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BROUWER'S FIXED POINT THEOREM WITH SEQUENTIALLY AT MOST ONE FIXED POINT: A CONSTRUCTIVE ANALYSIS

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Abstract. In this paper we present a constructive proof of Brouwer's fixed point theorem with sequentially at most one fixed point, and apply it to the mini-max theorem of zero-sum games.

Keywords: Brouwer's fixed point theorem; constructive mathematics; sequentially at most one fixed point; minimax theorem.

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1. Introduction

It is well known that Brouwer's fixed point theorem can not be constructively proved. See [3] or [8].

> [6] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics à la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive.

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Sperner's lemma which is used to prove Brouwer's theorem, however, can be constructively proved. Some authors have presented an approximate version of Brouwer's theorem using Sperner's lemma. See [8] and [9]. Thus, Brouwer's fixed point theorem is constructively, in the sense of constructive mathematics à la Bishop, proved in its approximate version.

Also Dalen in [8] states a conjecture that a uniformly continuous function f from a simplex into itself, with property that each open set contains a point x such that $x \neq f(x)$, which means $|x - f(x)| > 0$, and also at every point x on the boundaries of the simplex $x \neq f(x)$, has an exact fixed point. We present a partial answer to Dalen's conjecture.

Recently [2] showed that the following theorem is equivalent to Brouwer's fan theorem.

> Each uniformly continuous function φ from a compact metric space X into itself with at most one fixed point and approximate fixed points has a fixed point.

By reference to the notion of *sequentially at most one maximum* in [1] we require a stronger condition that a function φ has *sequentially at most one fixed point*, and will show the following result.

> Each uniformly continuous function φ from a compact metric space X into itself with *sequentially at most one fixed point* and approximate fixed points has a fixed point,

without the fan theorem. Orevkov in [7] constructed a computably coded continuous function f from the unit square into itself, which is defined at each computable point of the square, such that f has no computable fixed point. His map consists of a retract of the computable elements of the square to its boundary followed by a rotation of the boundary of the square. As pointed out by Hirst in [5], since there is no retract of the square to its boundary, his map does not have a total extension.

In the next section we present our theorem and its proof. In Section 3, as an application of the theorem we consider the mini-max theorem of two-person zero-sum games.

2. Theorem and proof

Let $\mathbf{p}$ be a point in a compact metric space X, and consider a uniformly continuous function φ from X into itself. According to [8] and [9] φ has an approximate fixed point. It means

$$\text{For each } \varepsilon > 0 \text{ there exists } \mathbf{p} \in X \text{ such that } |\mathbf{p} - \varphi(\mathbf{p})| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\inf_{\mathbf{p} \in X} |\mathbf{p} - \varphi(\mathbf{p})| = 0.$$

The notion that φ has at most one fixed point is defined as follows;

Definition 2.1. *For all* $\mathbf{p}, \mathbf{q} \in X$*, if* $\mathbf{p} \neq \mathbf{q}$*, then* $\varphi(\mathbf{p}) \neq \mathbf{p}$ *or* $\varphi(\mathbf{q}) \neq \mathbf{q}$.

Next by reference to the notion of *sequentially at most one maximum* in [1], we define the notion that φ has *sequentially at most one fixed point* as follows;

Definition 2.2. *All sequences* $(\mathbf{p}_n)_{n \geq 1}$*,* $(\mathbf{q}_n)_{n \geq 1}$ *in* X *such that* $|\varphi(\mathbf{p}_n) - \mathbf{p}_n| \longrightarrow 0$ *and* $|\varphi(\mathbf{q}_n) - \mathbf{q}_n| \longrightarrow 0$ *are eventually close in the sense that* $|\mathbf{p}_n - \mathbf{q}_n| \longrightarrow 0$.

Now we show the following lemma, which is based on Lemma 2 of [1].

Lemma 2.1. *Let* φ *be a uniformly continuous function from a compact metric space* X *into itself. Assume* $\inf_{\mathbf{p} \in X} |\mathbf{p} - \varphi(\mathbf{p})| = 0$*. If the following property holds,*

> *For each* $\delta > 0$ *there exists* $\varepsilon > 0$ *such that if* $\mathbf{p}, \mathbf{q} \in X$*,* $|\varphi(\mathbf{p}) - \mathbf{p}| < \varepsilon$ *and* $|\varphi(\mathbf{q}) - \mathbf{q}| < \varepsilon$*, then* $|\mathbf{p} - \mathbf{q}| \leq \delta$*,*

then, there exists a point $\mathbf{r} \in X$ *such that* $\varphi(\mathbf{r}) = \mathbf{r}$*, that is,* φ *has a fixed point.*

Proof.

Choose a sequence $(\mathbf{p}_n)_{n \geq 1}$ in X such that $|\varphi(\mathbf{p}_n) - \mathbf{p}_n| \longrightarrow 0$. Compute N such that $|\varphi(\mathbf{p}_n) - \mathbf{p}_n| < \varepsilon$ for all $n \geq N$. Then, for $m, n \geq N$ we have $|\mathbf{p}_m - \mathbf{p}_n| \leq \delta$. Since $\delta > 0$ is arbitrary, $(\mathbf{p}_n)_{n \geq 1}$ is a Cauchy sequence in X, and converges to a limit $\mathbf{r} \in X$. The continuity of φ yields $|\varphi(\mathbf{r}) - \mathbf{r}| = 0$, that is, $\varphi(\mathbf{r}) = \mathbf{r}$.

This completes the proof.

Next we show the following theorem, which is based on Proposition 3 of [1].

Theorem 2.1. *Each uniformly continuous function* φ *from a compact metric space* X *into itself with sequentially at most one fixed point and approximate fixed points has a fixed point.*

Proof.

Choose a sequence $(\mathbf{r}_n)_{n\geq 1}$ in X such that $|\varphi(\mathbf{r}_n)-\mathbf{r}_n|\longrightarrow 0$. In view of Lemma 2.1 it is enough to prove that the following condition holds.

For each $\delta>0$ there exists $\varepsilon>0$ such that if $\mathbf{p},\mathbf{q}\in X$, $|\varphi(\mathbf{p})-\mathbf{p}|<\varepsilon$ and $|\varphi(\mathbf{q})-\mathbf{q}|<\varepsilon$, then $|\mathbf{p}-\mathbf{q}|\leq\delta$.

Assume that the set

$$K=\{(\mathbf{p},\mathbf{q})\in X\times X:\ |\mathbf{p}-\mathbf{q}|\geq\delta\}$$

is nonempty and compact (See Theorem 2.2.13 of [4]). Since the mapping $(\mathbf{p},\mathbf{q})\longrightarrow\max(|\varphi(\mathbf{p})-\mathbf{p}|,|\varphi(\mathbf{q})-\mathbf{q}|)$ is uniformly continuous, we can construct an increasing binary sequence $(\lambda_n)_{n\geq 1}$ such that

$$\lambda_n=0\Rightarrow\inf_{(\mathbf{p},\mathbf{q})\in K}\max(|\varphi(\mathbf{p})-\mathbf{p}|,|\varphi(\mathbf{q})-\mathbf{q}|)<2^{-n},$$

$$\lambda_n=1\Rightarrow\inf_{(\mathbf{p},\mathbf{q})\in K}\max(|\varphi(\mathbf{p})-\mathbf{p}|,|\varphi(\mathbf{q})-\mathbf{q}|)>2^{-n-1}.$$

It suffices to find n such that $\lambda_n=1$. In that case, if $|\varphi(\mathbf{p})-\mathbf{p}|<2^{-n-1}$, $|\varphi(\mathbf{q})-\mathbf{q}|<2^{-n-1}$, we have $(\mathbf{p},\mathbf{q})\notin K$ and $|\mathbf{p}-\mathbf{q}|\leq\delta$. Assume $\lambda_1=0$. If $\lambda_n=0$, choose $(\mathbf{p}_n,\mathbf{q}_n)\in K$ such that $\max(|\varphi(\mathbf{p}_n)-\mathbf{p}_n|,|\varphi(\mathbf{q}_n)-\mathbf{q}_n|)<2^{-n}$, and if $\lambda_n=1$, set $\mathbf{p}_n=\mathbf{q}_n=\mathbf{r}_n$. Then, $|\varphi(\mathbf{p}_n)-\mathbf{p}_n|\longrightarrow 0$ and $|\varphi(\mathbf{q}_n)-\mathbf{q}_n|\longrightarrow 0$, so $|\mathbf{p}_n-\mathbf{q}_n|\longrightarrow 0$. Computing N such that $|\mathbf{p}_N-\mathbf{q}_N|<\delta$, we must have $\lambda_N=1$.

This completes the proof.

3. Application: Minimax theorem of zero-sum games

Consider a two person zero-sum game. There are two players A and B. Player A has m alternative pure strategies, and the set of his pure strategies is denoted by $S_A=\{a_1,a_2,\dots,a_m\}$. Player B has n alternative pure strategies, and the set of his pure strategies is denoted by $S_B=\{b_1,b_2,\dots,b_n\}$. m and n are finite natural numbers. The payoff of player A when a combination of players' strategies is (a_i,b_j) is denoted by $M(a_i,b_j)$. Since we consider a zero-sum game, the payoff of player B is equal to $-M(a_i,b_j)$. Let p_i be a probability that A chooses his strategy a_i, and q_j be a probability that B chooses his strategy b_j. A mixed strategy of A is represented by a probability distribution over S_A, and is denoted by $\mathbf{p} = (p_1, p_2, \dots, p_m)$ with $\sum_{i=1}^m p_i = 1$. Similarly, a mixed strategy of B is denoted by $\mathbf{q} = (q_1, q_2, \dots, q_n)$ with $\sum_{j=1}^n q_j = 1$. A combination of mixed strategies $(\mathbf{p}, \mathbf{q})$ is called a *profile*. The expected payoff of player A at a profile $(\mathbf{p}, \mathbf{q})$ is written as follows,

$$M(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i M(a_i, b_j) q_j.$$

We assume that $M(a_i, b_j)$ is finite. Then, since $M(\mathbf{p}, \mathbf{q})$ is linear with respect to probability distributions over the sets of pure strategies of players, it is a uniformly continuous function. The expected payoff of A when he chooses a pure strategy a_i and B chooses a mixed strategy $\mathbf{q}$ is $M(a_i, \mathbf{q}) = \sum_{j=1}^n M(a_i, b_j) q_j$, and his expected payoff when he chooses a mixed strategy $\mathbf{p}$ and B chooses a pure strategy b_j is $M(\mathbf{p}, b_j) = \sum_{i=1}^m p_i M(a_i, b_j)$. The set of all mixed strategies of A is denoted by P, and that of B is denoted by Q. P is an $m-1$-dimensional simplex, and Q is an $n-1$-dimensional simplex.

We call $v_A(\mathbf{p}) = \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q})$ the *guaranteed payoff* of A at $\mathbf{p}$. And we define v_A^* as follows,

$$v_A^* = \sup_{\mathbf{p}} \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q})$$

This is a constructive version of the maximin payoff. Similarly, we call $v_B(\mathbf{q}) = \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q})$ the guaranteed payoff of player B at $\mathbf{q}$, and define v_B^* as follows,

$$v_B^* = \inf_{\mathbf{q}} \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q}).$$

This is a constructive version of the minimax payoff. For a fixed $\mathbf{p}$ we have $\inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q}) \le M(\mathbf{p}, \mathbf{q})$ for all $\mathbf{q}$, and so

$$\sup_{\mathbf{p}} \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q}) \le \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q}) \text{ for all } \mathbf{q}$$

holds. Then, we obtain $\sup_{\mathbf{p}} \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q}) \le \inf_{\mathbf{q}} \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q})$. This is rewritten as

$$v_A^* \le v_B^*. \tag{1}$$

Define a function $\Gamma = (\bar{\mathbf{p}}(\mathbf{p},\mathbf{q}), \bar{\mathbf{q}}(\mathbf{p},\mathbf{q}))$ as follows;

$$\bar{p}_i(\mathbf{p},\mathbf{q}) = \frac{p_i + \max(M(a_i,\mathbf{q}) - M(\mathbf{p},\mathbf{q}),0)}{1+\sum_{k=1}^m \max(M(a_k,\mathbf{q}) - M(\mathbf{p},\mathbf{q}),0)},$$

$$\bar{q}_j(\mathbf{p},\mathbf{q}) = \frac{q_j + \max(M(\mathbf{p},\mathbf{q}) - M(\mathbf{p},b_j),0)}{1+\sum_{k=1}^n \max(M(\mathbf{p},\mathbf{q}) - M(\mathbf{p},b_k),0)}.$$

We assume the following condition;

Assumption 3.1. *All sequences* $((\mathbf{p}_n,\mathbf{q}_n))_{n\geq 1}$, $((\mathbf{p}'_n,\mathbf{q}'_n))_{n\geq 1}$ *in* $P\times Q$ *such that* $\max(M(a_i,\mathbf{q}_n)-M(\mathbf{p}_n,\mathbf{q}_n),0) \longrightarrow 0$, $\max(M(\mathbf{p}_n,\mathbf{q}_n)-M(\mathbf{p}_n,b_j),0) \longrightarrow 0$, $\max(M(a_i,\mathbf{q}'_n)-M(\mathbf{p}'_n,\mathbf{q}'_n),0) \longrightarrow 0$ *and* $\max(M(\mathbf{p}'_n,\mathbf{q}'_n) - M(\mathbf{p}'{}_n,b_j),0) \longrightarrow 0$ *for all* i *and* j *are eventually close in the sense that* $|(\mathbf{p}_n,\mathbf{q}_n) - (\mathbf{p}'_n,\mathbf{q}'_n)| \longrightarrow 0$.

Since $M(\mathbf{p}_n,\mathbf{q}_n) = \sum_{i=1}^m p_i M(a_i,\mathbf{q}_n)$, it is impossible that $\max(M(a_i,\mathbf{q}_n)-M(\mathbf{p}_n,\mathbf{q}_n),0) > 0$ for all i such that $p_i > 0$. Similarly, it is impossible that $M(\mathbf{p}_n,\mathbf{q}_n)-\max(M(\mathbf{p}_n,b_j),0) > 0$ for all j such that $q_j > 0$. $|\Gamma((\mathbf{p}_n,\mathbf{q}_n)) - (\mathbf{p}_n,\mathbf{q}_n)| \longrightarrow 0$ means $|\bar{p}_i - p_i| \longrightarrow 0$ for all i and $|\bar{q}_j - q_j| \longrightarrow 0$ for all j. Therefore, we must have $\max(M(a_i,\mathbf{q}'_n)-M(\mathbf{p}'_n,\mathbf{q}'_n),0) \longrightarrow 0$ and $\max(M(\mathbf{p}'_n,\mathbf{q}'_n) - M(\mathbf{p}'{}_n,b_j),0) \longrightarrow 0$ for all i and j, and so under Assumption 3.1 we find

> All sequences $((\mathbf{p}_n,\mathbf{q}_n))_{n\geq 1}$, $((\mathbf{p}'_n,\mathbf{q}'_n))_{n\geq 1}$ in $P\times Q$ such that $|\Gamma((\mathbf{p}_n,\mathbf{q}_n))-(\mathbf{p}_n,\mathbf{q}_n)| \longrightarrow 0$ and $|\Gamma((\mathbf{p}'_n,\mathbf{q}'_n)) - (\mathbf{p}'_n,\mathbf{q}'_n)| \longrightarrow 0$ are eventually close in the sense that $|(\mathbf{p}_n,\mathbf{q}_n) - (\mathbf{p}'_n,\mathbf{q}'_n)| \longrightarrow 0$.

Thus, Γ has sequentially at most one fixed point.

Summing up $\bar{p}_i$ from 1 to m, for each i

$$\sum_{i=1}^m \bar{p}_i(\mathbf{p},\mathbf{q}) = \frac{\sum_{i=1}^m p_i + \sum_{i=1}^m \max(M(a_i,\mathbf{q}) - M(\mathbf{p},\mathbf{q}),0)}{1+\sum_{k=1}^m \max(M(a_k,\mathbf{q}) - M(\mathbf{p},\mathbf{q}),0)} = 1.$$

Similarly, summing up $\bar{q}_j$ from 1 to n, for each j

$$\sum_{j=1}^n \bar{q}_j(\mathbf{p},\mathbf{q}) = \frac{\sum_{j=1}^n q_j + \sum_{j=1}^n \max(M(\mathbf{p},\mathbf{q}) - M(\mathbf{p},b_j),0)}{1+\sum_{k=1}^n \max(M(\mathbf{p},\mathbf{q}) - M(\mathbf{p},b_k),0)} = 1.$$

Let $\bar{\mathbf{p}}(\mathbf{p},\mathbf{q}) = (\bar{p}_1,\bar{p}_2,\dots,\bar{p}_m)$, $\bar{\mathbf{q}}(\mathbf{p},\mathbf{q}) = (\bar{q}_1,\bar{q}_2,\dots,\bar{q}_n)$. Then, $\Gamma = (\bar{\mathbf{p}}(\mathbf{p},\mathbf{q}), \bar{\mathbf{q}}(\mathbf{p},\mathbf{q}))$ is a uniformly continuous function from $P\times Q$ into itself. There are $m+n-2$ independent vectors in $P\times Q$, and so $P\times Q$ is an $m+n-2$-dimensional space. Since it is a product of two simplices, it is a compact subset of a metric space. Therefore, Γ has a fixed point. Let $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ be the fixed point, and $\lambda = \sum_{k=1}^{n} \max(M(a_k, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}), 0)$, $\lambda' = \sum_{k=1}^{m} \max(M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, b_k), 0)$. Then,

$$\frac{\tilde{p}_i + \max(M(a_i, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}), 0)}{1 + \lambda} = \tilde{p}_i,$$

$$\frac{\tilde{q}_j + \max(M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, b_j), 0)}{1 + \lambda'} = \tilde{q}_j.$$

Thus, we have

$$\max(M(a_i, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}), 0) = \lambda \tilde{p}_i,$$

and

$$\max(M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, b_j), 0) = \lambda' \tilde{q}_j.$$

Since $M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) = \sum_{i=1}^{m} p_i M(a_i, \tilde{\mathbf{q}})$, it is impossible that $\max(M(a_i, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}), 0) = M(a_i, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) > 0$ for all i such that $\tilde{p}_i > 0$. Therefore, $\lambda = 0$, and we have $\sup_{\mathbf{p}} M(\mathbf{p}, \tilde{\mathbf{q}}) = M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$. Similarly, we obtain $\lambda' = 0$ and $\inf_{\mathbf{q}} M(\tilde{\mathbf{p}}, \mathbf{q}) = M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$. Then,

$$v_B^* = \inf_{\mathbf{q}} \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q}) \leq M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) \leq \sup_{\mathbf{p}} \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q}) = v_A^*.$$

With (1)

$$v_A^* = v_B^* = M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}).$$

Therefore, the value of the game is determined at the fixed point of Γ.

		Player 2	
		X	Y
Player 1	X	1, -1	-1, 1
	Y	-1, 1	1, -1

TABLE 1. Example of game

Consider an example. See a game in Table 1. It is an example of the so-called Matching-Pennies Game. Pure strategies of Player 1 and 2 are X and Y. The left side number in each cell represents the payoff of Player 1 and the right side number represents the payoff of Player 2. Let p_X and $1-p_X$ denote the probabilities that Player 1 chooses, respectively, X and Y, and q_X and $1 - q_X$ denote the probabilities for Player 2. Denote the expected payoff of Player 1 by $M(p_X, q_X)$. Since we consider a zero-sum game, the expected payoff of Player 2 is $-M(p_X, q_X)$. Then,

$$M(p_X, q_X) = p_X q_X - (1-p_X)q_X - p_X(1-q_X) + (1-p_X)(1-q_X)$$
$$= (2p_X - 1)(2q_X - 1)$$

Denote the payoff of Player 1 when he chooses X by $M(X, q_X)$, and that when he chooses Y by $M(Y, q_X)$. Similarly for Player B. Then,

$$M(X, q_X) = 2q_X - 1,\ M(Y, q_X) = 1 - 2q_X,\ -M(p_X, X) = 1 - 2p_X,\ -M(p_X, Y) = 2p_X - 1,$$

$$M(X, q_X) - M(p_X, q_X) = 2(1-p_X)(2q_X - 1),\ M(Y, q_X) - M(p_X, q_X) = -2p_X(2q_X - 1),$$

$$-M(p_X, X) + M(p_X, q_X) = 2(q_X - 1)(2p_X - 1),\ -M(p_X, Y) + M(p_X, q_X) = 2q_X(2p_X - 1).$$

And we have

$$\text{When } q_X > \frac{1}{2},\ M(X, q_X) > M(Y, q_X) \text{ and } M(X, q_X) > M(p_X, q_X) \text{ for } p_X < 1,$$

$$\text{When } q_X < \frac{1}{2},\ M(Y, q_X) > M(X, q_X) \text{ and } M(Y, q_X) > M(p_X, q_X) \text{ for } p_X > 0,$$

$$\text{When } p_X > \frac{1}{2},\ -M(p_X, Y) > -M(p_X, X) \text{ and } -M(p_X, Y) > -M(p_X, q_X) \text{ for } q_X > 0,$$

$$\text{When } p_X < \frac{1}{2},\ -M(p_X, X) > -M(p_X, Y) \text{ and } -M(p_X, X) > -M(p_X, q_X) \text{ for } q_X < 1.$$

Consider sequences $(p_X(n))_{n\geq 1}$ and $(q_X(n))_{n\geq 1}$, and let $0 < \varepsilon < \frac{1}{2}$, $0 < \delta < \varepsilon$. There are the following cases.

(1) (a) If $p_X(n) > \frac{1}{2} + \delta$ and $q_X(n) > \frac{1}{2} + \delta$, or
  (b) $p_X(n) > \frac{1}{2} + \delta$ and $q_X(n) < \frac{1}{2} - \delta$, or
  (c) $p_X(n) < \frac{1}{2} - \delta$ and $q_X(n) < \frac{1}{2} - \delta$, or
  (d) $p_X(n) < \frac{1}{2} - \delta$ and $q_X(n) > \frac{1}{2} + \delta$, or
  (e) $p_X(n) > \frac{1}{2} + \delta$ and $\frac{1}{2} - \varepsilon < q_X(n) < \frac{1}{2} + \varepsilon$, or
  (f) $p_X(n) < \frac{1}{2} - \delta$ and $\frac{1}{2} - \varepsilon < q_X(n) < \frac{1}{2} + \varepsilon$, or
  (g) $\frac{1}{2} - \varepsilon < p_X(n) < \frac{1}{2} + \varepsilon$, and $q_X(n) > \frac{1}{2} + \delta$ or

(h) $\frac{1}{2}-\varepsilon < p_X(n) < \frac{1}{2}+\varepsilon$, and $q_X(n) < \frac{1}{2}-\delta$,

then there exists no pair of $(p_X(n), q_X(n))$ such that $M(X, q_X(n)) - M(p_X(n), q_X(n)) \longrightarrow 0$, $M(Y, q_X(n)) - M(p_X(n), q_X(n)) \longrightarrow 0$, $-[M(p_X(n), X) - M(p_X(n), q_X(n))] \longrightarrow 0$ and $-[M(p_X(n), Y) - M(p_X(n), q_X(n))] \longrightarrow 0$.

(2) If $\frac{1}{2}-\varepsilon < p_X(n) < \frac{1}{2}+\varepsilon$ and $\frac{1}{2}-\varepsilon < q_X(n) < \frac{1}{2}+\varepsilon$ with $0 < \varepsilon < \frac{1}{2}$, $M(X, q_X(n)) - M(p_X(n), q_X(n)) \longrightarrow 0$, $M(Y, q_X(n)) - M(p_X(n), q_X(n)) \longrightarrow 0$, $-[M(p_X(n), X) - M(p_X(n), q_X(n))] \longrightarrow 0$ and $-[M(p_X(n), Y) - M(p_X(n), q_X(n))] \longrightarrow 0$, then $(p_X(n), q_X(n)) \longrightarrow (\frac{1}{2}, \frac{1}{2})$.

Therefore, the payoff functions satisfy Assumption 3.1.

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