $\{x_n\}$ is a Cauchy sequence. From (3.9), we see that that for any $n\geq n_0$, any $n\geq n_1$ and any

$q_1 \in F$

$$\begin{aligned} d(x_{n+m}, q_1) &\leq d(x_{n+m-1}, q_1) + l_{n+m-1} \\ &\leq d(x_{n+m-2}, q_1) + l_{n+m-1} + l_{n+m-2} \\ &\leq d(x_{n+m-3}, q_1) + l_{n+m-1} + l_{n+m-2} + l_{n+m-3} \\ &\vdots \\ &\leq d(x_n, q_1) + \sum_{j=n}^{n+m-1} l_j. \end{aligned} \tag{3.12}$$

By (3.12), we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, q_1) + d(x_n, q_1) \\ &\leq 2d(x_n, q_1) + \sum_{j=n}^{n+m-1} l_j. \end{aligned} \tag{3.13}$$

By the arbitrariness of $q_1 \in F$ and from (3.13), we have that

$$d(x_{n+m}, x_n) \leq 2d(x_n, F) + \sum_{j=n}^{n+m-1} l_j, \quad \forall n \geq n_0. \tag{3.14}$$

For all $\varepsilon > 0$, there exists a positive number $n_1 \geq n_0$, such that for $n \geq n_1$, $(x_n, F) < \frac{\varepsilon}{4}$ and $\sum_{j=n}^{n+m-1} l_j < \frac{\varepsilon}{2}$, It follows from (3.13) that

$$\begin{gathered} d(x_{n+m}, x_n) \leq 2d(x_n, F) + \sum_{j=n}^{n+m-1} l_j < \frac{2\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon \\ \text{and } \lim_{n\to\infty} d(x_{n+m}, x_n) = 0, m \geq 1. \end{gathered} \tag{3.15}$$

Hence, $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset of X and it is complete, we have that there exists a $q \in C$ such that $x_n \to q$ as $n \to \infty$. Next, we show that $q \in F$. Assume that $q \notin F$. Note that F is closed set, $d(q, F) > 0$. Thus for all $q \in F$, we have

$$d(q, q_1) \leq d(q, x_n) + d(x_n, q_1). \tag{3.16}$$

This implies that

$$d(q, F) \leq d(q, x_n) + d(x_n, F). \tag{3.17}$$

From (3.16) and (3.17), we have that $d(p, F) \leq 0$. This is a contradiction. Hence $q \in F$. This completes the proof.

Corollary 3.3. *Let C be a bounded closed convex subset of a complete uniformly convex metric space X and P be the nonexpansive retraction of X onto C. Let $T: C \to X$ be a uniformly total quasi-asymptotically nonexpansive nonself mapping. Let $T_i: C \to X,\ i = 1,2,$ uniformly total quasi-asymptotically nonexpansive nonself mappings with sequences $\left\{k_n^{(i)}\right\}$ and $\left\{u_n^{(i)}\right\}$ satisfying $\lim_{n\to\infty} k_n^{(i)} = 0$ and $\lim_{n\to\infty} u_n^{(i)} = 0$ and strictly increasing function $\xi^{(i)}: [0,\infty) \to [0,\infty)$ with $\xi^{(i)}(0) = 0,\ i = 1,2.$ For arbitrarily chosen $x_1 \in C,$ the sequence $\{x_n\}$ is defined as follows:*

$$\begin{aligned} x_{n+1} &= P\left(W\left(x_n, T_1(PT_1)^{n-1}y_n, \frac{1}{2}\right)\right) \\ y_n &= P\left(W\left(x_n, T_2(PT_2)^{n-1}x_n, \frac{1}{2}\right)\right), \end{aligned} \tag{3.18}$$

with $\left\{k_n^{(1)}\right\}, \left\{k_n^{(2)}\right\}, \left\{u_n^{(1)}\right\}, \left\{u_n^{(2)}\right\}, \xi^{(1)}, \xi^{(2)}$ satisfy the following conditions:
(a) $\sum_{n=1}^{\infty} k_n^{(i)} < \infty,\ \sum_{n=1}^{\infty} u_n^{(i)} < \infty, i = 1,2,$ (b) there exists a constant $M^* > 0$ such that $\xi^{(i)}(r) \le M^* r, r \ge 0,\ i = 1,2$ and $F := F(T_1) \cap F(T_2) = \{x \in C : T_1 x = T_2 x = x\} \neq \emptyset$. Then, $\{x_n\}$ converges to converges strongly to a common fixed point of $T_i, i = 1,2$.

Conflict of Interests

The author declares that there is no conflict of interests.

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