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CONVERGENCE THEOREMS OF A HYBRID ITERATION METHOD FOR FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

ESRA YOLACAN*, HUKMI KIZILTUNC

Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey

Abstract. In this paper, a hybrid iteration method is studied. Convergence theorems for a fixed point of asymptotically nonexpansive mapping are established in Banach spaces.

Keywords: asymptotically nonexpansive mapping; fixed point; hybrid iteration scheme.

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1. Introduction and preliminaries

Let K be a nonempty closed convex subset of a real Banach space E. Let$T : K \to K$ be a mapping. Recall that T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x \in K.$$

T is said to be *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x \in K.$$

*Corresponding author
E-mail addresses: yolacanesra@gmail.com (E. Yolacan), hukmu@atauni.edu.tr (H. Kiziltunc)
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T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x-y\| \quad \forall x \in K, n \geq 1.$$

Iterative techniques for approximating fixed points of nonexpansive mappings and asymptotically nonexpansive mappings have been studied by various authors; see, e.g., [1-16]. In 2007, Wang [17] introduced an explicit hybrid iteration method for nonexpansive mappings in Hilbert space. In the same year, Osilike *et al.* [18] extended Wang's results to arbitrary Banach spaces without the strong monotonicity assumption imposed on the hybrid operator. In 2012, Qiu *et al.* [13] improved and extended the results in [11] and studied the strong convergence.

Inspired and motivated by this facts, we study a hybrid iteration scheme for approximating fixed points of asymptotically nonexpansive mappings. Convergence theorems for a fixed point of asymptotically nonexpansive mapping are established in uniformly convex Banach spaces.

Let K be a nonempty closed convex subset of a real uniformly convex Banach space and $T : K \to K$ be asymptotically nonexpansive. This scheme is defined as follows.

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \left[T^n x_n - \lambda_{n+1}\mu A\left(T^n x_n\right)\right], \quad \forall n \geq 0, \tag{1.1}$$

where $u \in K$ and $x_0 \in K$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are real sequences in $[0,1)$ and $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$.

Recall the following definitions.

Definition 1.1. [19] A norm on Banach space E is uniformly convex if for all $\{x_n\}$, $\{y_n\} \subset \{z \in E : \|z\| = 1\}$ such that $\left\|\frac{1}{2}(x+y)\right\| \to 1$, we have $\|x_n - y_n\| \to 0$.

A Banach space E is said to satisfy the *Opial's condition* [20] if, for all sequences $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to some $x \in E$, the inequality $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ holds for all $y \neq x$ in E.

Definition 1.2. [21] Let C and K be two Banach spaces and let T be a mapping from C into K. Then the mapping T is said to be

(i) demiclosed if $x_n \rightharpoonup x$ in C and $Tx_n \to y$ in K imply $Tx = y$;

(ii) demicompact if any bounded sequence $\{x_n\}$ in C such that $\{x_n - Tx_n\}$ converges strongly has a convergent subsequence;

(iii) completely continuous if it is continuous and compact.

T is said to satisfy *condition (A)* [5] if $F(T) \neq \emptyset$ and there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in D(T)$, where $d(x, F(T)) := \inf\{\|x - p\| : p \in F(T)\}$.

In the sequel, we need the following useful known lemmas to prove our main results.

Lemma 1.1. [22] *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \ \ \forall n \geq 1$$

If $\sum_{n=1}^{\infty} b_n < \infty$ *and* $\sum_{n=1}^{\infty} \delta_n < \infty$*, then*

(i) $\lim_{n\to\infty} a_n$ *exists;*

(ii) In particular, if $\{a_n\}$ *has a subsequence* $\{a_{nk}\}$ *converging to* 0, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2. [14] *Let E be a uniformly convex Banach space and let a, b be two constants with* $0 < a < b < 1$. *Suppose that* $\{t_n\} \subset [a,b]$ *is a real sequence and* $\{x_n\}$, $\{y_n\}$ *are two sequences in E. Then the conditions*

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d, \ \limsup_{n\to\infty} \|x_n\| \leq d, \ \limsup_{n\to\infty} \|y_n\| \leq d$$

imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, *where $d \geq 0$ is a constant.*

Lemma 1.3. [23] *Let E be a real uniformly convex Banach space. Let K be a nonempty closed convex subset of E and let $T : K \to K$ be nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

2. Main results

Theorem 2.1. *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $T : K \to K$ be an asymptotically nonexpansive mapping with sequence* $\{k_n\} \subset [1,\infty)$ *satisfying* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$*; $A : K \to K$ is an L-Lipschitzian mapping. Let the* *hybrid iteration* $\{x_n\}$ *be defined by (1.1) and* $F(T) \neq \emptyset$, *where* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\lambda_n\}$ *are real sequences in* $[0,1)$ *and satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Then

(1) $\lim_{n\to\infty} \|x_n - p\| = 0$ *exists,* $\forall p \in F$;

(2) $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$;

(3) $\{x_n\}$ *converges strongly to a fixed point of* T *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. (1) Let $p \in F(T)$. In view of arbitrary $u \in K$, we have

$$M = \max\{\|u - p\|\}. \tag{2.1}$$

By using (1.1), we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
= \;& \|\alpha_n u + \beta_n x_n + \gamma_n [T^n x_n - \lambda_{n+1}\mu A(T^n x_n)] - p\| \\
\leq \;& \alpha_n \|u - p\| + \beta_n \|x_n - p\| + \gamma_n \|T^n x_n - p\| + \gamma_n \lambda_{n+1}\mu \|A(T^n x_n)\| \\
\leq \;& \alpha_n \|u - p\| + \beta_n \|x_n - p\| + \gamma_n \|T^n x_n - p\| + \gamma_n \lambda_{n+1}\mu \|A(T^n x_n) - A(p)\| + \gamma_n \lambda_{n+1}\mu \|A(p)\| \\
\leq \;& \alpha_n \|u - p\| + \beta_n \|x_n - p\| + \gamma_n k_n \|x_n - p\| \\
& + \gamma_n \lambda_{n+1}\mu k_n L \|x_n - p\| + \gamma_n \lambda_{n+1}\mu \|A(p)\|. \qquad (2.2)
\end{aligned}$$

Since $\lim_{n\to\infty} k_n = 1\,(n \to \infty)$, we know that $\{k_n\}$ is bounded and there exists $Q_1 \geq 1$ such that $k_n \leq Q_1$. Let $h_n = k_n - 1$, by $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$ we have $\sum\limits_{n=1}^{\infty} h_n < \infty$. Hence we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
\leq \;& \alpha_n M + k_n(\beta_n + \gamma_n)\|x_n - p\| + \gamma_n \lambda_{n+1}\mu Q_1 L \|x_n - p\| + \gamma_n \lambda_{n+1}\mu \|A(p)\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n M + (1+h_n)(1-\alpha_n)\|x_n - p\| + \gamma_n \lambda_{n+1}\mu Q_1 L \|x_n - p\| + \gamma_n \lambda_{n+1}\mu \|A(p)\| \\
&\leq \alpha_n M + (1-\alpha_n+h_n)\|x_n - p\| + \gamma_n \lambda_{n+1}\mu Q_1 L \|x_n - p\| + \gamma_n \lambda_{n+1}\mu \|A(p)\| \\
&\leq (1+h_n+\lambda_{n+1}\mu Q_1 L)\|x_n - p\| + \lambda_{n+1}\mu \|A(p)\| + M \\
&\leq (1+\delta_n)\|x_n - p\| + b_n, \qquad (2.3)
\end{aligned}
$$

where $\delta_n = h_n + \lambda_{n+1}\mu Q_1 L$ and $b_n = \lambda_{n+1}\mu \|A(p)\| + M$. Since condition (iii) and $\sum_{n=1}^{\infty} h_n < \infty$, we have $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$ by Lemma 1.1 we get $\lim_{n\to\infty}\|x_n - p\|$ exists. This completes the proof of (1).

(2) Since $\{\|x_n - p\|\}$ is bounded, there exists $Q_2 > 0$ such that

$$\|x_n - p\| \leq Q_2, \ \ \forall n \geq 1. \qquad (2.4)$$

We can assume

$$\lim_{n\to\infty}\|x_n - p\| = c, \qquad (2.5)$$

where $c \geq 0$ is some number. Since $\{x_n - p\}$ is a convergent sequence, so $\{x_n\}$ is bounded sequence in K.

By (2.5) we have that

$$\limsup_{n\to\infty}\|x_n - p\| = c. \qquad (2.6)$$

By condition (iii) $k_n \leq Q_1$, $\sum_{n=1}^{\infty} h_n < \infty$ and (2.4), (2.5), (2.6), we have that

$$
\begin{aligned}
&\limsup_{n\to\infty}\|[T^n x_n - \lambda_{n+1}\mu \|A(T^n x_n)\|] - p\| \\
&\leq \limsup_{n\to\infty}\{\|T^n x_n - p\| + \lambda_{n+1}\mu \|A(T^n x_n)\|\} \\
&\leq \limsup_{n\to\infty}\{k_n\|x_n - p\| + \lambda_{n+1}\mu (L k_n \|x_n - p\| + \|A(p)\|)\} \\
&\leq \limsup_{n\to\infty}\{k_n\|x_n - p\| + \lambda_{n+1}\mu (L Q_1 Q_2 + \|A(p)\|)\} \\
&\leq \limsup_{n\to\infty}\{(1+h_n)\|x_n - p\| + \lambda_{n+1}\mu (L Q_1 Q_2 + \|A(p)\|)\} \\
&\leq c. \qquad (2.7)
\end{aligned}
$$

Therefore (2.5), (2.6), (2.7) and Lemma 1.2 we know that

$$\lim_{n\to\infty} \|[T^n x_n - \lambda_{n+1}\mu \|A(T^n x_n)\|] - x_n\| = 0. \quad (2.8)$$

From condition (ii) and (2.8), we have

$$\begin{aligned} & \|x_{n+1} - x_n\| \\ = \;& \|\alpha_n u + \beta_n x_n + \gamma_n [T^n x_n - \lambda_{n+1}\mu A(T^n x_n)] - x_n\| \\ \leq \;& \alpha_n \|u - p\| + \gamma_n \|[T^n x_n - \lambda_{n+1}\mu A(T^n x_n)] - x_n\| \\ \to \;& 0, \quad \text{as } (n \to \infty). \end{aligned} \quad (2.9)$$

It follows from condition (iii) and (2.8) that

$$\begin{aligned} & \|x_n - T^n x_n\| \\ \leq \;& \|x_n - [T^n x_n - \lambda_{n+1}\mu A(T^n x_n)]\| \\ & + \|[T^n x_n - \lambda_{n+1}\mu A(T^n x_n)] - T^n x_n\| \\ \leq \;& \|x_n - [T^n x_n - \lambda_{n+1}\mu A(T^n x_n)]\| + \lambda_{n+1}\mu (LQ_1Q_2 + \|A(p)\|) \\ \to \;& 0 \text{ as } (n \to \infty). \end{aligned} \quad (2.10)$$

By (2.9) and (2.10), we have that

$$\begin{aligned} & \|x_n - Tx_n\| \\ \leq \;& \|x_n - T^n x_n\| + \|T^n x_n - Tx_n\| \\ \leq \;& \|x_n - T^n x_n\| + k_n \left\|T^{n-1} x_n - x_n\right\| \\ \leq \;& \|x_n - T^n x_n\| \\ & + k_n \left(\left\|T^{n-1} x_n - T^{n-1} x_{n-1}\right\| + \left\|T^{n-1} x_{n-1} - x_{n-1}\right\| + \|x_{n-1} - x_n\|\right) \\ \leq \;& \|x_n - T^n x_n\| \\ & + k_n \left(k_{n-1} \|x_n - x_{n-1}\| + \left\|T^{n-1} x_{n-1} - x_{n-1}\right\| + \|x_{n-1} - x_n\|\right) \\ \to \;& 0, \quad \text{as } (n \to \infty). \end{aligned} \quad (2.11)$$

This completes the proof of (2).

(3) From (2.3), we obtain

$$\begin{aligned} & \|x_{n+1}-p\| \\ \leq & (1+\delta_n)\|x_n-p\|+b_n \\ = & \|x_n-p\|+\omega_n, \end{aligned} \tag{2.12}$$

where $\omega_n=\delta_n\|x_n-p\|+b_n$. Since $\{x_n-p\}$ is bounded and $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}b_n<\infty$, we get $\sum_{n=1}^{\infty}\omega_n<\infty$. Therefore, (2.3) implies

$$d(x_{n+1},F(T))\leq d(x_n,F(T))+\omega_n. \tag{2.13}$$

By Lemma 1.1 (i), it follows from (2.13) that $\lim_{n\to\infty}d(x_n,F(T))$ exists. Noticing

$$\liminf_{n\to\infty}d(x_n,F(T))=0,$$

it follows from (2.13) and Lemma 1.1 (ii) that we have $\lim_{n\to\infty}d(x_n,F(T))=0$. For arbitrary $\varepsilon>0$, there exists a positive integer N_1 such that $d(x_n,F(T))<\frac{\varepsilon}{4}$ for all $n\geq N_1$. In addition, $\sum_{n=1}^{\infty}\omega_n<\infty$ implies that there exists a positive integer N_2 such that $\sum_{j=1}^{\infty}\omega_j<\frac{\varepsilon}{4}$ for all $n\geq N_2$. Choose $N=\max\{N_1,N_2\}$, then $d(x_n,F(T))<\frac{\varepsilon}{4}$ and $\sum_{j=N}^{\infty}\omega_j<\frac{\varepsilon}{4}$. This means that there exists a $x^*\in F(T)$ such that $\|x_N-x^*\|\leq\frac{\varepsilon}{4}$. It follows from (2.12) that for all n, $m\geq N$,

$$\begin{aligned} \|x_n-x_m\| &\leq \|x_n-x^*\|+\|x_m-x^*\| \\ &\leq \|x_N-x^*\|+\sum_{j=N+1}^{n}\omega_j+\|x_N-x^*\|+\sum_{j=N+1}^{m}\omega_j \\ &\leq 2\left(\|x_N-x^*\|+\sum_{j=N}^{\infty}\omega_j\right) \\ &< \varepsilon. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Suppose $\lim_{n\to\infty}x_n=q$, then since $\lim_{n\to\infty}\|Tx_n-x_n\|=0$, we have $q\in F(T)$. This completes the proof of (3).

Theorem 2.2. *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $T:K\to K$ be an asymptotically nonexpansive mapping with sequence*

$\{k_n\} \subset [1,\infty)$ *satisfying* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$; $A : K \to K$ *is an L-Lipschitzian mapping. Let the hybrid iteration* $\{x_n\}$ *be defined by (1.1) and* $F(T) \neq \emptyset$, *where* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\lambda_n\}$ *are real sequences in* $[0,1)$ *and satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$.

If T *is demicompact, then the sequence* $\{x_n\}$ *converges strongly to a fixed point of* T .

Proof. By Theorem 2.1 (1), $\{x_n\}$ is bounded. Since T is demicompact from the fact that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Then there exists a subsequence$\{x_{n_j}\}$ of $\{x_n\}$ that converges strongly to $q \in K$ as $j \to \infty$. Therefore it follows from (2.11) $Tx_{n_j} \to q$ as $j \to \infty$. Using the continity of T we get that $Tq = q$ as so $q \in F(T)$. It follows from (2.3) and Theorem 3.1 and $\lim_{n\to\infty} x_{n_j} = q$ that $\{x_n\}$ converges strongly to $q \in F(T)$. This completes the proof.

Theorem 2.3. *Let E be a real uniformly convex Banach space satisfying Opial's condition, let K be a nonempty closed convex subset of E, and let* $T : K \to K$ *be an asymptotically nonexpansive mapping with sequence* $\{k_n\} \subset [1,\infty)$ *satisfying* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$; $A : K \to K$ *is an L-Lipschitzian mapping. Let the hybrid iteration* $\{x_n\}$ *be defined by (1.1) and* $F(T) \neq \emptyset$, *where* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\lambda_n\}$ *are real sequences in* $[0,1)$ *and satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Then $\{x_n\}$ *converges weakly to a fixed point of* T .

Proof. From Lemma 1.3, $I - T$ demiclosed at zero and since Theorem 2.1 (2) and E satisfies *Opial's condition,* it follows from standart argument that $\{x_n\}$ converges weakly to a fixed point of T. This completes the proof.

Remark 2.4. By using Theorem 2.1, we can prove that $\{x_n\}$ converges strongly to a fixed point of T if T is completely continuous or satisfies condition (A). Therefore the results presented in this paper extend and improve the corresponding results in [10-13] and [18].

Conflict of Interests

The authors declare that there is no conflict of interests.

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REFERENCES

[1] F.E. Browder, W. Petryshyn, Construction of fixed points of nonlinear mappings. J. Math. Anal. Appl. 20 (1967), 197-228.

[2] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73 (1967), 957-961.

[3] S.H. Khan, U.D. Fukhar, Weak and strong convergence of a scheme with errors for two nonepansive mappings, Nonlinear Anal. 61 (2005), 1295-1301.

[4] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003), 372-379.

[5] H.F. Senter, W.G. Dotson, Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. 44 (1974), 375-380.

[6] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigoups withhout Bochner integrals, J. Math. Anal. Appl. 305 (2005), 305.

[7] W. Takahashi, T. Tamura, Convergence theorems for a pair of nonexpansive mappings, J. Convex Anal. 5 (1998), 45-58.

[8] W. Takahashi, Nonlinear Functional Analysis, Fixed Point Theory Appl. Yokohama Publisher, Yokoma, Japan, 2000.

[9] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 58 (1992), 486-491.

[10] J.S. Jung, Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 302 (2005), 509-520.

[11] Y. Song, X. Chai, Halpern iteration for firmly type nonexpansive mappings, Nonlinear Anal. 71 (2009), 4500-4506.

[12] L. Wang, S. Yao, Hybrid iteration method for fixed points of nonexpansive mapings, Thai J. Math. 5 (2007), 183-190.

[13] L. Qiu, S. Yao, Hybrid iteration method for fixed points of nonexpansive mapings, Inter. Math. . Forum 7 (2012), 251-258.

[14] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull Aust Math Soc. 43 (1991), 153-159.

[15] F. Gu, A new hybrid iteration method for a finite family of asymptotically nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. 2013 (2013), Article ID 10.

[16] H.K. Xu, Another control condition in an iterative method for nonexpansive mappings, Bull. Aust. Math. Soc. 65 (2002), 109-113.

[17] L. Wang, An iteration method for nonexpansive mappings in Hilbert spaces, Fixed Point Theory Appl. 2007 (2007), Article ID 28619.

[18] M. Osilike, F. Isiogugu, P. Nwokoro, Hybrid iteration method for fixed points of nonexpansive mappings in arbitrary Banach spaces, Fixed Point Theory Appl. 2007 (2007), Article ID 64309.

[19] J.A. Clarkson, Uniformly convex spaces, Trans. Amer. Math. Soc. 40 (1939), 396-414.

[20] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 591-597.

[21] R.P. Agarwal, D. O'Regan, D.R. Sahu, Fixed Point Theory for Lipschitzian-type Mappings with Applications, Springer, New York, 2009.

[22] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993), 301-308.

[23] J. Gornicki, Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces. Comment Math. Univ. Carolinae. 30 (1989), 249-252.