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WEAK AND STRONG CONVERGENCE OF HYBRID ITERATIVE METHODS FOR FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, a composite hybrid iteration method is investigated for approximating fixed points of asymptotically nonexpansive mappings. Weak and strong convergence theorems are established in arbitrary Banach spaces. The results presented in this paper mainly improved the corresponding results in Miao and Li [Weak and strong convergence an iterative method for nonexpansive mappings in Hilbert spaces, Appl. Anal. Discrete Math. 2 (2008), 197-204].

Keywords: asymptotically nonexpansive mapping; fixed points; hybrid iteration method; iterative approximation; arbitrary Banach spaces.

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1. Introduction

Let E be an arbitrary Banach space and let $T : E \longrightarrow E$ be a mapping. Recall that T is said to be L-Lipschitzian if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in E. \tag{1.1}$$

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T is said to be nonexpansive if $L=1$ in (1.1). T is s and asymptotically nonexpansive if there exists a sequence $\{k_n\} \subseteq [1,\infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x-y\|, \quad \forall x,y \in E. \tag{1.2}$$

T is uniformly L-Lipschitizian if there exists $L>0$ such that

$$\|T^n x - T^n y\| \le L\|x-y\|, \quad \forall n \ge 1 \text{ and } \forall x,y \in E. \tag{1.3}$$

The class of asymptotically nonexpansive mappings was introduced by Geobel and Kirk [1]. They proved that if K is a nonempty closed convex and bounded subset of a uniformly convex Banach space, then every assymptotically nonexpansive self-mapping of K has a fixed point. It is obvious that every nonexpansive mapping is asymptotically nonexpansive with the constant sequence $\{k_n\} = \{1\}$ and every asymptotically nonexpansive map is uniformly L-Lipschitizian.

Let H be a Hilbert space, A mapping $T : H \longrightarrow H$ is said to be η-strongly monotone if there exists $\eta > 0$ such that

$$\langle Tx - Ty, x-y\rangle \le \eta\|x-y\|^2, \quad \forall x,y \in H. \tag{1.4}$$

Note that the class of asymptotically nonexpansvie mappings is a natural generalization of the important class of nonexpansive mappings. Iterative techniques for approximating fixed points of asymptotically nonexpansive mappings have been extensively studied based on the Mann [2] iteration by several authors; see, for example, [3, 4, 5, 6]). Recently, Xu and Kim [7], Yamada [8] and Wang [9] introduced the hybrid iteration method which has been used in solving certain variational inequalities.

Let H be a Hilbert space, $T : H \longrightarrow H$ a nonexpansive mapping with $F(T) = \{x \in H : Tx = x\} \ne \emptyset$ and $F : H \longrightarrow H$ an $\eta-$strongly monotone and Lipschitz mapping. Let $\{a_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{\lambda_n\}_{n=1}^{\infty}$ be real sequences in $[0,1)$, and $\mu > 0$. The sequence $\{x_n\}_{n=1}^{\infty}$ is generated from an arbitrary $x_1 \in H$ by

$$x_{n+1} = a_n x_n + (1-a_n)T^{\lambda_{n+1}} x_n, \quad n \ge 1, \tag{1.5}$$

where $T^{\lambda_{n+1}}x = Tx - \lambda_{n+1}\mu F(Tx), \mu > 0$. Using the result, Wang [9] obtained weak and strong convergence of (1.5) to the fixed point of T. Observe that if either $\lambda_n = 0, \forall n \ge 1$ or $F \equiv 0$, then

(1.5) reduces to the well known Mann iteration method

$$x_{n+1} = a_n x_n + (1 - a_n) T x_n, \quad n \geq 1, \tag{1.6}$$

which has been used by several authors for the approximation of fixed points of operators or operator equations.

Motivated by the work of Wang [9] and earlier results of Xu and Kim [7] and Yamada [8], Miao and Li generalized (1.5) by developing the following composite hybrid iteration process:

Let H be a Hilbert space, $T : H \longrightarrow H$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f(resp.g) : H \longrightarrow H$ an $\eta_f(resp.\eta_g)-$strongly monotone and $k_f(resp.k_g)-$Lipschitzian mappings. For any $x_1 \in H, \{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = & a_n x_n + (1 - a_n) T_f^{\lambda_{n+1}} y_n, \\ y_n = & b_n x_n + (1 - b_n) T_g^{\beta_n} x_n, \quad n \geq 1, \end{cases} \tag{1.7}$$

where

$$\begin{aligned} T_f^{\lambda_{n+1}} x &= Tx - \lambda_{n+1} \mu_f f(Tx), \quad \mu_f \geq 0, \quad \forall x \in H, \\ T_g^{\beta_n} x &= Tx - \beta_n \mu_g g(Tx), \quad \mu_g \geq 0, \quad \forall x \in H, \end{aligned}$$

and $\{a_n\} \in (0,1), \{b_n\} \in (0,1)$ and $\lambda_n \in [0,1), \beta_n \in [0,1)$ satisfy the following conditions:

(i) $\alpha \leq a_n \leq 1 - \alpha, \beta \leq b_n \leq 1 - \beta$, for some $\alpha, \beta \in (0, \frac{1}{2})$;

(ii) $\Sigma_{n=1}^{+\infty} \lambda_n < +\infty, \quad \Sigma_{n=1}^{+\infty} \beta_n < +\infty$;

(iii) $0 < \mu_f < \frac{2\eta_f}{k_f^2}, 0 < \mu_g < \frac{2\eta_g}{k_g^2}$.

Observe if $b_n = 1, \forall n \geq 1$, (1.7) reduces to (1.5) and if $b_n = 1, \forall n \geq 1, \lambda_n = 0$ or $f \equiv 0$, (1.7) reduces to the well known Mann iteration method. Using (1.7), Miao and Li [10] proved the following:

The iterative process $\{x_n\}$ as in (1.7) satisfies:

(1) $\lim_{n \to +\infty} \|x_n - p\|$ exists for each $p \in F(T)$,

(2) $\lim_{n \to +\infty} \|x_n - Tx_n\| = 0$.

The sequence $\{x_n\}$ converges weakly to a fixed point of T. Let T be completely continuous or demicompact, The iterative process $\{x_n\}$ converges strongly to a fixed point of T.

It is our purpose in this paper to extend the above results from a Hilbert spaces to arbitrary Banach spaces and from nonexpansive mappings to the more general asymptotically nonexpansive mappings. Our results are much more general and also more applicable than the results of Miao and Li [10] because the strong monotonicity condition imposed on f and g is not required in our results.

2. Preliminaries

Let E be a Banach space. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at a point $p \in D(T)$ if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in E which converges weakly to a point $x \in E$ and $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to p, then $Tx = p$. Furthermore, T is said to be demicompact if whenever $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $D(T)$ such that $\{x_n - Tx_n\}_{n=1}^{\infty}$ converges strongly, then $\{x_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly. T is said to satisfy condition (A) if $F(T) \neq \emptyset$ and there exist nondecreasing functions $f : [0,\infty) \to [0,\infty)$ and $g : [0,\infty) \to [0,\infty)$ with $f(0) = 0, g(0) = 0, f(t) > 0$ and $g(t) > 0$ $\forall t \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x,F(T)))$ and $\|x - Tx\| \geq g(d(x,F(T)))$ for all $x \in D(T)$ where $d(x,F(T)) := inf\{\|x-p\| : p \in F(T)\}$.

Lemma 2.1 [6] *Let E be a uniformly convex Banach space and K a nonempty closed convex (not neccessarily bounded) subset of E. $T : K \longrightarrow K$ a asymptotically nonexpansive mapping with $\{k_n\} \subseteq [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ and $F(T)$ nonempty fixed point set. Then $(I-T)$ is demiclosed at 0, i.e. for any sequence $\{x_n\}$ in K s.t. $\{x_n\}$ converges weakly to p and $\{x_n - Tx_n\}$ converges strongly to 0, Then $(I-T)(p) = 0$.*

Lemma 2.2 ([6], see also [11]). *Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of non-negative real numbers satisfying the inequality $a_{n+1} \leq (1+\delta_n)a_n + b_n, n \geq 1$. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.3 ([12], see also [13]). *Let E be an arbitrary normed space and let $\{t_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions:*

(i) $0 \leq t_n \leq t \leq 1$, *and for some* $t \in (0,1)$,

(ii) $\Sigma_{n=1}^{+\infty} t_n = +\infty$,

Let $\{u_n\}_{n=1}^{\infty}$ *and* $\{v_n\}_{n=1}^{\infty}$ *be two sequences in* E *such that*

(iii) $u_{n+1} = (1-t_n)u_n + t_n v_n, \quad \forall n \geq 1$

(iv) $\lim_{n\to\infty} \|u_n\| = d$ *for some* $d \in [0,\infty)$,

(v) $\limsup \|v_n\| \leq d$,

(vi) $\{\Sigma_{j=1}^{n} t_j v_j\}_{n=1}^{+\infty}$ *is bounded. Then* $d = 0$.

Definition 2.1. A bounded convex subset K of a normed space E is said to have normal structure if every non-trivial convex subset C of K contains at least one non-diametrical point, that is, there exists $x^0 \in E$ such that $\sup\{\|x^0 - x\| : x \in C\} < \sup\{\|x-y\| : x,y \in C\} = d(C)$ where $d(C)$ is the diametre of C.

Every uniformly convex Banach space E has a normal structure and every compact convex subset K of a Banach space E has normal structure.

Lemma 2.4 [14] *Let* E *be a real Banach space with normal structure* $N(E) > \max(1,\varepsilon_0)$, $\varepsilon_0 > 0$, K *a nonempty bounded closed convex subset of* E *and* $T : K \to K$ *a uniformly* $L-$*Lipschitzian mapping with* $L < \alpha$, $\alpha > 1$. *Then* T *has a fixed point.*

3. Main Results

Lemma 3.1. *Let* E *be a normed space and let* $T : E \to E$ *be a uniformly* $L-$*Lipschitzian mapping with nonempty fixed point set* $F(T)$. *Let* $f(resp.g) : E \longrightarrow E$ *be* $L_f(resp.L_g)-$*Lipschitzian mappings. For any* $x_1 \in E$, *generate* $\{x_n\}$ *by*

$$\begin{cases} x_{n+1} &=& a_n x_n + (1-a_n) T_f^{\lambda_{n+1}} y_n, \\ y_n &=& b_n x_n + (1-b_n) T_g^{\beta_n} x_n, \quad n \geq 1, \end{cases}$$

where

$$\begin{aligned} T_f^{\lambda_{n+1}} x &= T^n x - \lambda_{n+1} \mu_f f(T^n x), \quad \mu_f \geq 0, \quad \forall x \in E, \\ T_g^{\beta_n} x &= T^n x - \beta_n \mu_g g(T^n x), \quad \mu_g \geq 0, \quad \forall x \in E, \end{aligned}$$

and $\{a_n\}, \{b_n\}$ *are sequences in* $(0,1)$ *and* $\{\lambda_n\}, \{\beta_n\}$ *are sequences in* $[0,1)$ *such that* $\Sigma_{n=1}^{+\infty}\lambda_n < +\infty$ *and* $\Sigma_{n=1}^{+\infty}\beta_n < +\infty$. *Then*

$$\begin{aligned}\|x_n - Tx_n\| \leq\ & \|x_n - T^n x_n\| + \{L^2 + (1+L)L^2(1+\ell\mu_f\lambda_n)(1+\mu_g\ell\beta_{n-1}) \\ & +\ell(1+L)L\mu_f\lambda_n\}\|x_{n-1} - T^{n-1}x_{n-1}\| \\ & +\{(1+L)L^2(1+\ell\mu_f\lambda_n)\mu_g[\ell\|x_{n-1}-p\| + \|g(p)\|]\}\beta_{n-1} \\ & +\{(1+L)L\mu_f[\ell\|x_{n-1}-p\| + \|f(p)\|]\}\lambda_n.\end{aligned}$$

Proof. For $p \in F(T)$, set $\Phi_n = \|x_n - T^n x_n\|$ and $\ell = \max\{L_f, L_g\}$. Then

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - T^n x_n\| + L\|T^{n-1}x_n - x_n\| \\ &\leq \Phi_n + L^2\|x_n - x_{n-1}\| + L\|x_n - T^{n-1}x_{n-1}\| \\ &\leq \Phi_n + L^2\Phi_{n-1} + L^3\|y_{n-1} - x_{n-1}\| + \ell L^3\lambda_n\mu_f\|y_{n-1} - x_{n-1}\| \\ &\quad + L^2\lambda_n\mu_f\|f(T^{n-1}x_{n-1})\| + (1-b_{n-1})L^2\|T_g^{\beta_{n-1}}x_{n-1} - x_{n-1}\| \\ &\quad + \ell L^2\lambda_n\mu_f\|y_{n-1} - x_{n-1}\| + L\lambda_n\mu_f\|f(T^{n-1}x_{n-1})\| \\ &\leq \Phi_n + L^2\Phi_{n-1} + L^2(1+L)(1+\ell\lambda_n\mu_f)\|T_g^{\beta_{n-1}}x_{n-1} - x_{n-1}\| \\ &\quad + (1+L)L\lambda_n\mu_f\|f(T^{n-1}x_{n-1})\|.\end{aligned} \tag{3.1}$$

On the other hand, we have

$$\begin{aligned}\|T_g^{\beta_{n-1}}x_{n-1} - x_{n-1}\| &= \|T^{n-1}x_{n-1} - \beta_{n-1}\mu_g g(T^{n-1}x_{n-1}) - x_{n-1}\| \\ &\leq \Phi_{n-1} + \beta_{n-1}\mu_g\|g(T^{n-1}x_{n-1})\| \\ &\leq \Phi_{n-1} + \beta_{n-1}\mu_g\ell\|T^{n-1}x_{n-1} - x_{n-1}\| + \beta_{n-1}\mu_g\|g(x_{n-1})\| \\ &\leq (1+\beta_{n-1}\mu_g\ell)\Phi_{n-1} + \beta_{n-1}\mu_g\ell\|x_{n-1} - p\| + \beta_{n-1}\mu_g\|g(p)\|\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\|f(T^{n-1}x_{n-1})\| &\leq \ell\Phi_{n-1} + \|f(x_{n-1})\| \\ &\leq \ell\Phi_{n-1} + \ell\|x_{n-1} - p\| + \|f(p)\|.\end{aligned} \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we have

$$\begin{aligned}\|x_n - Tx_n\| &\leq \Phi_n + L^2\Phi_{n-1} + L^2(1+L)(1+\ell\lambda_n\mu_f)\{(1+\beta_{n-1}\mu_g\ell)\Phi_{n-1} \\ &\quad + \beta_{n-1}\mu_g\ell\|x_{n-1}-p\| + \beta_{n-1}\mu_g\|g(p)\|\} \\ &\quad + (1+L)L\lambda_n\mu_f\{\ell\Phi_{n-1} + \ell\|x_{n-1}-p\| + \|f(p)\|\} \\ &= \|x_n - T^n x_n\| + \{L^2 + (1+L)L^2(1+\ell\mu_f\lambda_n)(1+\mu_g\ell\beta_{n-1}) \qquad (3.4) \\ &\quad + \ell(1+L)L\mu_f\lambda_n\}\|x_{n-1} - T^{n-1}x_{n-1}\| \\ &\quad + \{(1+L)L^2(1+\ell\mu_f\lambda_n)\mu_g[\ell\|x_{n-1}-p\| + \|g(p)\|]\}\beta_{n-1} \\ &\quad + \{(1+L)L\mu_f[\ell\|x_{n-1}-p\| + \|f(p)\|]\}\lambda_n.\end{aligned}$$

Theorem 3.1. *Let E be an arbitrary Banach space, $T : E \longrightarrow E$ a asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and sequence $\{k_n\} \subseteq [1,\infty)$ such that $\Sigma_{n=1}^{+\infty}(k_n - 1) < +\infty$, and $f(resp.g) : E \longrightarrow E$ are $L_f(resp.L_g)-$Lipschitzian mappings. For any $x_1 \in E, \{x_n\}$ is generate by*

$$\begin{cases} x_{n+1} = & a_n x_n + (1-a_n)T_f^{\lambda_{n+1}} y_n \\ y_n = & b_n x_n + (1-b_n)T_g^{\beta_n} x_n, \end{cases} \qquad (3.5)$$

where

$$\begin{aligned} T_f^{\lambda_{n+1}}x &= T^n x - \lambda_{n+1}\mu_f f(T^n x), \quad \mu_f \geq 0, \quad \forall x \in E, \\ T_g^{\beta_n}x &= T^n x - \beta_n\mu_g g(T^n x), \quad \mu_g \geq 0, \quad \forall x \in E, \end{aligned}$$

and $\{a_n\} \in (0,1), \{b_n\} \in (0,1)$ and $\lambda_n \in [0,1), \beta_n \in [0,1)$ satisfy the following conditions:

(i) $0 < \alpha < a_n < 1$, *for some* $\alpha \in (0,1)$*;*

(ii) $\Sigma_{n=1}^{+\infty}(1-a_n) = +\infty$*;*

(iii) $\Sigma_{n=1}^{+\infty}(1-b_n) < +\infty$*;*

(iv) $\Sigma_{n=1}^{+\infty}\lambda_n < +\infty$*;*

(v) $\Sigma_{n=1}^{+\infty}\beta_n < +\infty$.

Then we have

(a) $\lim_{n\to+\infty}\|x_n - p\|$ *exists for each* $p \in F(T)$*;*

(b) $\lim_{n\to+\infty}\|x_n - Tx_n\| = 0$*;*

(c) $\{x_n\}$ *converges strongly to a fixed point of T if and only if* $\liminf_{n\to+\infty} d(x_n, F(T)) = 0$.

Proof. Let $p \in F(T)$ be an arbitrary and set $\ell := \max\{L_f, L_g\}$. Then

$$\begin{aligned} \|x_{n+1}-p\| &= \|a_n(x_n-p)+(1-a_n)(T^ny_n-p)-(1-a_n)\lambda_{n+1}\mu_f f(T^ny_n)\| \\ &\le a_n\|x_n-p\|+(1-a_n)\|T^ny_n-p\|+(1-a_n)\lambda_{n+1}\mu_f\|f(T^ny_n)\| \\ &\le a_n\|x_n-p\|+(1-a_n)k_n\|y_n-p\|+(1-a_n)\lambda_{n+1}\mu_f\{\|f(T^ny_n)-f(p)\|+\|f(p)\|\} \\ &\le a_n\|x_n-p\|+\{(1-a_n)k_n+(1-a_n)\lambda_{n+1}\mu_f k_n\ell\}\|y_n-p\| \\ &\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \|y_n-p\| &= \|b_n(x_n-p)+(1-b_n)(T^nx_n-p)-(1-b_n)\beta_n\mu_g g(T^nx_n)\| \\ &\le b_n\|x_n-p\|+(1-b_n)\|T^nx_n-p\|+(1-b_n)\beta_n\mu_g\|g(T^nx_n)\| \\ &\le b_n\|x_n-p\|+(1-b_n)k_n\|x_n-p\|+(1-b_n)\beta_n\mu_g\{\|g(T^nx_n)-g(p)\|+\|g(p)\|\} \\ &\le \{b_n+(1-b_n)k_n+(1-b_n)\beta_n\mu_g k_n\ell\}\|x_n-p\|+(1-b_n)\beta_n\mu_g\|g(p)\|. \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), we find that

$$\begin{aligned} &\|x_{n+1}-p\| \\ &= a_n\|x_n-p\|+\{(1-a_n)k_n+(1-a_n)\lambda_{n+1}\mu_f k_n\ell\}\{b_n+(1-b_n)k_n \\ &\quad +(1-b_n)\beta_n\mu_g k_n\ell\}\|x_n-p\| \\ &\quad +\{(1-a_n)k_n+(1-a_n)\lambda_{n+1}\mu_f k_n\ell\}(1-b_n)\beta_n\mu_g\|g(p)\|]+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\ &= [1-(1-a_n)+(1-a_n)b_nk_n+(1-a_n)(1-b_n)k_n^2+(1-a_n)b_n\lambda_{n+1}\mu_f k_n\ell \qquad (3.8) \\ &\quad +(1-a_n)(1-b_n)k_n^2\lambda_{n+1}\mu_f\ell+(1-a_n)(1-b_n)k_n^2\beta_n\mu_g\ell \\ &\quad +(1-a_n)(1-b_n)\lambda_{n+1}\beta_n\mu_f\mu_g k_n^2\ell^2]\|x_n-p\| \\ &\quad +\{(1-a_n)k_n+(1-a_n)\lambda_{n+1}\mu_f k_n\ell\}(1-b_n)\beta_n\mu_g\|g(p)\|]+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\ &= [1+\delta_n]\|x_n-p\|+\sigma_n, \end{aligned}$$

where

$$\delta_n = (1-a_n)(k_n-1)\{(1-b_n)k_n+1\}+(1-a_n)b_n\lambda_{n+1}\mu_f k_n \ell$$
$$+(1-a_n)(1-b_n)k_n^2\lambda_{n+1}\mu_f\ell+(1-a_n)(1-b_n)k_n^2\beta_n\mu_g\ell$$
$$+(1-a_n)(1-b_n)\lambda_{n+1}\beta_n\mu_f\mu_g k_n^2\ell^2$$

and

$$\sigma_n = \{(1-a_n)k_n+(1-a_n)\lambda_{n+1}\mu_f k_n\ell\}(1-b_n)\beta_n\mu_g\|g(p)\|]+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|$$
$$=(1-a_n)(1-b_n)\beta_n\mu_g k_n\|g(p)\|+(1-a_n)(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_g k_n\ell\|g(p)\|$$
$$+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|.$$

Since $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}\sigma_n<\infty$, it follows from Lemma 2.2 that $\lim_{n\to\infty}\|x_n-p\|$ exists. This completes the proof of (a). Consequently, $\{\|x_n-p\|\}$ is bounded. Since $\{x_n\}$ is bounded, then there exists $M>0$ such that $\|x_n-p\|\leq M$. Observe that

$$\begin{aligned}\|x_{n+1}-T^nx_{n+1}\| &\leq a_n\|x_n-T^nx_{n+1}\|+(1-a_n)\|T^ny_n-T^nx_{n+1}\|+(1-a_n)\lambda_{n+1}\mu_f\|f(T^ny_n)\|\\
&\leq a_n\|x_n-T^nx_{n+1}\|+(1-a_n)k_n\|y_n-x_{n+1}\|+(1-a_n)\lambda_{n+1}\mu_f k_n\ell\|y_n-p\|\\
&\quad+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|\\
&\leq [a_n+(1-a_n)k_n]\|x_n-x_{n+1}\|+a_n\|x_{n+1}-T^nx_{n+1}\|\\
&\quad+(1-a_n)\|(1-b_n)(T^nx_n-x_n)-(1-b_n)\beta_n\mu_g g(T^nx_n)\|\\
&\quad+(1-a_n)\lambda_{n+1}\mu_f k_n\ell\|y_n-p\|+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|\\
&\leq [a_n+(1-a_n)k_n]\|x_n-x_{n+1}\|+a_n\|x_{n+1}-T^nx_{n+1}\|\\
&\quad+(1-a_n)(1-b_n)\|T^nx_n-x_n\|+(1-a_n)(1-b_n)\beta_n\mu_g k_n\ell\|x_n-p\|\\
&\quad+(1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\|+(1-a_n)\lambda_{n+1}\mu_f k_n\ell\|y_n-p\|\\
&\quad+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|.\end{aligned} \tag{3.9}$$

Substitute (3.7) into (3.9) to obtain

$$\begin{aligned}\|x_{n+1}-T^nx_{n+1}\| \le{}& [a_n+(1-a_n)k_n]\|x_n-x_{n+1}\|+a_n\|x_{n+1}-T^nx_{n+1}\| \\ &+(1-a_n)(1-b_n)\|T^nx_n-x_n\|+(1-a_n)(1-b_n)\beta_n\mu_gk_n\ell\|x_n-p\| \\ &+(1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\|+(1-a_n)\lambda_{n+1}\mu_fk_n\ell\{b_n+(1-b_n)k_n \\ &+(1-b_n)\beta_n\mu_gk_n\ell\}\|x_n-p\| \\ &+(1-a_n)(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_gk_n\ell\|g(p)\|+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-T^nx_{n+1}\| \le{}& \frac{[a_n+(1-a_n)k_n]}{(1-a_n)}\|x_n-x_{n+1}\|+(1-b_n)\|T^nx_n-x_n\| \\ &+(1-b_n)\beta_n\mu_gk_n\ell\|x_n-p\|+(1-b_n)\beta_n\mu_g\|g(p)\| \\ &+\lambda_{n+1}\mu_fk_n\ell\{b_n+(1-b_n)k_n+(1-b_n)\beta_n\mu_gk_n\ell\}\|x_n-p\| \\ &+(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_gk_n\ell\|g(p)\|+\lambda_{n+1}\mu_f\|f(p)\|.\end{aligned} \tag{3.10}$$

Note that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le (1-a_n)\|T^ny_n-x_n\|+(1-a_n)\lambda_{n+1}\mu_f\|f(T^ny_n)\| \\ &\le (1-a_n)(1-b_n)k_n\|x_n-T^nx_n\|+(1-a_n)\|x_n-T^nx_n\| \\ &\quad+(1-a_n)(1-b_n)\beta_n\mu_gk_n\ell\|x_n-p\|+(1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\| \\ &\quad+(1-a_n)\lambda_{n+1}\mu_fk_n\ell\|y_n-p\|+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|.\end{aligned} \tag{3.11}$$

Substitute (3.7) into (3.11) to obtain

$$\begin{aligned}\|x_{n+1}-x_n\| \le{}& (1-a_n)(1-b_n)k_n\|x_n-T^nx_n\|+(1-a_n)\|x_n-T^nx_n\| \\ &+(1-a_n)(1-b_n)\beta_n\mu_gk_n\ell\|x_n-p\|+(1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\|\end{aligned}$$

$$
\begin{aligned}
&+(1-a_n)\lambda_{n+1}\mu_f k_n\ell[\{b_n+(1-b_n)k_n+(1-b_n)\beta_n\mu_g k_n\ell\}\|x_n-p\| \\
&+(1-b_n)\beta_n\mu_g\|g(p)\|]+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
=&\,(1-a_n)[(1-b_n)k_n+1]\|x_n-T^nx_n\|+(1-a_n)(1-b_n)\beta_n\mu_g k_n\ell\|x_n-p\| \\
&+(1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\| \\
&+(1-a_n)\lambda_{n+1}\mu_f k_n\ell\{b_n+(1-b_n)k_n+(1-b_n)\beta_n\mu_g k_n\ell\}\|x_n-p\| \\
&+(1-a_n)(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_g k_n\ell\|g(p)\|+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|.
\end{aligned} \tag{3.12}
$$

Substituting (3.12) into (3.10), we find that

$$
\begin{aligned}
&\|x_{n+1}-T^nx_{n+1}\| \\
&\le[1+(1-b_n)a_nk_n+(1-a_n)(1-b_n)k_n^2+(1-a_n)(k_n-1)+(1-b_n)]\|x_n-T^nx_n\| \\
&\quad+[a_n+(1-a_n)k_n+1]\{(1-b_n)\beta_n\mu_g k_n\ell+\lambda_{n+1}\mu_f k_n\ell\{b_n+(1-b_n)k_n \\
&\quad+(1-b_n)\beta_n\mu_g k_n\ell\}\}\|x_n-p\|+\beta_n\mu_g\{1+\lambda_{n+1}\mu_f k_n\ell\}[(1-b_n)[a_n+(1-a_n)k_n]+1]\|g(p)\| \\
&\quad+\lambda_{n+1}\mu_f[a_n+(1-a_n)k_n+1]\|f(p)\|.
\end{aligned}
$$

It follows that

$$\|x_{n+1}-T^nx_{n+1}\|\le[1+\omega_n]\|x_n-T^nx_n\|+\varphi_n,$$

where $\omega_n=(1-b_n)a_nk_n+(1-a_n)(1-b_n)k_n^2+(1-a_n)(k_n-1)+(1-b_n)$ and

$$
\begin{aligned}
\varphi_n=&\,[a_n+(1-a_n)k_n+1]\{(1-b_n)\beta_n\mu_g k_n\ell+\lambda_{n+1}\mu_f k_n\ell\{b_n+(1-b_n)k_n \\
&+(1-b_n)\beta_n\mu_g k_n\ell\}\}M+\beta_n\mu_g\{1+\lambda_{n+1}\mu_f k_n\ell\}[(1-b_n)[a_n+(1-a_n)k_n]+1]\|g(p)\| \\
&+\lambda_{n+1}\mu_f[a_n+(1-a_n)k_n+1]\|f(p)\|.
\end{aligned}
$$

From conditions (iii)- (vi), it follows that $\sum_{n=1}^{\infty}\omega_n<\infty$ and $\sum_{n=1}^{\infty}\varphi_n<\infty$. Also it follows from Lemma 2.2 that $\lim_{n\to\infty}\|x_n-T^nx_n\|$ exists. Let $\lim_{n\to\infty}\|x_n-T^nx_n\|=d$ and set $u_n=x_n-T^nx_n$. It follows that

$$u_{n+1}=(1-t_n)u_n+t_nv_n, \tag{3.13}$$

where $t_n = 1 - a_n$, $\quad v_n = \frac{1}{(1-a_n)}(T^n x_n - T^n x_{n+1}) + (T^n y_n - T^n x_n) - \lambda_{n+1}\mu_f f(T^n y_n)$.

$$\begin{aligned}
\|v_n\| &\le \frac{1}{(1-a_n)}\|T^n x_n - T^n x_{n+1}\| + \|T^n y_n - T^n x_n\| + \lambda_{n+1}\mu_f\|f(T^n y_n)\| \\
&\le \frac{k_n}{(1-a_n)}\|x_n - x_{n+1}\| + k_n\|y_n - x_n\| + \lambda_{n+1}\mu_f\|f(T^n y_n) - f(p) + f(p)\| \\
&\le \frac{k_n}{(1-a_n)}\|x_n - x_{n+1}\| + k_n\|x_n - p\| + (1+\lambda_{n+1}\mu_f\ell)k_n\|y_n - p\| + \lambda_{n+1}\mu_f\|f(p)\|.
\end{aligned} \tag{3.14}$$

Substitute (3.7) and (3.12) into (3.14) to obtain

$$\begin{aligned}
\|v_n\| \quad \le \quad & [1 + (1-b_n)k_n^2 + (k_n - 1)]\|x_n - T^n x_n\| \\
& + \{[\lambda_{n+1}\mu_f k_n^2\ell + (1+\lambda_{n+1}\mu_f)k_n]\{b_n + (1-b_n)k_n + (1-b_n)\beta_n\mu_g k_n\ell\} \\
& + (1-b_n)\beta_n\mu_g k_n^2\ell + k_n\}\|x_n - p\| + (1-b_n)\beta_n\mu_g k_n[2 + \lambda_{n+1}\mu_f\ell \\
& + \lambda_{n+1}\mu_f k_n\ell]\|g(p)\| + \lambda_{n+1}\mu_f(k_n+1)\|f(p)\|.
\end{aligned}$$

It follows that

$$\|v_n\| \le [1 + \psi_n]\|x_n - T^n x_n\| + \vartheta_n, \tag{3.15}$$

where $\psi_n = (1-b_n)k_n^2 + (k_n - 1)$ and

$$\begin{aligned}
\vartheta_n \quad = \quad & \{[\lambda_{n+1}\mu_f k_n^2\ell + (1+\lambda_{n+1}\mu_f)k_n]\{b_n + (1-b_n)k_n + (1-b_n)\beta_n\mu_g k_n\ell\} \\
& + (1-b_n)\beta_n\mu_g k_n^2\ell + k_n\}M + (1-b_n)\beta_n\mu_g k_n[2 + \lambda_{n+1}\mu_f\ell \\
& + \lambda_{n+1}\mu_f k_n\ell]\|g(p)\| + \lambda_{n+1}\mu_f(k_n+1)\|f(p)\|.
\end{aligned}$$

From conditions (iii)- (vi), $\lim_{n\to\infty}\psi_n = 0$ and $\lim \vartheta_n$ exists. Since $\lim_{n\to\infty}\|x_n - T^n x_n\|$ exists,

$$\|v_n\| \le \|x_n - T^n x_n\| + \psi_n D + Q, D > 0, Q > 0.$$

Therefore, $\limsup_{n\to\infty}\|v_n\| \le d$. Observe that

$$\|\sum_{j=1}^{n} t_j v_j\| \le\le 2k_n\|x_1 - p\| + k_n\|y_1 - p\| + \mu_f K(k_n\sum_{j=1}^{n}\lambda_{j+1} + \mu_g\sum_{j=1}^{n}\beta_j) \le W,$$

$\forall n \ge 1$ and for some $W > 0$. Hence $\{\sum_{j=1}^{n} t_j v_j\}_{n=1}^{\infty}$ is bounded. It follows from Lemma 2.3 that $\lim_{n\to\infty}\|x_n - T^n x_n\| = 0$. Since $\{\|x_n - p\|\}$ is bounded, it also follows from Lemma 3.1 that

$\lim_{n\to\infty}\|v_n\| = \lim_{n\to\infty}\|x_n - Tx_n\| = 0$. This completes the proof of (b). From (3.8), we obtain that $\|x_{n+1} - p\| \le \|x_n - p\| + \xi_n$ where $\xi_n = \delta_n M + \sigma_n$. Hence $d(x_{n+1}, F(T)) \le d(x_n, F(T)) + \xi_n$. Since $\sum_{n=1}^{\infty}\xi_n < \infty$, it follows from Lemma 2.2 that $\lim\limits_{n\to\infty} d(x_n, F(T))$ exists. If x_n converges strongly to a fixed point p of T then $\lim_{n\to\infty}\|x_n - p\| = 0$. Since $0 \le d(x_n, F(T)) \le \|x_n - p\|$, we have $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Conversely, suppose $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, then we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus for arbitrary $\varepsilon > 0$, there exists a positive integer N_1 such that $d(x_n, F(T)) < \frac{\varepsilon}{4}, \forall n \ge N_1$. Furthermore, $\sum_{n=1}^{\infty}\xi_n < \infty$ implies that there exists a positive integer N_2 such that $\sum_{j=n}^{\infty}\xi_j < \frac{\varepsilon}{4}, \forall n \ge N_2$. Choose $N = \max\{N_1, N_2\}$, then $d(x_N, F(T)) < \frac{\varepsilon}{4}$ and $\sum_{j=N}^{\infty}\xi_j < \frac{\varepsilon}{4}$. It follows from Lemma 2.2 that $\forall n, m \ge N$ and for all $p \in F(T)$, we have

$$\begin{aligned}
\|x_n - x_m\| &\le \|x_n - p\| + \|x_m - p\| \\
&\le \|x_N - p\| + \sum_{j=N+1}^{\infty}\xi_j + \|x_N - p\| + \sum_{j=N+1}^{\infty}\xi_j \\
&\le 2\|x_N - p\| + 2\sum_{j=N+1}^{\infty}\xi_j .
\end{aligned}$$

Taking infimum over all $p \in F(T)$, we obtain

$$\|x_n - x_m\| \le 2d(x_N, F(T)) + 2\sum_{j=N+1}^{\infty}\xi_j, \forall n, m \ge N.$$

Thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Suppose $\lim\limits_{n\to\infty} x_n = u$, then since $\lim\limits_{n\to\infty}\|x_n - Tx_n\| = 0$, we have $u \in F(T)$. This completes the proof of Theorem 3.1.

Corollary 3.1. *Let E be a real Banach space with normal structure $N(E) > \max(1, \varepsilon_0)$, $\varepsilon_0 > 0$, K a nonempty bounded closed convex subset of E, and $T : E \longrightarrow E$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subseteq [1, \infty)$ such that $\sum\limits_{n=1}^{+\infty}(k_n - 1) < +\infty$, and $f(resp.g) : E \longrightarrow E$ are $L_f(resp.L_g)-$Lipschitzian mappings, under the hypothesis of Theorem* 3.1, *the iteration scheme (3.5) converges strongly to a fixed point $F(T)$ of T if and only if $\liminf\limits_{n\to+\infty} d(x_n, F(T)) = 0$.*

Proof. Since T is asymptotically nonexpansive uniformly $L-$Lipschitzian, the proof follows from Lemma 3.4 and Theorem 3.1.

Corollary 3.2. *Let E be an arbitrary Banach spaces, $T : E \longrightarrow E$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f(resp.g) : E \longrightarrow E$ are $L_f(resp.L_g)-$Lipschitzian mappings. For any*

$x_1 \in E, \{x_n\}$ is generate by (1.7) satisfying the following conditions; (i) $0 < \alpha < a_n < 1$, for some $\alpha \in (0,1)$; (ii) $\Sigma_{n=1}^{+\infty}(1-a_n) = +\infty$ (iii) $\Sigma_{n=1}^{+\infty}(1-b_n) < +\infty$ (iv) $\Sigma_{n=1}^{+\infty}\lambda_n < +\infty$ (v) $\Sigma_{n=1}^{+\infty}\beta_n < +\infty$. Then $\{x_n\}$ converges strongly to a fixed point $F(T)$ of T if and only if $\liminf_{n\to+\infty} d(x_n, F(T)) = 0$.

Theorem 3.2. *Let E be a uniformly convex Banach space and K a nonempty closed convex (not neccessarily bounded) subset of E. Let $T : K \to K$ an asymptotically nonexpansive mapping with $\{k_n\} \subseteq [1,\infty)$, $\lim_{n\to\infty} k_n = 1$, under the hypothesis of Theorem 3.1, the iteration scheme (3.5) converges weakly to a fixed point of T.*

Proof. From Lemma 2.1, $(I-T)$ is demiclosed at zero, and since the $\lim_{n\to+\infty} \|x_n - p\| = 0$, it follows from standard argument that $\{x_n\}_{n=1}^{\infty}$ converges weakly to a fixed point of T.

Remark 3.1. It follows from Lemma 2.2 and Theorem 3.1 that under the hypothesis of Theorem 3.1, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point p of T if and only if $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ which converges strongly to p . Thus, under the hypothesis of Theorem 3.1, if T is in addition completely continuous or demicompact, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Furthermore, if T satisfies condition (A), then $\liminf_{n\to+\infty} d(x_n, F(T)) = 0$. So under the conditions of Theorem 3.1, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Remark 3.2. Theorems 3.1 and 3.2 and Remark 3.1 extend the results of [10] from Hilbert spaces to arbitrary Banach spaces and respectively from nonexpansive operator to more general asymptotically nonexpansive maps as considered here. Furthermore, the strong monotonicity condition imposed on f and g in [10] is not required in our results.

Remark 3.3. If $b_n = 1$ in (3.5), the results of Osilike, Isiogugu, and Nwokoro [15] become special cases of our results.

Conflict of Interests

The authors declare that there is no conflict of interests.

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