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SOME FIXED POINT RESULTS OF INTEGRAL TYPE AND APPLICATIONS

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Abstract. In this paper, an integral type of Suzuki-type mappings is investigated for generalizing the Banach contraction theorem on a metric space. As an application, the existence of a continuous solution for an integral equation is obtained.

Keywords: Suzuki type mapping; fixed point; integral equation; integral type mapping.

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1. Introduction and preliminaries

The most important result on fixed points for contractive-type mappings is the well-known Banach contraction theorem, which was established in 1922; see [1] and the references therein.

Theorem 1.1. *Let (X,d) be a complete metric space, $\beta \in (0,1)$ and let $T : X \to X$ be a mapping such that for each $x,y \in X$,*

$$d(Tx,Ty) \leq \beta d(x,y).$$

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Then T has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = a$.

Since 1922, many authors have introduced various types of contraction inequalities to generalized the well-known Banach contraction theorem. In 2002 Branciari proved the following result; see [2].

Theorem 1.2. *Let (X,d) be a complete metric space, $\beta \in (0,1)$ and $T : X \longrightarrow X$ a mapping such that for each $x,y \in X$,*

$$\int_0^{d(Tx,Ty)} f(t)dt \leq \beta \int_0^{d(x,y)} f(t)dt,$$

where $f : [0,\infty) \to (0,\infty)$ is a Lebesgue integrable mapping which is summable (i.e., with finite integral on each compact subset of $[0,\infty)$) and for each $\varepsilon > 0$, $\int_0^{\varepsilon} f(t)dt > 0$. Then T has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = a$.

In 2008, Suzuki [3] introduced a new method on the problem and the method was further extended by some authors; see, for example, [4-7]. The following result was proved in [8].

Theorem 1.3. *Let (X,d) be a complete metric space and let $T : X \longrightarrow X$ be a mapping. Suppose that there exist $\alpha \in (0,\frac{1}{2}]$, $\beta \in (0,1)$ such that $\alpha d(x,Tx) \leq d(x,y)$ implies $d(Tx,Ty) \leq \beta d(x,y)$ for all $x,y \in X$. Then T has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = a$.*

The aim of this paper is to provide a new condition for T which guarantees the existence of its fixed point based on Suzuki and Branciari's idea. In order to obtain our main results, we need the following lemmas.

Lemma 1.4. *Let $a,b \in [0,\infty)$ and $f : [0,\infty) \to (0,\infty)$ a Lebesgue integrable mapping which is summable and for each $\varepsilon > 0$, $\int_0^{\varepsilon} f(t)dt > 0$. Then*
i) $a = 0$ whenever $\int_0^a f(t)dt = 0$,
ii) $a < b$ whenever $\int_0^a f(t)dt < \int_0^b f(t)dt$.

Lemma 1.5. *Let $L > 0$, $\alpha(x),\beta(x) \in C([a,b])$ and $f : [0,\infty) \to (0,\infty)$ a Lebesgue integrable mapping which is summable and for each $\varepsilon > 0$, $\int_0^{\varepsilon} f(t)dt > 0$. Then $\int_0^{\|\alpha\|_\infty} f(t)dt < L\int_0^{\|\beta\|_\infty} f(t)dt$ whenever $\int_0^{|\alpha(x)|} f(t)dt < L\int_0^{|\beta(x)|} f(t)dt$.*

2. Main results

The following theorem is the main result of this paper.

Theorem 2.1 *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ a mapping. Suppose that there exist $\alpha \in (0,\frac{1}{2}]$, $\beta \in (0,1)$ such that $\alpha d(x,Tx) \leq d(x,y)$ implies*

$$\int_0^{d(Tx,Ty)} f(t)dt \leq \beta \int_0^{d(x,y)} f(t)dt$$

for all $x,y \in X$ and $f : [0,\infty) \to (0,\infty)$ is a Lebesgue integrable mapping which is summable and for each $\varepsilon > 0$, $\int_0^{\varepsilon} f(t)dt > 0$. Then T has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = a$.

Proof. Fix arbitrary $1 > r > \beta$, $x_0 \in X$ and $x_1 = Tx_0$. We have $\alpha d(x_0,Tx_0) < d(x_0,x_1)$. Hence,

$$\int_0^{d(Tx_0,Tx_1)} f(t)dt \leq \beta \int_0^{d(x_0,x_1)} f(t)dt < r \int_0^{d(x_0,x_1)} f(t)dt.$$

Since $r < 1$, we have $\int_0^{d(x_1,Tx_1)} f(t)dt < \int_0^{d(x_0,x_1)} f(t)dt$. Let $x_2 = Tx_1$. By lemma 1.4, we have $d(x_1,Tx_1) < d(x_0,x_1)$. Therefore, we find $\alpha d(x_1,Tx_1) < d(x_1,x_2)$ and

$$\int_0^{d(Tx_1,Tx_2)} f(t)dt \leq \beta \int_0^{d(x_1,x_2)} f(t)dt < r \int_0^{d(x_1,x_2)} f(t)dt < r^2 \int_0^{d(x_0,x_1)} f(t)dt.$$

Now let $x_3 = Tx_2$. By lemma 1.4, $d(x_2,x_3) < d(x_1,x_2) < d(x_0,x_1)$. Since $\alpha d(x_2,Tx_2) < d(x_2,x_3)$, we have

$$\int_0^{d(Tx_2,Tx_3)} f(t)dt \leq \beta \int_0^{d(x_2,x_3)} f(t)dt < r \int_0^{d(x_2,x_3)} f(t)dt < r^3 \int_0^{d(x_0,x_1)} f(t)dt.$$

By continuing this process, we obtain a sequence $\{x_n\}_{n\geq 1}$ in X such that $x_{n+1} = Tx_n$, $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$ and

$$\int_0^{d(x_n,x_{n+1})} f(t)dt < r^n \int_0^{d(x_0,x_1)} f(t)dt.$$

We claim that for any $y \in X$, one of the flowing relations hold:

$$\alpha d(x_n,Tx_n) \leq d(x_n,y) \text{ or } \alpha d(x_{n+1},Tx_{n+1}) \leq d(x_{n+1},y). \tag{2.1}$$

Otherwise, if $\alpha d(x_n,Tx_n) > d(x_n,y)$ and $\alpha d(x_{n+1},Tx_{n+1}) > d(x_{n+1},y)$, we have

$$d(x_n,x_{n+1}) \leq d(x_n,y) + d(x_{n+1},y) < \alpha d(x_n,Tx_n) + \alpha d(x_{n+1},Tx_{n+1})$$

$$= \alpha d(x_n,x_{n+1}) + \alpha d(x_{n+1},x_{n+2}) \leq 2\alpha d(x_n,x_{n+1}) \leq d(x_n,x_{n+1}),$$

which is a contradiction. Now let $a_n = d(x_n, x_{n+1})$ for all $n \geq 1$. It is obvious that $\{a_n\}_{n\geq 1}$ is monotone non-increasing and so there exists $a \geq 0$ such that $\lim_{n\to\infty} a_n = a$. Since

$$\int_0^a f(t)dt = \lim_{n\to\infty} \int_0^{a_n} f(t)dt \leq \lim_{n\to\infty} r^n \int_0^{d(x_0,x_1)} f(t)dt = 0,$$

we have $a = 0$. We claim $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X,d), i.e,

$$\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N} \mid \forall m,n \in \mathbb{N}, m > n > N_\varepsilon\ d(x_m, x_n) < \varepsilon.$$

Suppose that there exists an $\varepsilon > 0$ such that for each $N \in \mathbb{N}$ there are $m_N, n_N \in \mathbb{N}$, with $m_N > n_N > N$, such that $d(x_{m_N}, x_{n_N}) \geq \varepsilon$. We choose the sequences $\{m_N\}_{N\geq 1}$ and $\{n_N\}_{N\geq 1}$ such that for each $N \in \mathbb{N}$, m_N is minimal in the sense that $d(x_{m_N}, x_{n_N}) \geq \varepsilon$ but $d(x_h, x_{n_N}) < \varepsilon$ for each $h \in \{n_N + 1, ..., m_N - 1\}$. Now we analyze the properties of $d(x_{m_N}, x_{n_N})$, $d(x_{m_N+1}, x_{n_N+1})$ and $d(x_{m_N+2}, x_{n_N+1})$. Since

$$\begin{aligned}\varepsilon &\leq d(x_{m_N}, x_{n_N})\\ &\leq d(x_{m_N}, x_{m_N-1}) + d(x_{m_N-1}, x_{n_N})\\ &< d(x_{m_N}, x_{m_N-1}) + \varepsilon,\end{aligned}$$

we have $d(x_{m_N}, x_{n_N}) \to \varepsilon^+$ as $N \to \infty$. We claim that there exists $k \in \mathbb{N}$ such that for each natural number $N > k$ we have $d(x_{m_N+1}, x_{n_N+1}) < \varepsilon$ and $d(x_{m_N+2}, x_{n_N+1}) < \varepsilon$. Suppose there exists a subsequence $\{N_k\}_{k\geq 1} \subseteq \mathbb{N}$ such that $d(x_{m_{N_k}+1}, x_{n_{N_k}+1}) \geq \varepsilon$ or $d(x_{m_{N_k}+2}, x_{n_{N_k}+1}) \geq \varepsilon$. If $d(x_{m_{N_k}+1}, x_{n_{N_k}+1}) \geq \varepsilon$,

$$\varepsilon \leq d(x_{m_{N_k}+1}, x_{n_{N_k}+1}) \leq d(x_{m_{N_k}+1}, x_{m_{N_k}}) + d(x_{m_{N_k}}, x_{n_{N_k}}) + d(x_{n_{N_k}}, x_{n_{N_k}+1})$$

and then $d(x_{m_{N_k}+1}, x_{n_{N_k}+1}) \to \varepsilon$, as $k \to \infty$. If $d(x_{m_{N_k}+2}, x_{n_{N_k}+1}) \geq \varepsilon$,

$$\begin{aligned}\varepsilon \leq d(x_{m_{N_k}+2}, x_{n_{N_k}+1}) &\leq d(x_{m_{N_k}+2}, x_{m_{N_k}+1}) + d(x_{m_{N_k}+1}, x_{m_{N_k}})\\ &+ d(x_{m_{N_k}}, x_{n_{N_k}}) + d(x_{n_{N_k}}, x_{n_{N_k}+1})\end{aligned}$$

and then $d(x_{m_{N_k}+2}, x_{n_{N_k}+1}) \to \varepsilon$, when $k \to \infty$. In view of

$$d(x_{m_{N_k}+1}, x_{n_{N_k}}) \leq d(x_{m_{N_k}+1}, x_{m_{N_k}}) + d(x_{m_{N_k}}, x_{n_{N_k}}),$$

we have $\lim_{k\to\infty} d(x_{m_{N_k}+1}, x_{n_{N_k}}) \leq \varepsilon$. From relation 2.1, we have

$$\int_0^{d(x_{m_{N_k}+1}, x_{n_{N_k}+1})} f(t)dt \leq \beta \int_0^{d(x_{m_{N_k}}, x_{n_{N_k}})} f(t)dt$$

or

$$\int_0^{d(x_{m_{N_k}+2},x_{n_{N_k}+1})} f(t)dt \leq \beta \int_0^{d(x_{m_{N_k+1}},x_{n_{N_k}})} f(t)dt.$$

As $k \to \infty$, we have $\int_0^{\varepsilon} f(t)dt \leq \beta \int_0^{\varepsilon} f(t)dt$, which is a contradiction. So there exists $k \in \mathbb{N}$ such that for each natural number $N > k$ one has $d(x_{m_N+1}, x_{n_N+1}) < \varepsilon$ and $d(x_{m_N+2}, x_{n_N+1}) < \varepsilon$. Now we claim that there exist a $\delta_\varepsilon \in (0,\varepsilon)$ and $N_\varepsilon \in \mathbb{N}$ such that for each natural number $N > N_\varepsilon$. Note that

$$d(x_{m_N+1},x_{n_N+1}) < \varepsilon - \delta_\varepsilon \; or \; d(x_{m_{N_k}+2},x_{n_{N_k}+1}) < \varepsilon - \delta_\varepsilon.$$

Suppose that exist a subsequence $\{N_k\}_{k\geq 1} \subseteq \mathbb{N}$ such that $d(x_{m_{N_k}+1}, x_{n_{N_k}+1}) \to \varepsilon$ and

$$d(x_{m_{N_k}+2},x_{n_{N_k}+1}) \to \varepsilon$$

as $k \to \infty$. Now by relation 2.1, we have

$$\int_0^{d(x_{m_{N_k}+1},x_{n_{N_k}+1})} f(t)dt \leq \beta \int_0^{d(x_{m_{N_k}},x_{n_{N_k}})} f(t)dt$$

or

$$\int_0^{d(x_{m_{N_k}+2},x_{n_{N_k}+1})} f(t)dt \leq \beta \int_0^{d(x_{m_{N_k+1}},x_{n_{N_k}})} f(t)dt,$$

which is a contradiction. If $d(x_{m_N+1}, x_{n_N+1}) < \varepsilon - \delta_\varepsilon$, then

$$\varepsilon \leq d(x_{m_N},x_{n_N}) \leq d(x_{m_N},x_{m_N+1}) + d(x_{m_N+1},x_{n_N+1}) + d(x_{n_N+1},x_{n_N})$$

$$< d(x_{m_N},x_{m_N+1}) + (\varepsilon - \delta_\varepsilon) + d(x_{n_N},x_{n_N+1}).$$

If $d(x_{m_N+2}, x_{n_N+1}) < \varepsilon - \delta_\varepsilon$, then

$$\varepsilon \leq d(x_{m_N},x_{n_N}) \leq d(x_{m_N},x_{m_N+1}) + d(x_{m_N+1},x_{m_N+2})$$

$$+d(x_{m_N+2},x_{n_N+1}) + d(x_{n_N+1},x_{n_N})$$

$$< d(x_{m_N},x_{m_N+1}) + d(x_{m_N+1},x_{m_N+2}) + (\varepsilon - \delta_\varepsilon) + d(x_{n_N},x_{n_N+1}).$$

It follows that $\varepsilon \leq \varepsilon - \delta_\varepsilon$ as $N \to \infty$, which is a contradiction. This proves our claim that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X,d). Let $\lim_{n\to\infty} x_n = x$. By relation 2.1, for each $n \geq 1$ either i)$\int_0^{d(Tx_n,Tx)} f(t)dt \leq \beta \int_0^{d(x_n,x)} f(t)dt$ or ii)$\int_0^{d(Tx_{n+1},Tx)} f(t)dt \leq \beta \int_0^{d(x_{n+1},x)} f(t)dt$ hold. Then

$\int_0^{d(Tx_n,Tx)} f(t)dt \to 0$ or $\int_0^{d(Tx_{n+1},Tx)} f(t)dt \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} d(Tx_n,Tx) = 0$ or $\lim_{n\to\infty} d(Tx_{n+1},Tx) = 0$. In case (i), since

$$d(x,Tx) \le d(x,Tx_n) + d(Tx_n,Tx) = d(x,x_{n+1}) + d(Tx_n,Tx),$$

we obtain $d(x,Tx) = 0$ and so $Tx = x$. We obtain $Tx = x$. Now we show that this fixed point is unique. Suppose that there are two distinct points $a,b \in X$ such that $Ta = a$ and $Tb = b$. Since $d(a,b) > 0 = \alpha d(a,Ta)$, we have the contradiction

$$0 < \int_0^{d(a,b)} f(t)dt = \int_0^{d(Ta,Tb)} f(t)dt \le \beta \int_0^{d(a,b)} f(t)dt.$$

To prove that $\lim_{n\to\infty} T^n x = a$, let x be arbitrary and $a \in F(T)$. Note that $d(a,T^{n-1}x) \ge 0 = \alpha d(a,Ta)$ for every $n \in N$, we have

$$\int_0^{d(a,T^n x)} f(t)dt \le \beta \int_0^{d(a,T^{n-1}x)} f(t)dt \le \beta^2 \int_0^{d(a,T^{n-2}x)} f(t)dt \le \dots \le \beta^n \int_0^{d(a,x)} f(t)dt.$$

3. Example and applications

In this section, we give some remarks and examples which clarify the connection between our result and the classical ones in the literature. As an application, the existence of a continuous solution for an integral equation is obtained.

Remark 3.1 Theorem 2.1 is a generalization of theorem 1.3. Letting $f(t) = 1$ for each $t \ge 0$ in theorem 2.1, we have $\int_0^{d(Tx,Ty)} f(t)dt = d(Tx,Ty) \le \beta d(x,y) = \beta \int_0^{d(x,y)} f(t)dt$. The converse is not true as we will see in example 3.1.

Remark 3.2 Theorem 2.1 is a generalization of theorem 1.2. The converse is not true as we will see the example 3.1.

Example 3.1 Let $X : \{(0,0),(5,6),(5,4),(0,4)\} \cup \{(n,0) : n \in \mathbb{N}\} \cup \{(n+12,n+13) : n \in \mathbb{N}\}$ and its metric defined by $d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2,y_2|$. Define a mapping T on X by

$$T((x_1,x_2)) = \begin{cases} (x_1,0), & x_1 \leqslant x_2 \\ \\ (0,x_2), & x_2 < x_1 \end{cases} \tag{1}$$

Then T satisfies the assumptions of Theorem 2.1 with $f(t) = t^t(1+\ln t)$ for $t > 0$, $f(0) = 0$, $\alpha = 5/12$ $\beta = 1/2$ while T is not satisfies the assumptions of Theorem 1.2. First note that, $\int_0^{d(Tx,Ty)} f(t)dt \leq \frac{1}{2}\int_0^{d(x,y)} f(t)dt$ if $(x,y) \neq ((5,6),(5,4))$ and $(x,y) \neq ((5,4),(5,6))$. In this context one has $\int_0^x f(t)dt = x^x$. Let $d(Tx,Ty) = n$ and $d(x,y) = m$. It is clear that $n < m$ if $(x,y) \neq ((5,6),(5,4))$ and $(x,y) \neq ((5,4),(5,6))$. Then we have

$$\int_0^{d(Tx,Ty)} f(t)dt = \int_0^n f(t)dt = n^n < \frac{1}{2}m^m = \frac{1}{2}\int_0^m f(t)dt = \frac{1}{2}\int_0^{d(x,y)} f(t)dt,$$

because

$$\frac{n^n}{m^m} = \frac{n^n}{m^{n+k}} = (\frac{n}{m})^n \frac{1}{m^k} < \frac{1}{2}.$$

On the other hands, since $\alpha d((5,6),T(5,4)) > 5/2 > 2$ and $\alpha d((5,4),T(5,6)) > 25/12 > 2$, T satisfies the assumption in Theorem 2.1.

Remark 3.3 Let $x = (n+12, n+13)$ and $y = (n,0)$. Then in exampel 3.1, we have $\frac{d(Tx,Ty)}{d(x,y)} = \frac{n+12}{n+25}$ and so $\sup_{x,y \in X \setminus \{(5,6),(5,4)\}} \frac{d(Tx,Ty)}{d(x,y)} = 1$. Thus T is not a contraction mapping.

Let us consider the following integral equation:

$$x(t) = g(t) + \int_0^t K(s,x(s))ds,\ t \in [0,1]\ (2)$$

we are going to give existence and uniqueness results for the solution of the integral equation using theorem 2.1. Let us consider $X := (C([0,1], \|.\|_\infty)$.

Theorem 3.1 *Consider the integral equation (2). Suppose*

i) $K : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : [0,1] \to \mathbb{R}^n$ *are continuous;*

ii) there exist $\alpha \in (0, \frac{1}{2}]$, $0 < L < 1$ *such that* $\alpha |x(t) - g(t) - \int_0^t K(s,x(s))ds| \leq |x(t) - y(t)|$ *implies*

$$\int_0^{|K(t,x(t)) - K(t,y(t))|} f(\lambda)d\lambda \leq L \int_0^{|x(t)-y(t)|} f(\lambda)d\lambda$$

for all $x, y \in X$ *and* $f : [0,\infty) \to (0,\infty)$ *is a Lebesgue integrable mapping which is summable and for each* $\varepsilon > 0$, $\int_0^\varepsilon f(\lambda)d\lambda > 0$. *Then the integral equation (2), have a unique solution.*

Proof. Let $T : X \to X$, $x \mapsto T(x)$, where

$$T(x)(t) = \int_0^t K(s,x(s))ds + g(t),\ t \in [0,1].$$

In this way, the integral equation (2) can be rewritten as $x = T(x)$. Next, we show that T satisfies the conditions of Theorem 2.1. Let $x, y \in X$ and $\alpha|x(t) - g(t) - \int_0^t K(s, x(s))ds| \leq |x(t) - y(t)|$. Then

$$\alpha\|x - Tx\|_\infty \leq \|x - y\|_\infty$$

implies

$$\begin{aligned}
\int_0^{\|Tx-Ty\|_\infty} f(\lambda)d\lambda &= \int_0^{\max_{t\in[0,1]}|Tx(t)-Ty(t)|} f(\lambda)d\lambda \\
&\leq \int_0^{\max_{t\in[0,1]}\int_0^t|K(s,x(s))-K(s,y(s))|ds} f(\lambda)d\lambda \\
&\leq \int_0^{\max_{s\in[0,1]}|K(s,x(s))-K(s,y(s))|} f(\lambda)d\lambda \\
&\leq L\int_0^{\|x-y\|_\infty} f(\lambda)d\lambda.
\end{aligned}$$

Now Theorem 2.1 shows that there exists $x_0 \in X$ such that $Tx_0 = x_0$ and so

$$x_0(t) = Tx_0(t) = \int_0^t K(s, x(s))ds + g(t).$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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