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WEAK AND STRONG CONVERGENCE OF THE ISHIKAWA ITERATIVE SEQUENCE TO FIXED POINTS OF LIPSCHITZ PSEUDOCONTRACTIVE MAPS IN HILBERT SPACES

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Abstract. In this paper, we study the weak and strong convergence of the Ishikawa iterative sequence to a fixed point of a Lipschitz pseudocontrative mapping in a Hilbert space. We do not require any compactness type assumptions either on the mapping or its domain. Furthermore, we do not need to compute for closed convex subsets, C_n, of the Hilbert space.

Keywords: Fixed Points; Pseudocontractive Maps; Ishikawa Scheme; Strong Convergence; Hilbert Spaces.

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1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to C$ is called:

(i) Lipschitzian if there exists $L > 0$ such that

$$||Tx - Ty|| \leq L||x - y||,$$

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for all $x, y \in C$. If $L < 1$, then T is called a contraction. If $L = 1$, then T is called nonexpansive.
(ii) Pseudocontractive if

$$||x-y|| \leq ||(1+s)(x-y) - s(Tx-Ty)||$$

for all $x, y \in C$ and $s > 0$, or equivalently

$$\langle Tx - Ty, x - y\rangle \leq ||x-y||^2 \tag{1.1}$$

for all $x, y \in C$. Let $A : D(A) \subseteq E \to E$ be a map. Then A is called accretive if

$$||x-y|| \leq ||x-y+s(Ax-Ay)||. \tag{1.2}$$

for all $x, y \in D(A)$ and $s > 0$. We immediately observe from (1.1) and (1.2) that A is accretive if and only if $T := I - A$ is pseudocontractive, where I denotes the identity operator. Thus, the zeros of accretive operators corresponds to the fixed points of pseudocontrative maps.

2. Preliminaries

The notion of accretive operators was independently introduced in 1967 by Browder [2] and Kato [1]. An early fundamental result due to Browder, states that the initial value problem

$$\frac{du}{dt} + Au = 0, u(0) = u_0$$

is solvable if A is locally Lipschitzian and accretive on E. Therefore, the importance of research into the fixed point theory of pseudocontractive maps cannot be over-emphasized (given its firm connection with accretive maps). Many authors (see e.g [3], [5], [10], [11], [14], [17]) have contributed considerably to this end.

It is well known that if $T : E \to E$ is a contraction map, then the Picard's iterative sequence, starting from an arbitrary $x_0 \in E$, given by

$$x_{n+1} = Tx_n \tag{2.1}$$

for all $n \geq 0$, converges to the unique fixed point of T. If however, T is a nonexpansive map, then (2.1) is not guaranteed to converge to a fixed point of T, even on a compact subset of E. Observe that if C is the unit disc in $\Re^2$ (which is compact) and T is its rotation about the origin, then T is easily shown to be nonexpansive and has 0 as its unique fixed point. Starting from an x_0 on the circumference, (2.1) does not converge to the fixed point of T. Krasnoselskii [18] showed that instead of (2.1), if we consider the averaging sequence

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n)$$

for all $n \geq 0$, then starting from an arbitrary x_0 on the unit disc, we achieve convergence to the fixed point of T. A further generalization due to Schaefer [19] for the fixed points of nonexpansive mappings is

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n$$

for all $n \geq 0,\ \lambda \in (0,1)$. The most general iteration sequence for nonexpansive mappings which has been studied by many authors is due to Mann [16] and is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n \tag{2.2}$$

for all $n \geq 0,\ \{\alpha_n\} \subset (0,1)$, satisfying certain conditions. This iteration sequence, however, does not generally converge to a fixed point of T (when it exists), without additional conditions imposed either on T, the domain of T or the range of T. Without any of these conditions, the best we can get is that $\{x_n\}$ is an approximate fixed point sequence, i.e $\lim ||x_n - Tx_n|| = 0$. To get weak convergence, we need the additional condition that T be demiclosed at zero, together with the fact that E be an Opial space, while to get strong convergence, we need some compactness type assumptions on T, domain of T or range of T.

The natural question that arises is the following: Can the Mann iterative sequence converge to the fixed points of the more general class of pseudocontractive maps? To this end, we quickly submit that all attempts to use the Mann iteration sequence for Lipschitz pseudocontractive maps have proven abortive. In [20], Chidume and Mutangadura gave an example of a Lipschitz pseudocontractive self map of a compact convex subset of a Hilbert space with a unique fixed point, for which the Mann iterative sequence fails to converge.

The next natural question is the following: What iterative sequence can we employ for the convergence to fixed points of Lipschitz pseudocontractive maps? In [17], Ishikawa introduced an iteration sequence, which in some sense is more general than the Mann iterative sequence, which he used for the convergence to fixed points of Lipschitz pseudocontractive maps. More precisely, he proved the following.

Theorem 2.1. *If C is a compact convex subset of a Hilbert space H and $T: C \to C$ is a Lipschitz pseudocontractive mapping and x_0 is any point of C, then the sequence $\{x_n\}_{n\geq 0}$ converges strongly to a fixed point of T, where $\{x_n\}$ is defined iteratively for each integer $n \geq 0$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n; \;\; y_n = (1-\beta_n)x_n + \beta_n T x_n,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers satisfying the conditions

$$(i)\; 0 \leq \alpha_n \leq \beta_n < 1; (ii)\; \lim \beta_n = 0; (iii)\; \sum \alpha_n \beta_n = \infty.$$

The Ishikawa iterative sequence actually leads to an approximate fixed point sequence for Lipschitz pseudocontractive maps, such that the imposition of compactness type assumptions on T or domain of T or range of T, yields convergence to fixed points of T.

In order to obtain strong convergence to fixed points of pseudocontractive maps without the compactness type assumptions, many authors (see e.g [21], [22]) have defined what they call hybrid Mann and Ishikawa algorithms. However, these hybrid schemes are hinged on some special subsets, C_n and Q_n of the Banach space, whose computations are non-trivial.

More recently, Zegeye *et al.* [25] proved the following results.

Theorem 2.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \to C, i = 1,2,...,N$ be a finite family of Lipschitz pseudocontractive mappings with Lipschitzian constants L_i, for $i = 1,2,...,N$ respectively. Assume that the interior of $F := \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n; \;\; y_n = (1-\beta_n)x_n + \beta_n T x_n,$$

where $T_n := T_{n(mod)N}$ and $\{\alpha_n\}$, $\{\beta_n\}$ $\in (0,1)$ satisfying the following conditions. (i) $\alpha_n \leq \beta_n$ $\forall n \geq 0$; (ii) $\liminf \alpha_n = \alpha > 0$; (iii) $sup_{n\geq 0}\beta_n \leq \beta < \frac{1}{\sqrt{1+L^2}+1}$ for $L := max\{L_i : i = 1,2,...,N\}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, ..., T_N\}$.

Although the results of Zegeye *et al.* are plausible, pseudocontraction maps abound whose fixed point sets are finite and as such have empty interiors. So, the question that still remains to be answered is: Is it possible to obtain strong convergence of the Ishikawa iteration sequence (not hybrid) to fixed points of Lipschitz pseudocontraction maps, without the compactness type assumptions on either T or its domain and without the assumption that the interior of the fixed point set be nonempty?

We now state some results in the Literature which will help us answer the above question to a reasonable extent. The first one and its proof is given by Zhou in [21], as Tool 2.3.

Tool 2.3 *Let C be a closed convex subset of a real Hilbert space H and $T : C \to C$ be a demicontinuous pseudocontractive self mapping from C into itself. Then $F(T)$ is a closed convex subset of C and $I - T$ is demiclosed at zero.*

Lemma 2.4. [26] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \leq (1+\delta_n)a_n + b_n, n \geq 1$. If $\sum \delta_n < \infty$ and $\sum b_n < \infty$, then $lim\ a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero, then $lim\ a_n = 0$.*

Definition 2.5. Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $||.||$ and let C be a nonempty closed convex subset of H. The orthogonal projection $P_C x$ of x onto C is defined by $P_C x = \arg\min_{y \in C} ||x - y||$, and has the following properties:

(i) $\langle x - P_C x, z - P_C x \rangle \leq 0,$ for all $z \in C$

(ii) $||P_C x - P_C y||^2 \leq \langle P_C x - P_C y, x - y \rangle$, for all $x, y \in H$.

Theorem 2.6. [23] *Let $T : C \to C$ be a nonlinear mapping with $F(T) \neq \emptyset$, where C is a closed convex subset of a real Hilbert space H. Suppose the following conditions are satisfied:*

(i) $I - T$ is demiclosed at 0

(ii) T is demicontractive with constant k, or equivalently T satisfies condition A with $\lambda = \frac{1-k}{2}$

(iii) $0 < a \leq \alpha_n \leq b < 2\lambda = 1 - k$

Then the Mann iteration sequence converges weakly to a fixed point of $F(T)$, for any starting x_0.

Theorem 2.7. [23] *Suppose T satisfies the conditions of Theorem 2.6. If in addition there exists $0 \neq h \in H$ such that*

$$\langle x - Tx, h \rangle \leq 0 \tag{2.3}$$

for all $x \in D(T)$, then starting from a suitable x_0, the Mann iteration sequence (2.2) converges strongly to an element of $F(T)$.

In [24], Maruster and Maruster noted that if T satisfies the positivity type condition $\langle Tx, x\rangle \geq \|x\|^2$, then it is sufficient to find a non-zero solution of the variational inequality (2.3). This motivates our choice of monotonicity type condition.

It is our purpose in this article to prove weak and strong convergence of the Ishikawa iterative sequence to a fixed point of a Lipschitz pseudocontractive map in a nonempty closed convex subset of a Hilbert space. We do not need any compactness type assumption on T or its domain. Neither do we require the interior of the fixed points set of T to be nonempty.

3. Main results

Theorem 3.1. *If C is a closed convex subset of a Hilbert space H and $T : C \to C$ is a Lipschitz pseudocontractive mapping and x_0 is any point of C, then the sequence $\{x_n\}_{n\geq 0}$ converges weakly to a fixed point of T, where $\{x_n\}$ is defined iteratively for each integer $n \geq 0$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n; \;\; y_n = (1-\beta_n)x_n + \beta_n T x_n,$$

where $\{\alpha_n\}$, $\{\beta_n\} \in (0,1)$ satisfying the following conditions. $(i)\alpha_n \leq \beta_n \; \forall n \geq 0; (ii) inf \;\alpha_n = \alpha > 0; (iii) sup_{n\geq 0}\beta_n \leq \beta < \frac{1}{\sqrt{1+L^2}+1}$.

Proof. As in the proof of Theorem 2.1 in [17], we have

$$||x_{n+1} - p||^2 \leq ||x_n - p||^2 - \alpha_n\beta_n(1-2\beta_n - L^2\beta_n^2)||x_n - Tx_n||. \tag{3.1}$$

This, together with the conditions imposed on $\{\alpha_n\}$ and $\{\beta_n\}$ yields $\lim ||x_n - Tx_n|| = 0$. From (3.1) and Lemma 2.4, we have that $\lim\{||x_n - p||\}$ exists. It follows that $\{||x_n - p||\}$ is bounded. Therefore, $\{x_n\}$ is norm bounded. Hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakyly to $x^* \in C$. These, together with Tool 2.3 implies $x^* \in F(T)$. Since H is an Opial space, a well known standard argument yields that $\{x_n\}$ converges weakly to x^*.

Theorem 3.2. *If C is a closed convex subset of a Hilbert space H and $T : C \to C$ is a Lipschitz pseudocontractive mapping and x_0 is any point of C, then the sequence $\{x_n\}_{n\geq 0}$ converges* *weakly to a fixed point of* T, *where* $\{x_n\}$ *is defined iteratively for each integer* $n \geq 0$ *by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n; \;\; y_n = (1-\beta_n)x_n + \beta_n T x_n,$$

where $\{\alpha_n\}, \{\beta_n\} \in (0,1)$ *satisfy the following conditions.* $(i)\alpha_n \leq \beta_n \; \forall n \geq 0; (ii)\sum \alpha_n\beta_n = \infty; (iii) sup_{n\geq 0}\beta_n \leq \beta < \frac{1}{\sqrt{1+L^2}+1}$.

Proof. As in the proof of our theorem 2.1 in [17], we have

$$||x_{n+1}-p||^2 \leq ||x_n-p||^2 - \alpha_n\beta_n(1-2\beta_n-L^2\beta_n^2)||x_n-Tx_n|| \tag{3.2}$$

This, together with the conditions imposed on $\{\alpha_n\}$ and $\{\beta_n\}$ yield $\lim inf||x_n - Tx_n|| = 0$. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim ||x_{n_k} - Tx_{n_k}|| = 0$. From (3.2) and lemma 2.4, we have that $\lim\{||x_n - p||\}$ exists. It follows that $\{||x_n - p||\}$ is bounded. Therefore, $\{x_n\}$ is norm bounded. Since $\{x_n\}$ is norm bounded, so is $\{x_{n_k}\}$ and as such, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakyly to $x^* \in C$. These, together with Tool 2 implies $x^* \in F(T)$. Since H is an Opial space, a well known standard argument yields that $\{x_n\}$ converges weakly to x^*.

Theorem 3.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space,* H*. Let* $T : C \to C$ *be a Lipschitz pseudocontractive with* $F(T) = \{x \in C : Tx = x\} \neq \emptyset$*. Suppose* T *satisfies the conditions of either theorem* 5 *or* 6 *and the monotonicity condition* $\langle Tx, x\rangle \geq ||x||^2, \forall x \in C$*. Then starting from a suitable* $x_0 \in C$*, the Ishikawa iteration sequence converges strongly to an element of* $F(T)$*.*

Proof. Let $p \in F(T)$ and choose $x_0 \in C$, such $\langle x_0, p\rangle \geq \langle p, p\rangle$. Then, there exists $\varepsilon_0 > 0$ such that $\langle x_0 - p, p\rangle \geq \varepsilon_0||x_0 - p||^2$. Assume $\langle x_n - p, p\rangle \geq \varepsilon_0||x_n - p||^2$. Then using (1.1) and the monotonicity condition in our theorem, we have

$$\begin{aligned}
\langle x_{n+1}-p,p\rangle &= \langle[(1-\alpha_n)x_n+\alpha_n Ty_n-p],p\rangle \\
&= \langle(1-\alpha_n)[x_n-p]+\alpha_n[Ty_n-p],p\rangle \\
&= (1-\alpha_n)\langle x_n-p,p\rangle+\alpha_n\langle Ty_n-p,p\rangle \\
&= (1-\alpha_n)\langle x_n-p,p\rangle+\alpha_n[\langle y_n-p,p\rangle+\langle Ty_n-y_n,p\rangle] \\
&= (1-\alpha_n)\langle x_n-p,p\rangle+\alpha_n\langle(1-\beta_n)[x_n-p]+\beta_n[Tx_n-p],p\rangle \\
&\quad +\alpha_n\langle Ty_n-y_n,p\rangle \\
&= (1-\alpha_n)\langle x_n-p,p\rangle+\alpha_n(1-\beta_n)\langle x_n-p,p\rangle \\
&\quad +\alpha_n\beta_n\langle Tx_n-p,p\rangle+\alpha_n\langle Ty_n-y_n,p\rangle \\
&= (1-\alpha_n)\langle x_n-p,p\rangle++\alpha_n(1-\beta_n)\langle x_n-p,p\rangle \\
&\quad +\alpha_n\beta_n[\langle x_n-p,p\rangle+\langle Tx_n-x_n,p\rangle]+\alpha_n\langle Ty_n-y_n,p\rangle \\
&= \langle x_n-p,p\rangle+\alpha_n\beta_n\langle Tx_n-x_n,p\rangle+\alpha_n\langle Ty_n-y_n,p\rangle \\
&= \langle x_n-p,p\rangle+\alpha_n\beta_n[\langle Tx_n-x_n,x_n\rangle+\langle Tx_n-x_n,p-x_n\rangle] \\
&\quad +\alpha_n[\langle Ty_n-y_n,y_n\rangle+\langle Ty_n-y_n,p-y_n\rangle] \\
&= \langle x_n-p,p\rangle+\alpha_n\beta_n\langle Tx_n-x_n,x_n\rangle \\
&\quad -\alpha_n\beta_n[\langle Tx_n-p,x_n-p\rangle+\langle p-x_n,x_n-p\rangle] \\
&\quad +\alpha_n\langle Ty_n-y_n,y_n\rangle-\alpha_n[\langle Ty_n-p,y_n-p\rangle+\langle p-y_n,y_n-p\rangle] \\
&\geq \langle x_n-p,p\rangle+\alpha_n\beta_n\langle Tx_n-x_n,x_n\rangle \\
&+ \alpha_n\langle Ty_n-y_n,y_n\rangle \\
&\geq \langle x_n-p,p\rangle \\
&\geq \varepsilon_0||x_n-p||^2 \\
&\geq \varepsilon_0||x_{n+1}-p||^2.
\end{aligned}$$

So that, since $x_n \rightharpoonup p$ from theorems 5 and 6, then $x_n \to p$.

Example 3.4. *Let $H = R$ (reals) and $C = [1,2]$ be a nonempty closed convex subset of H. Define $T : C \to C$ by $Tx = 1$. Then $F(T) = \{1\}$ and it is easily verifiable that T satisfies all the conditions of theorem 3.3. Therefore, the class of maps for which our results hold is non-void.*

Remark 3.5. We note that one of the ways of choosing x_0 is as follows: For any $\beta > 1$, choose $x_0 = P_C(\beta p)$, where $p \in F(T)$ and $P : H \to C$ is the metric projection from H onto C. This follows since it is well known (see Definition 1) that P is firmly nonexpansive (*i.e* satisfies condition (ii) of Definition 2.5), so that

$$\begin{aligned} ||x_0 - p||^2 &= ||P_C(\beta p) - P_(p)||^2 \\ &\leq \langle P_C(\beta p) - P_C(p), \beta p - p\rangle \\ &= \langle x_0 - p, (\beta - 1)p\rangle \\ &= (\beta - 1)\langle x_0 - p, p\rangle. \end{aligned}$$

This implies $\langle x_0 - p, p\rangle \geq \varepsilon_0 ||x_0 - p||^2$, where $\varepsilon_0 = \frac{1}{\beta - 1}$.

Remark 3.6. In [24], Maruster and Maruster noted that if T satisfies the positivity type condition $\langle Tx, x\rangle \geq ||x||^2$, then it is sufficient to find a non-zero solution of the variational inequality (2.3), where T is a demicontractive map. This motivates the condition in our theorem. As a matter of fact, our theorem is a necessity result, for the class of lipschitz pseudocontractive maps.

Remark 3.7. Observe that prior to the work embodied herein, the methods employed in [24] have been restricted to the class of demicontractive maps.

Conflict of Interests

The author declares that there is no conflict of interests.

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