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NEW TYPE OF FIXED POINT THEOREMS IN GENERALIZED METRIC SPACES

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Abstract. In this paper, we prove some new type of fixed point theorems in generalized complete metric spaces. The results presented in this paper mainly improve the corresponding results announced by Wardowski [D. Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012 (2012), Article ID 94] from metric spaces to generalized metric spaces.

Keywords: F-contraction; fixed point; complete metric space.

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1. Introduction and Preliminaries

In 1992, Dhage [2], introduced the notion of generalized metric or D-metric spaces and claimed that D-metric convergence define a Hausdorff topology and that D-metric is sequentially continuous in all the three variables. Many authors have taken these claims for granted and used them in proving fixed point theorems in D-metric spaces. Also in 1996, Rhoades [1], generalized Dhages contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self map in D-metric space. D. Wardowski in [3] has

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introduced a new type of contraction and proved a new fixed point theorem. On the other hand, Suzuki generalized the notion of contractive mappings in 2008 (see, [7],[8],[9]). After this time, some authors published many results by using the Suzuki's method for mappings and multifunctions (see for example, [10] and [11] and the references therein). In this paper we prove the result obtained by Wardowski in generalized metric spaces. Also by combining Samet's and Wardowski's methods, (see [5], [3]) and by using the result obtained by Karapinar in [4] we get a new result in generalized metric spaces. Again by combining results of Suzuki and Wardowski we obtain a new result in the generalized metric spaces.

Let X be a nonempty set. A generalized D^*-metric on X is a function, $D^*: X^3 \to \mathbb{R}^+$ that satisfies the following conditions for all $x,y,z,a \in X$,

(D1) $D^*(x,y,z) \geq 0$,

(D2) $D^*(x,y,z) = 0$ if and only if $x = y = z$,

(D3) $D^*(x,y,z) = D^*(p\{x,y,z\})$,(symmetry) where p is a permutation function,

(D4) $D^*(x,y,z) \leq D^*(x,a,a) + D^*(a,y,z)$,

the function D^* is called a generalized D^*-metric and the pair (X,D^*) is called a generalized D^*-metric space.

Note that every D^*-metric on X induces a metric d_{D^*} on X defined by

$$d_{D^*}(x,y) = D^*(x,y,y) + D^*(y,x,x), \ \forall x,y \in X. \tag{1}$$

Remark 1.1. *In a D^*-metric space, we prove that $D^*(x,x,y) = D^*(x,y,y)$*

(i) $D^*(x,x,y) \leq D^*(x,x,x) + D^*(x,y,y) = D^*(x,y,y)$,

(ii) $D^*(y,y,x) \leq D^*(y,y,y) + D^*(y,x,x) = D^*(y,x,x)$,

Hence by (i),(ii) we get $D^*(x,x,y) = D^*(x,y,y)$.

Definition 1.2. [6] *Let (X,D^*) be a D^*-metric space, and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is D^*-convergent to $x \in X$ if*

$$\lim_{n,m\to+\infty} D^*(x,x_n,x_m) = 0.$$

that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $D^*(x,x_n,x_m) < \varepsilon$, for all $n,m \geq N$.

Proposition 1.3. [6] *Let (X,D^*) be a D^*-metric space. The following are equivalent*

(i) $\{x_n\}$ *is* D^**-convergent to* x*,*

(ii) $D^*(x_n, x_n, x) \to 0$ *as* $n \to \infty$*,*

(iii) $D^*(x_n, x, x) \to 0$ *as* $n \to \infty$*,*

(vi) $D^*(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$*.*

Definition 1.4. [6] *Let* (X, D^*) *be a* D^**-metric space. A sequence* $\{x_n\}$ *is called a* D^**-Cauchy sequence if for any* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $D^*(x_n, x_m, x_l) < \varepsilon$ *for all* $m, n, l \geq N$*, that is,* $D^*(x_n, x_m, x_l) \to 0$ *as* $n, m, l \to \infty$*.*

Proposition 1.5. [6] *Let* (X, D^*) *be a* D^**-metric space. Then the following are equivalent*

(1) *the sequence* $\{x_n\}$ *is* D^**-Cauchy,*

(2) *for any* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $D^*(x_n, x_m, x_m) < \varepsilon$ *, for all* $m, n \geq N$*.*

Definition 1.6. [6] *A* D^**-metric space* (X, D^*) *is called* D^**-complete if every* D^**-Cauchy sequence is* D^**-convergent in* (X, D^*)*.*

Note that in D^*-metric space a nonempty set $A \subset X$ is D^*-closed in the D^*-metric space (X, D^*) if $A = \overline{A}$.

Proposition 1.7. *Let* (X, D^*) *be a* D^**-metric space and* A *be a nonempty subset of* X*.* A *is* D^**-closed if for any* D^**-convergent sequence* $\{x_n\}$ *in* A *with limit* x*, one has* $x \in A$*.*

Definition 1.8. [2] *A* D^**-metric space* X *is said to be compact if every* τ*-open cover of* X *has a finite subcover.*

Theorem 1.9. [2] *In a* D^**-metric space* X*, the following statement are equivalent.*

(a) *X is compact,*

(b) *X is countably compact,*

(c) *X has Bolzano-Weierstrass property,*

(d) *X is sequentially compact.*

Theorem 1.10. [2] *In a* D^**-metric space* X*,*

(a) *a compact subset of a* D^**-metric space is closed and bounded,*

(b) a D^*-metric space X is a compact if and only if it is complete and totally bounded,

(c) a subset S of a complete D^*-metric space is compact if and only if it is closed and totally bounded.

Theorem 1.11. [2] *Every real-valued continuous function on a compact D^*-metric space X is bounded and attains its supremum and infimum on X.*

Wardowski has defined F-contraction as the following (see [3]).

Definition 1.12. *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a mapping satisfying,*

(F_1) F is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta, F(\alpha) < F(\beta)$;

(F_2) for each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive real numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F_3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$\forall\, x,y \in X \; d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \le F(d(x,y))).$$

Example 1.13. *Let $F(\alpha) = -\frac{1}{\sqrt{\alpha}}, \alpha > 0$. It is clear that F satisfies (F_1)-(F_3). In this case, each F-contraction T satisfies*

$$d(Tx,Ty) \le \frac{1}{(1+\tau\sqrt{d(x,y)})^2} d(x,y), \; for\ all\ x,y \in X,\ Tx \neq Ty.$$

Example 1.14. *If $F(\alpha) = -\frac{1}{\alpha^2}$, $\alpha > 0$ then F satisfies (F_1)-(F_3). In this case, each F-contraction T satisfies*

$$\frac{d(Tx,Ty)}{d(x,y)} \le \sqrt{\frac{d(x,y)^2}{1+\tau d(x,y)^2}}.$$

Example 1.15. *Consider $F(\alpha) = \tan(\alpha + \frac{\pi}{2})$. F satisfies conditions (F_1)-(F_3).*

Wardowski has stated modified Banach contraction theorem as the following.

Theorem 1.16. [3] *Let (X,d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.*

Now state and prove the main results.

2. Main results

We define modified F-contraction as the following.

Definition 2.17. *A mapping $T : X \to X$ is said to be a GF-contraction if there exists $\tau > 0$ such that for all $x, y \in X$,*

$$(D^*(Tx,Ty,Tz) > 0 \Rightarrow \tau + F(D^*(Tx,Ty,Tz)) \leq F((D^*(x,y,z)))), \tag{2}$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ satisfies the following conditions:

(GF_1) F is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta, F(\alpha) < F(\beta)$;

(GF_2) for each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive real numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(GF_3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Theorem 2.18. *Let (X, D^*) be a D^*complete D^*-metric space and $T : X \to X$ be a GF-contraction mapping. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.*

Proof. Take an arbitrary $x_0 \in X$ and define the sequence $\{x_n\}$ as

$$x_n = Tx_{n-1},\ n = 1,2,3,\ldots.$$

If $x_{n_0+1} = x_{n_0}$ for some $n \in \mathbb{N}$, then obviously, the fixed point of T is x_{n_0}. Assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Put $x = x_{n-1}$ and $y = z = x_n$ in (2). Then

$$\begin{aligned} F(D^*(Tx_{n-1},Tx_n,Tx_n)) &\leq F((D^*(x_{n-1},x_n,x_n))) - \tau \\ &\leq F(D^*(x_{n-2},x_{n-1},x_{n-1})) - \tau - \tau \\ &= F(D^*(x_{n-2},x_{n-1},x_{n-1})) - 2\tau \end{aligned}$$

$$\leq F(D^*(x_{n-2},x_{n-1},x_{n-1}))-3\tau$$

$$\vdots$$

$$\leq F(D^*(x_0,x_1,x_1))-n\tau,$$

tending $n\to\infty$, we have

$$\lim_{n\to\infty} F(D^*(x_n,x_{n+1},x_{n+1}))=-\infty.$$

Thus from (GF_2), we obtain

$$\lim_{n\to\infty} D^*(x_n,x_{n+1},x_{n+1})=0.$$

On the other hand, by symmetry (D3) and the rectangle (D4), we have

$$D^*(x,y,y)=D^*(y,y,x)\leq D^*(y,x,x)+D^*(x,y,x)=2D^*(y,x,x). \tag{3}$$

The inequality (3) with $x=x_n$ and $y=x_{n-1}$ becomes,

$$D^*(x_n,x_{n-1},x_{n-1})\leq 2D^*(x_{n-1},x_n,x_n).$$

Hence, we get

$$\lim_{n\to\infty} D^*(x_n,x_{n-1},x_{n-1})=0.$$

On the other hand if put $\gamma_n=D^*(x_{n-1},x_n,x_n)$, then by using (2), we obtain,

$$(\gamma_n)^kF(\gamma_n) \quad \leq (\gamma_n)^kF(\gamma_0)-(\gamma_n)^kn\tau. \tag{4}$$

Thus

$$(\gamma_n)^kF(\gamma_n)-(\gamma_n)^kF(\gamma_0)\leq(\gamma_n)^k(F(\gamma_0)-n\gamma)-(\gamma_n)^kF(\gamma_0)=-(\gamma_n)^kn\tau\leq 0.$$

By attention to, $\lim_{n\to\infty}\gamma_n^k F(\gamma_n)=0$ and by $\lim_{n\to\infty}\gamma_n=0$ and Letting $n\to\infty$ in (4), we get

$$\lim_{n\to\infty}(\gamma_n)^k n=0. \tag{5}$$

So there exists $n_1\in\mathbb{N}$ such that $(\gamma_n)^k n\le 1$ for all $n\ge n_1$. Consequently we have

$$\gamma_n\le\frac{1}{n^{\frac{1}{k}}},\ \forall n\ge n_1.$$

Now, we show next that the sequence $\{x_n\}$ is a cauchy sequence in the metric space (X,d_{D^*}) where d_{D^*} is given in (1). Let $n,l\in\mathbb{N}$ with $n>l>n_1$ we obtain

$$\begin{aligned}
d_{D^*}(x_n,x_l) &\le d_{D^*}(x_n,x_{n-1})+d_{D^*}(x_{n-1},x_{n-2})+\ldots+d_{D^*}(x_{l+1},x_l)\\
&= D^*(x_n,x_{n-1},x_{n-1})+D^*(x_{n-1},x_n,x_n)\\
&+D^*(x_{n-1},x_{n-2},x_{n-2})+D^*(x_{n-2},x_{n-1},x_{n-1})+\ldots\\
&+D^*(x_{l+1},x_l,x_l)+D^*(x_l,x_{l+1},x_{l+1})\\
&=\sum_{i=l+1}^{n}[D^*(x_i,x_{i-1},x_{i-1})+D^*(x_{i-1},x_i,x_i)]. \qquad (6)
\end{aligned}$$

By using of (3), we get

$$\begin{aligned}
0\le d_{D^*}(x_n,x_l) &\le \sum_{i=l+1}^{n}[2D^*(x_{i-1},x_i,x_i)+D^*(x_{i-1},x_i,x_i)]\\
&=\sum_{i=l+1}^{n}3D^*(x_{i-1},x_i,x_i). \qquad (7)
\end{aligned}$$

Hence for $n>l>n_1$ we have,

$$0\le d_{D^*}(x_n,x_l)\le\sum_{i=l+1}^{n}3\gamma_n\le 3\sum_{i=l+1}^{n}\frac{1}{i^{\frac{1}{k}}}.$$

From the above and from the convergence of the series $\sum_{i=l+1}^{n}\frac{1}{i^{\frac{1}{k}}}$ we receive that $\{x_n\}$ is a cauchy sequence in (X,d_{D^*}). Since (X,d) is D^*-complete then (X,d_{D^*}) is complete (see proposition 10 in [6]) and hence $\{x_n\}$ converges to a number say, $u\in X$. Suppose that $u\ne Tu$ or

$d_{D^*}(u,Tu)>0$, then we have,

$$\begin{aligned} 0 \le d_{D^*}(x_n,Tu) &= D^*(x_n,Tu,Tu)+D^*(Tu,x_n,x_n)\\ &= D^*(Tx_{n-1},Tu,Tu)+D^*(Tu,Tx_{n-1},Tx_{n-1})\\ &\le D^*(Tx_{n-1},Tu,Tu)+2D^*(Tx_{n-1},Tu,Tu)\\ &\le 3D^*(Tx_{n-1},Tu,Tu)=3D^*(x_n,u,u). \end{aligned}$$

Passing to Limit as $n\to\infty$, we end up with $0\le d_{D^*}(u,Tu)\le 0$ which contradicts the assumption $d_{D^*}(u,Tu)>0$, Hence $u=Tu$. therefore $u\in X$ is a fixed point of T. To prove the uniqueness, we assume that $v\in X$ is another fixed point of T such that $u\neq v$. we can substitute $x=u$ and $y=z=v$ in (2). This yields

$\tau+F(D^*(u,v,v))\le F((D^*(u,v,v)))$, which is contradiction.

Definition 2.19. *Let $T:X\to X$ and $\alpha:X\times X\times X\to[0,+\infty)$. We say that T is α-admissible mapping if*

$x,y\in X,\ \alpha(x,y,z)\ge 1 \implies \alpha(Tx,Ty,Tz)\ge 1.$

Denote with Ψ the family of nondecreasing functions $\psi:[0,+\infty)\to[0,+\infty)$ such that $\psi(t)<t$.

Lemma 2.20. *For every function $\psi:[0,+\infty)\to[0,+\infty)$ the following holds:*
if ψ is nondecreasing then for each $t>0$, $\lim_{n\to+\infty}\psi^n(t)=0$ implies $\psi(t)<t$.

Theorem 2.21. *Let (X,D^*) be a D^*-complete D^*-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty D^*-closed subsets of X. Let $Y=\cup_{j=1}^m A_j$ and $T:Y\to Y$ be a $\alpha-$admissible mapping satisfying*

$T(A_j)\subseteq A_{j+1},\ j=1,\dots,m$, where $A_{m+1}=A_1$.

If there exist two functions $\alpha:Y\times Y\times Y\to[0,+\infty)$ and $\psi\in\Psi$ such that

$$\forall\, x,y\in X\ (D^*(Tx,Ty,Tz)>0\Rightarrow \tau+\alpha(x,y,Tz)F(D^*(Tx,Ty,Tz)) \le F(\psi(D^*(x,y,z)))). \tag{8}$$

holds for all $x\in A_j$ and $y,z\in A_{j+1},\ j=1,\dots,m$, and there exists $x_0\in Y$ such that $\alpha(x_0,Tx_0,T^2x_0)\ge 1$, then T has a unique fixed point in $\cap_{j=1}^m A_j$.

Proof Let $x_0 \in Y$ such that $\alpha(x_0, Tx_0, T^2x_0) \geq 1$ and without loss of generality assume that $x_0 \in A_1$. Define the sequence $\{x_n\}$ in Y as follows

$$x_n = Tx_{n-1} \; for \; all \; n \in \mathbb{N}.$$

Since T is cyclic, $x_0 \in A_1$, $x_1 = T(x_0) \in A_2, \ldots$ and so on. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then obviously, the fixed point of T is x_{n_0}. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is $\alpha-$admissible, we have

$$\alpha(x_0,x_1,x_2) = \alpha(x_0,Tx_0,T^2x_0) \geq 1 \Rightarrow \alpha(Tx_0,Tx_1,Tx_2) = \alpha(x_1,x_2,x_3) \geq 1.$$

By induction, We get

$$\alpha(x_{n-1},x_n,x_{n+1}) \geq 1, \; for \; all \; n \in \mathbb{N}. \tag{9}$$

Applying the inequality (8) with $x = x_{n-1}$ and $y = z = x_n$, and using (9), we obtain

$$\begin{aligned} 0 \leq F(D^*(x_n,x_{n+1},x_{n+1})) &= F(D^*(Tx_{n-1},Tx_n,Tx_n)) \\ &\leq \alpha(x_{n-1},x_n,Tx_n)F(D^*(Tx_{n-1},Tx_n,Tx_n)) \\ &\leq F(\psi(D^*(x_{n-1},x_n,x_n))) - \tau \\ &< F((D^*(x_{n-1},x_n,x_n))) - \tau. \end{aligned}$$

Therefore, by repetition of the above inequality, we have that

$$F(D^*(x_n,x_{n+1},x_{n+1})) \leq F(D^*(x_0,x_1,x_1)) - n\tau, \; for \; all \; n \in \mathbb{N}. \tag{10}$$

tending $n \to \infty$ we have

$$\lim_{n\to\infty} F(D^*(x_n,x_{n+1},x_{n+1})) = -\infty.$$

Thus,

$$\lim_{n\to\infty} D^*(x_n,x_{n+1},x_{n+1}) = 0.$$

By similar proof in Theorem (2.18) we get

$$\lim_{n\to\infty} D^*(x_n,x_{n-1},x_{n-1}) = 0.$$

On the other hand if put $\gamma_n = D^*(x_{n-1},x_n,x_n)$, then by using (2) we obtain,

$$(\gamma_n)^k F(\gamma_n) \quad \leq (\gamma_n)^k F(\gamma_0) - (\gamma_n)^k n\tau. \tag{11}$$

Thus

$$(\gamma_n)^k F(\gamma_n) - (\gamma_n)^k F(\gamma_0) \leq (\gamma_n)^k (F(\gamma_0) - n\gamma) - (\gamma_n)^k F(\gamma_0) = -(\gamma_n)^k n\tau \leq 0.$$

By attention to, $\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0$ and by $\lim_{n\to\infty} \gamma_n = 0$ and Letting $n\to\infty$ in (11), we get

$$\lim_{n\to\infty} (\gamma_n)^k n = 0. \tag{12}$$

Now, from let us observe that from (12) there exists $n_1 \in \mathbb{N}$ such that $(\gamma_n)^k n \leq 1$ for all $n \geq n_1$. Consequently we have

$$\gamma_n \leq \frac{1}{n^{\frac{1}{k}}},\ \forall n \geq n_1.$$

We show next that the sequence $\{x_n\}$ is a cauchy sequence in the metric space (X, d_{D^*}) where d_{D^*} is given in (1). Let $n, l \in \mathbb{N}$ with $n > l > n_1$ we obtain

$$\begin{aligned} d_{D^*}(x_n,x_l) &\leq d_{D^*}(x_n,x_{n-1}) + d_{D^*}(x_{n-1},x_{n-2}) + \ldots + d_{D^*}(x_{l+1},x_l) \\ &= D^*(x_n,x_{n-1},x_{n-1}) + D^*(x_{n-1},x_n,x_n) \\ &\quad + D^*(x_{n-1},x_{n-2},x_{n-2}) + D^*(x_{n-2},x_{n-1},x_{n-1}) + \ldots \\ &\quad + D^*(x_{l+1},x_l,x_l) + D^*(x_l,x_{l+1},x_{l+1}) \\ &= \sum_{i=l+1}^{n} [D^*(x_i,x_{i-1},x_{i-1}) + D^*(x_{i-1},x_i,x_i)]. \end{aligned} \tag{13}$$

Again by using a similar proof in Theorem (2.18) we obtain that $\{x_n\}$ is a Cauchy sequence in the (X, d_{D^*}). Since the space (X, D^*) is D^*-complete, hence, $\{x_n\}$ converges to a number say, $u \in X$. Moreover, $\{x_n\}$ is D^*-Cauchy in (X, D^*) (see Proposition 9 in [6]). Now we show that $u \in \cap_{j=1}^m A_j$. if $x_0 \in A_1$, then the subsequence $\{x_{m(n-1)}\}_{n=1}^\infty \in A_1$, the subsequence $\{x_{m(n-1)+1}\}_{n=1}^\infty \in A_2$, and continuing in this way, the subsequence $\{x_{mn-1}\}_{n=1}^\infty \in A_m$. All the m

subsequences are D^*-convergent and hence, they all converge to the same limit u. In addition, the sets A_j are D^*-closed, thus the limit $u \in \cap_{j=1}^{m} A_j$. We show that $u \in X$ is a fixed point of T, Consider now (1) and (8) with $x = x_n$, $y = z = Tu$ and suppose that $u \neq Tu$ or $d_{D^*}(u, Tu) > 0$, then we have,

$$
\begin{aligned}
0 \leq d_{D^*}(x_n, Tu) &= D^*(x_n, Tu, Tu) + D^*(Tu, x_n, x_n) \\
&= D^*(Tx_{n-1}, Tu, Tu) + D^*(Tu, Tx_{n-1}, Tx_{n-1}) \\
&\leq D^*(Tx_{n-1}, Tu, Tu) + 2D^*(Tx_{n-1}, Tu, Tu) \\
&= 3D^*(Tx_{n-1}, Tu, Tu) \\
&= 3(D^*(x_n, u, u)). \qquad (14)
\end{aligned}
$$

Passing to Limit as $n \to \infty$, we end up with $0 \leq d_{D^*}(u, Tu) \leq 0$ which contradicts the assumption $d_{D^*}(u, Tu) > 0$, Hence $u = Tu$. therefore $u \in X$ is a fixed point of T. To prove the uniqueness, We assume that $v \in X$ is another fixed point of T such that $v \neq u$. Both u and v lie in $\cap_{j=1}^{m} Aj$, thus we can substitute $x = u$ and $y = z = v$ in (8). This yields

$$F(D^*(Tu, Tv, Tv)) + \tau \leq \alpha(u, v, v) F(D^*(Tu, Tv, Tv)) + \tau \leq F(\psi(D^*(u, v, v))),$$

and hence

$$F(D^*(Tu, Tv, Tv)) \leq F(\psi(D^*(u, v, v))),$$

since F is strictly increasing therefore by 2.20 we get

$$D^*(Tu, Tv, Tv) \leq \psi(D^*(u, v, v)) < D^*(u, v, v).$$

This is a contradiction, Thus $u = v$, and the fixed point of T is unique.

Example 2.22. *If $F(\alpha) = ln\alpha,\ \alpha > 0$, then consider the sequence $\{S_n\}_{n\in\mathbb{N}}$ as follows,*

$$S_1 = 1,\ S_2 = 1 + 2, \qquad \cdots S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2},\ n \in \mathbb{N}, \qquad \cdots$$

Let $X = \{S_n : n \in \mathbb{N}\}$ and

$$D^*(x, y, z) = \begin{cases} 0 & x = y = z, \\ \max\{x, y, z\} & otherwise. \end{cases}$$

Then (X,D^*) is a D^*-complete D^*-metric space. Define the mapping $T:X\to X$ by the formulate:

$$T(S_n)=S_{n-1},\ for\ n>1, T(S_1)=1.$$

The mapping T is not the Banach contraction and is not the F-contraction. Indeed, for $n\neq m\neq k$, we get

$$\lim_{n\to\infty}\frac{D^*(T(S_n),T(S_m),T(S_k))}{D^*(S_n,S_m,S_k)}=\lim_{n\to\infty}\frac{\max\{S_{n-1},S_{m-1},S_{k-1}\}}{\max\{S_n,S_m,S_k\}}=1.$$

On the other side taking $F(\alpha)=ln\alpha+\alpha$ we obtain that T is F-contraction with $\tau=1$. To see this, let us consider the following calculation

$$\frac{D^*(T(S_n),T(S_m),T(S_k))}{D^*(S_n,S_m,S_k)}e^{D^*(T(S_n),T(S_m),T(S_k))-D^*(S_n,S_m,S_k)}$$

$$=\frac{\max\{S_{n-1},S_{m-1},S_{k-1}\}}{\max\{S_n,S_m,S_k\}}e^{\max\{S_{n-1},S_{m-1},S_{k-1}\}-\max\{S_n,S_m,S_k\}}$$

$<e^{-\max\{n,m,k\}}<e^{-1}$.

Clearly S_1 is a fixed point of T.

Definition 2.23. *A mapping $T:X\to X$ is said to be an $GF-$Suzuki-contraction if there exists $\tau>0$ such that for all $x,y,z\in X$ with $(D^*(Tx,Ty,Tz)>0$*

$$\frac{1}{2}D^*(x,x,Tx)<D^*(x,y,z)\Rightarrow \tau+F(D^*(Tx,Ty,Tz))\leq F(D^*(x,y,z))),$$

where $F:\mathbb{R}_+\to\mathbb{R}$ satisfies in $(GF_1)-(GF_3)$.

If F is satisfies conditions F_1 and F_2 in 1.12, we can prove the following theorem.

Theorem 2.24. *Let (X,D^*) be a D^*-compact D^*-metric space and let $T:X\to X$ be an $GF-$Suzuki-contraction mapping. Then T has a unique fixed point.*

Proof. We put

$$\beta=\inf\{D^*(x,x,Tx):x\in X\}$$

and choose a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty}D^*(x_n,x_n,Tx_n)=\beta$. Since X is compact. without loss of generality, we may assume that $\{x_n\}$ and $\{Tx_n\}$ converge to some elements

$v, w \in X$, respectively. We shall show $\beta = 0$. Arguing by contradiction, we assume $\beta > 0$. For every $\varepsilon > 0$ there exists $x_j \in X$ for some $j \in \mathbb{N}$, such that

$$D^*(x_j, x_j, Tx_j) < \beta + \varepsilon.$$

So from (GF_2), we have

$$F(D^*(x_j, x_j, Tx_j)) < F(\beta + \varepsilon).$$

On the other hand,

$$\frac{1}{2} D^*(x_j, x_j, Tx_j) < D^*(x_j, x_j, Tx_j),$$

therefore by assumption of theorem there exists $\tau > 0$ such that

$$\tau + F(D^*(Tx_j, Tx_j, T^2x_j)) \leq F(D^*(x_j, x_j, Tx_j)) < F(\beta + \varepsilon),$$

thus

$$F(D^*(Tx_j, Tx_j, T^2x_j)) < F(\beta + \varepsilon) - \tau$$

Similarly, since $\frac{1}{2} D^*(Tx_j, Tx_j, T^2x_j) < D^*(Tx_j, Tx_j, T^2x_j)$, thus

$$\begin{aligned} \tau + F(D^*(T^2x_j, T^2x_j, T^3x_j)) &< F(D^*(Tx_j, Tx_j, T^2x_j)) \\ &\leq F(D^*(x_j, x_j, Tx_j)) - \tau \\ &< F(\beta + \varepsilon) - \tau. \end{aligned}$$

So

$$F(D^*(T^2x_j, T^2x_j, T^3x_j)) < F(\beta + \varepsilon) - 2\tau.$$

Now by continuing similar method we obtain

$$F(D^*(T^nx_j, T^nx_j, T^{n+1}xj)) < F(\beta + \varepsilon) - n\tau.$$

This implies that

$$\lim_{n \to \infty} F(D^*(T^nx_j, T^nx_j, T^{n+1}xj)) = -\infty.$$

From (GF_2), we have $\lim_{n \to \infty} D^*(T^nx_j, T^nx_j, T^{n+1}xj) = 0$, so that there exists $n_0 \in \mathbb{N}$ such that

$$D^*(T^nx_j, T^nx_j, T^{n+1}xj) < \beta, \ \forall\, n \geq n_0.$$

This is a contradiction with definition of β. So, $\beta = 0$. We have

$$\lim_{n\to\infty} D^*(v,v,Tx_n) = D^*(v,v,w) = \lim_{n\to\infty} D^*(x_n,x_n,Tx_n) = \beta = 0,$$

which implies that $\{Tx_n\}$ also converges to v. Since, $\lim_{n\to\infty} D^*(x_n,x_n,Tx_n) = \beta = 0$, then $\lim_{n\to\infty} F(D^*(x_n,x_n,Tx_n)) = -\infty$ and we have

$$\lim_{n\to\infty} F(D^*(Tx_n,Tx_n,T^2x_n)) \leq \lim_{n\to\infty} F(D^*(x_n,x_n,Tx_n)) - \tau\ , \tau > 0.$$

Thus, $\lim_{n\to\infty} F(D^*(Tx_n,Tx_n,T^2x_n)) = -\infty$. Hence $\lim_{n\to\infty} D^*(Tx_n,Tx_n,T^2x_n) = 0$. Since

$$\lim_{n\to\infty} d(v,v,T^2x_n) \leq \lim_{n\to\infty} (D^*(v,v,Tx_n) + D^*(Tx_n,Tx_n,T^2x_n)) = 0.$$

And so $\{T^2x_n\}$ converges to v. If

$$\frac{1}{2}D^*(x_n,x_n,Tx_n) \geq D^*(x_n,x_n,v) \ and\ \frac{1}{2}D^*(Tx_n,Tx_n,T^2x_n) \geq D^*(Tx_n,Tx_n,v),$$

then there exists $0 < \tau < \infty$ such that

$$\tau + F(2D^*(Tx_n,Tx_n,v)) \leq \tau + F(D^*(Tx_n,Tx_n,T^2x_n)) \leq F(D^*(x_n,x_n,Tx_n)),$$

$$F(2D^*(Tx_n,Tx_n,v)) \leq F(D^*(x_n,x_n,Tx_n)) - \tau,$$

$$F(2D^*(Tx_n,Tx_n,v)) < F(D^*(x_n,x_n,Tx_n)).$$

From (GF_1), we have

$$2D^*(Tx_n,Tx_n,v) < D^*(x_n,x_n,Tx_n) < D^*(x_n,x_n,v) + D^*(v,v,Tx_n),$$

and so,

$$D^*(Tx_n,Tx_n,v) < \frac{1}{2}D^*(x_n,x_n,v) \leq \frac{1}{2}D^*(x_n,x_n,v) + \frac{1}{2}D^*(v,v,Tx_n).$$

Hence

$$\begin{aligned} D^*(x_n,x_n,v) &< \frac{1}{2}D^*(x_n,x_n,v) + \frac{1}{2}D^*(v,v,Tx_n) \\ &\leq \frac{1}{2}D^*(x_n,x_n,v) + \frac{1}{2}D^*(x_n,x_n,v) \\ &= D^*(x_n,x_n,v). \end{aligned}$$

This is a contradiction. Hence for every $n \in \mathbb{N}$, either

$$\frac{1}{2}D^*(x_n,x_n,Tx_n) < D^*(x_n,x_n,v) \ or \ \frac{1}{2}D^*(Tx_n,Tx_n,T^2x_n) < D^*(Tx_n,Tx_n,v),$$

holds. By assumption, either $\tau + F(D^*(Tx_n,Tx_n,Tv)) \leq F(D^*(x_n,x_n,v))$ or

$$\tau + F(D^*(T^2x_n,T^2x_n,Tv)) \leq F(D^*(Tx_n,Tx_n,v)),$$

holds. Hence one of the following holds:

* There exists an infinite subset I of $\mathbb{N}$ such that $\tau + F(D^*(Tx_n,Tx_n,Tv)) \leq F(D^*(x_n,x_n,v))$ for all $n \in I$.

* There exists an infinite subset J of $\mathbb{N}$ such that $\tau + F(D^*(T^2x_n,T^2x_n,Tv)) \leq F(D^*(Tx_n,Tx_n,Tv))$ for all $n \in J$.

In the first case, we obtain

$$F(D^*(Tx_n,Tx_n,Tv)) \leq F(D^*(x_n,x_n,v)) - \tau,$$

$$F(D^*(Tx_n,Tx_n,Tv)) < F(D^*(x_n,x_n,v)).$$

Hence from (GF_1), we have

$$D^*(Tx_n,Tx_n,Tv) < D^*(x_n,x_n,v),$$

$$D^*(v,v,Tv) = \lim_{n\in I, n\to\infty} D^*(Tx_n,Tx_n,Tv) \leq \lim_{n\in I, n\to\infty} D^*(x_n,x_n,v) = 0$$

Also, in the second case, we obtain

$$F(D^*(T^2x_n,T^2x_n,Tv)) \leq F(D^*(Tx_n,Tx_n,v)) - \tau,$$

therefore

$$F(D^*(T^2x_n,T^2x_n,Tv)) < F(D^*(Tx_n,Tx_n,v)),$$

So from (GF_1), we have

$$D^*(T^2x_n,T^2x_n,Tv) < D^*(Tx_n,Tx_n,v),$$

and

$$D^*(v,v,Tv) = \lim_{n\in J, n\to\infty} D^*(T^2x_n, T^2x_n, Tv)$$
$$\leq \lim_{n\in J, n\to\infty} D^*(Tx_n, Tx_n, v) = 0.$$

Hence, v is a fixed point of T. T has at most one fixed point. Indeed, if $x_1^*, x_2^* \in X, Tx_1^* = x_1^* \neq x_2^* = Tx_2^*$. Then we get

$$D^*(x_1^*, x_1^*, Tx_1^*) = 0,\ \tau \leq F(D^*(x_1^*, x_1^*, x_2^*)) - F(D^*(Tx_1^*, Tx_1^*, Tx_2^*)) = 0.$$

This is a contradiction.

Conflict of Interests

The authors declare that there is no conflict of interests.

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