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COMMON FIXED POINT OF EXPANSION TYPE MAPS IN CONE METRIC SPACE USING IMPLICIT RELATIONS

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Abstract: The main object of this paper is to proved a common fixed point theorem for surjective expansion type maps using implicit relations in cone metric spaces. Further a common fixed point theorem for expansion type maps has been established on compact cone metric space. Our results generalize and compactify the corresponding result in [8, 9, 10, 19, 21].

Keywords: Cone metric space, Implicit relations, Expansion type maps, Common fixed point.

Mathematics Subject Classification: 47H10, 54H25.

1. INTRODUCTION

In 2007, Huang and Zhang [4] introduced the cone metric space by substituting an ordered Banach space for the real numbers and proved some fixed point theorems in this space. Many authors study this subject and proved some fixed point theorems (see [1, 2, 3, 4, 5, 6, 7, 13, 14, 15, 20] and references therein).

In 1976, Rosenholtz [16] discussed local expansion as, f is a local expansion if every point in X has a neighborhood B on which f is expansion. Infect Rosenholdz proved the following theorem:

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Theorem 1.1: Let (X, d) be a complete metric space and f: X → X be a self map of X onto itself satisfying

$$d(fx, fy) > \lambda\, d(x, y) \tag{1.1}$$

for all x, y ∈ X with x ≠ y and λ > 1. Then "f" has a fixed point in X.

After this, a number of fixed point theorems for expansion mappings have been proved by Park [11], Wang, Li, Gao & Iseki [21], Khan et al [8] Park & Rhoades [12], and Taniguchi [19] etc. Actually the above mentioned theorem of Rosenholdz appears to be the generalization for expansion mappings of Banach contraction Principle.

Definition 1.1: A self mapping f of a metric space (X, d) is called an expansion mapping or simple, an expansion if there is a number k > 1 such that, for each x, y∈ X,

$$d(fx, fy) \geq k\, d(x, y) \tag{1.2}$$

2. PRELIMNERIES

Throughout this paper R and R^+ denote the set of real numbers and the set of non-negative real numbers. We use the following definitions in the proof of our main theorems.

Definition 2.1: Let E be a real Banach space. A subset P of E is called a cone if and only if the following hold:

(i) P is closed, nonempty, and P ≠ {0},

(ii) a, b ∈ R, a, b ≥ 0, and x, y ∈ P imply that ax + by ∈ P,

(iii) x ∈ P and −x ∈ P imply that x = 0.

Given a cone P ⊂ E, we define a partial ordering ≤ with respect to P by x ≤ y if and only if y − x ∈ P. We will write x < y to indicate that x ≤ y but x ≠ y, while x ≪ y will stand for y − x ∈ *int*P, where *int*P denotes the interior of P.

Definition 2.2: The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$, for all $x, y \in E$. The least positive number K satisfying above is called the normal constant [4].

Definition 2.3: The cone P is called regular if every increasing sequence which is bounded above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that

$$\lim_{n\to\infty}\|x_n - x\| = 0.$$

Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded below is convergent. Also every regular cone is normal [17]. In addition, there are some non normal cones.

Example 2.1: Suppose $E = C^2_R([0,1])$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and consider the cone $P = \{f \in E: f \geq 0\}$. For all $K \geq 1$, set $f(x) = x$ and $g(x) = x^{2K}$. Then $0 \leq g \leq f$, $\|f\| = 2$ and $\|g\| = 2K + 1$. Since $K\|f\| < \|g\|$, K is not normal constant of P. Therefore, P is non-normal cone.

Lemma 2.1[22]: Let E be a real Banach space with a cone P, then

(i)' If $x \leq y$ and $0 \leq a \leq b$, then $ax \leq by$ for $x, y \in P$,

(ii)' If $x \leq y$ and $u \leq v$, then $x + u \leq y + v$,

(iii)' If $x_n \leq y_n$ for each $n \in N$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ then $x \leq y$.

Lemma 2.2[17]: If P is a cone, $x \in P$, $\alpha \in R$, $0 \leq \alpha < 1$, and $x \leq \alpha x$, then $x = 0$.

Definition 2.4: Let E is a real Banach space, P is a cone in E with *int* $P \neq \emptyset$, and $\leq$ is partial ordering with respect to P. Let X be a nonempty set. Define a function d: $X \times X \to E$, called a cone metric on X if it satisfies the following conditions:

(i)" $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$, (ii)" $d(x, y)= d(y, x)$, for all $x, y \in X$,

(iii)" $d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Then (X, d) is called a cone metric space.

Example 2.2: Suppose $E = \ell^1$, $P = \{\{x_n\}_{n\in N} \in E : x_n \geq 0\}$, for all n, (X, ρ) is a metric space and $d: X\times X \to E$ is defined by $d(x, y) = \{\rho(x, y)/2^n\}_{n\in N}$. Then (X, d) is a cone metric space and the normal constant of P is equal to 1.

Example 2.3: Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset R^2$, $X = R^2$ and $d: X \times X \to E$ defined by

$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = (\max\{|x_1-y_1|, |x_2-y_2|\}, \alpha.\max\{|x_1-y_1|, |x_2-y_2|\}),$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

3. DEFINITIONS AND LEMMAS

Definition 3.1: Let (X, d) be a cone metric space. Let $\{x_n\}_{n\in N}$ be a sequence in X and $x \in X$. If for any $c \in E$ with $0 \ll c$, there is $n_0 \in N$ such that for all $n > n_0$,

$$d(x_n, x) \ll c,$$

then $\{x_n\}_{n\in N}$ is said to be convergent to x, and x is the limit of $\{x_n\}_{n\in N}$. We denote this by

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Definition 3.2: Let (X, d) be a cone metric space and $\{x_n\}_{n\in N}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$, there is $n_0 \in N$ such that for all $m, n > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}_{n\in N}$ is called a Cauchy sequence in X. We denote this by

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0.$$

Definition 3.3: Let (X, d) be a cone metric space and $\{x_n\}_{n\in N}$ a sequence in X. If $\{x_n\}_{n\in N}$ is convergent, then it is a Cauchy sequence.

Definition 3.4: Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Definition 3.5: Let (X, d) be a cone metric space. Let T be a self-map on X. If for all sequence $\{x_n\}_{n\in N}$ in X,

$$\lim_{n\to\infty} x_n \to x, \Rightarrow \lim_{n\to\infty} T(x_n) \to T(x),$$

then T is called continuous on X.

Lemma 3.1: Let (X, d) be a cone metric space. If $\{x_n\}$ is a convergent sequence in X, then the limit of $\{x_n\}$ is unique.

Lemma 3.2: Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_{nk}\}$ is any subsequence of $\{x_n\}$, then $\{x_{nk}\}$ converges to x.

Definition 3.7: Let E and F be reel Banach spaces and P and Q be cones on E and F, respectively. Let (X, d) and (Y, ρ) be cone metric spaces, where d: $X \times X \to E$ and ρ: $Y \times Y \to F$. A function f: $X \to Y$ is said to be continuous at $x_0 \in X$, if for every $c \in F$ with $0 \ll c$, there exists $b \in E$ with $0 \ll b$ such that for $x \in X$,

$$d(x, x_0) \ll b \Rightarrow \rho(f(x), f(x_0)) \ll c.$$

If f is continuous at every point of X, then it is said to be continuous on X.

Lemma 3.5: Let (X, d) and (Y, ρ) be cone metric spaces. A function f: $X \to Y$ is continuous at a point $x_0 \in X$ if and only if whenever a sequence $\{x_n\}$ in X converges to x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$.

Pathak and Tiwari [10] proved the following theorem:

Theorem 3.1: Let A and B be surjective mappings from a complete metric space (X, d) into itself satisfying

$$d(Ax, By) \geq \phi\{d(Ax, x), d(Ax, y), d(By, x), d(By, y), d(x, y)\}$$

for all x, y $\in$ X with $x \neq y$, where $\phi \in \Phi$, where Φ denote the family of all real valued functions $\phi: (R^+)^5 \to R$, satisfying the following conditions:

(C_1) ϕ is lower semi – continuous in each co-ordinate variable.

(C_2) ϕ is non – increasing in second and third coordinate variables.

(C_3) Let $\upsilon, w \in R^+$ be such that either $\upsilon \geq \phi(\upsilon, \upsilon + w, 0, w, w)$, or $\upsilon \geq \phi(w, 0, \upsilon + w, \upsilon, w)$. Then $\upsilon \geq hw$, where $h = \phi(1, 1, 1, 1, 1) > 1$.

Then A and B have a common fixed point in X.

Recently Sahin and Telci [17] prove a common fixed point theorem for expansion type mappings in complete cone metric spaces. They proved the theorem as follows:

Theorem 3.2: Let (X, d) be a complete cone metric space and P be a cone. Let f and g be surjective self-mappings of X satisfying the following inequalities

$$d(gfx, fx) \geq ad(fx, x), \tag{3.1}$$

$$d(fgx, gx) \geq bd(gx, x) \tag{3.2}$$

for all x in X, where a, b > 1. If either f or g is continuous, then f and g have a common fixed point.

4. MAIN RESULT

Let Φ denote the family of real valued functions $\phi: (R^+)^7 \to R^+$ satisfying the following conditions:

(M_1) ϕ is lower semi-continuous in each coordinate variable,

(M_2) ϕ is non-increasing in 2nd and 3rd coordinate variables,

(M_3) Let $\upsilon, w \in R^+$ be such that either

$$\upsilon \geq \phi(\upsilon, \upsilon + w, 0, w, w, w, w) \quad \text{or} \quad \upsilon \geq \phi(w, 0, \upsilon + w, \upsilon, w, w, w).$$

Then $\upsilon \geq hw$, where $h = \phi(1, 1, 1, 1, 1, 1, 1) > 1$.

Example 4.1: Define $\phi(t_1, t_2, t_3, t_4, t_5, t_6, t_7) = t_1 - 2t_2 - 2t_3 + 3t_4 + 4t_5 + 5t_6 + 6t_7$. Then for $\upsilon \geq \phi(\upsilon, \upsilon + w, 0, w, w, w, w) = \upsilon - 2(\upsilon + w) + 3w + 4w + 5w + 6w$, then $\upsilon \geq 8w \geq hw$, similarly, if $\upsilon \geq \phi(w, 0, \upsilon + w, \upsilon, w, w, w)$ then also $\upsilon \geq hw$.

Example 4.1: If $\phi(t_1, t_2, t_3, t_4, t_5, t_6, t_7) = p.t_1 - q.\max\{t_2, t_3\} + r.t_4 + s.t_5 + t.t_6 + t_7$. Then for $\upsilon \geq \phi(\upsilon, \upsilon + w, 0, w, w, w, w) = p.\upsilon - q.(\upsilon + w) + r.w + s.w + t.w + w$, or $\upsilon \geq ((s + t - r + 1) / (1 + q - p))w$, then $\upsilon \geq 8w \geq hw$, similarly, If $\upsilon \geq \phi(w, 0, \upsilon + w, \upsilon, w, w, w)$ then also $\upsilon \geq hw$.

Theorem 4.1: Let (X, d) be a complete cone metric space and let S and T be surjective self mappings such that

$$\begin{aligned} d(Sx, Ty) \geq \phi[&d(Sx, x), d(Sx, y), d(Ty, x), d(Ty, y), d(x, y), \\ &\max\{d(x, y), \alpha d(Sx, x) + \beta d(Sx, y)\}, \\ &\max\{d(x, y), \alpha d(Ty, x) + \beta d(Ty, y)\}] \end{aligned} \tag{4.1}$$

for all $x, y \in X$ with $x \neq y$, where $\phi \in \Phi$ and $0 \leq \alpha < 1$; $0 \leq \beta < 1$ are such that $\alpha + \beta \neq 0$ and $\min\left\{\frac{(1-\alpha)}{(\alpha+\beta)}, \frac{(1-\beta)}{(\alpha+\beta)}\right\} = k > 1$. Then S and T have a common fixed point in complete cone metric space.

Proof: Let x_0 be an arbitrary point in X. Since S and T are surjective, we choose a point x_1 in X such that $x_1 \in S^{-1}(x_0)$ and for this point x_1 there exists a point $x_2 \in T^{-1}(x_1)$. Continuing this way, we construct a sequence $\{x_n\}$ in X such that

$$x_{2n+1} = S^{-1}(x_{2n}) \quad \text{and} \quad x_{2n+2} = T^{-1}(x_{2n+1}), \quad n = 0,1, 2,..........$$

If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of S and T. Suppose if $x_{2n} = x_{2n+1}$ for some $n \geq 0$. Then x_{2n} is a fixed point of S. If $x_{2n+1} \neq x_{2n+2}$ then from (4.1) we have.

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d[(Sx_{2n+1}, Tx_{2n+2}) \\ &\geq \phi[d(Sx_{2n+1}, x_{2n+1}), d(Sx_{2n+1}, x_{2n+2}), d(Tx_{2n+2}, x_{2n+1}), d(Tx_{2n+2}, x_{2n+2}), d(x_{2n+1}, {}_{2n+2}), \\ &\qquad \max\{d(x_{2n+1}, x_{2n+2}), \alpha d(Sx_{2n+1}, x_{2n+1}) + \beta d(Sx_{2n+1}, x_{2n+2})\}, \\ &\qquad \max\{d(x_{2n+1}, x_{2n+2}), \alpha d(Tx_{2n+2}, x_{2n+1}) + \beta d(Tx_{2n+2}, x_{2n+2})\}] \\ &\geq \phi[d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), \\ &\qquad \max\{d(x_{2n+1}, x_{2n+2}), \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n}, x_{2n+2})\}, \\ &\qquad \max\{d(x_{2n+1}, x_{2n+2}), \alpha d(x_{2n+1}, x_{2n+1}) + \beta d(x_{2n+1}, x_{2n+2}))\}] \end{aligned}$$

$$\geq \phi[d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), 0, d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}),$$
$$\max\{d(x_{2n+1}, x_{2n+2}), \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n}, x_{2n+2})\},$$
$$\max\{d(x_{2n+1}, x_{2n+2}), \ \alpha.0 + \beta d(x_{2n+1}, x_{2n+2})\}] \qquad (4.2)$$

Now, in the **case I,** when

$$\max\{d(x_{2n+1}, x_{2n+2}), \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n}, x_{2n+2})\} = \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n}, x_{2n+2})$$

we have

$$\alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n}, x_{2n+2}) \geq d(x_{2n+1}, x_{2n+2})$$

yielding thereby

$$d(x_{2n}, x_{2n+1}) \geq \frac{(1-\beta)}{(\alpha+\beta)} d(x_{2n+1}, x_{2n+2}) \geq k d(x_{2n+1}, x_{2n+2}) \qquad (4.3)$$

In the other **case II** , when

$$\max\{d(x_{2n+1}, x_{2n+2}), \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$$

then condition (4.2) reduced to

$$d(x_{2n}, x_{2n+1}) \geq \phi[d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), 0, d(x_{2n+1}, x_{2n+2}),$$
$$d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})]$$

which, by (M_3), implies

$$d(x_{2n}, x_{2n+1}) \geq h d(x_{2n+1}, x_{2n+2}) \qquad (4.4)$$

Now, if we take $c = \min\{h, k\}$ then $c > 1$ and in either of the above cases we have

$$d(x_{2n+1}, x_{2n+2}) \leq c^{-1} d(x_{2n}, x_{2n+1}) \qquad (4.5)$$

This yields a contradiction and so $x_{2n+1} = x_{2n+2}$. Thus x_{2n} is a common fixed point of S and T. If $x_{2n+1} = x_{2n+2}$ for some $n \geq 0$ then it is similarity verified that x_{2n+1} is a common fixed point of S and T. Now we suppose $x_n \neq x_{n+1}$ for each $n \geq 0$. Then just as above, we have

$$d(x_{2n+1}, x_{2n+2}) \leq c^{-1} d(x_{2n}, x_{2n+1}).$$

Similarly

$$d(x_{2n+1}, x_{2n+2}) = d(Tx_{2n+2}, Sx_{2n+3})$$
$$\geq \phi[d(Sx_{2n+3}, x_{2n+3}), d(Sx_{2n+3}, x_{2n+2}), d(Tx_{2n+2}, x_{2n+3}), d(Tx_{2n+2}, x_{2n+2}), d(x_{2n+3}, x_{2n+2}),$$

$$\max\{d(x_{2n+3}, x_{2n+2}), \alpha d(Sx_{2n+3}, x_{2n+3}) + \beta d(Sx_{2n+3}, x_{2n+2})\},$$
$$\max(d(x_{2n+3}, x_{2n+2}), \alpha d(Tx_{2n+2}, x_{2n+3}) + \beta d(Tx_{2n+2}, x_{2n+2})\}]$$
$$\geq \phi[d(x_{2n+2}, x_{2n+3}), d(x_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+2}, x_{2n+3}),$$
$$\max\{d(x_{2n+2}, x_{2n+3}), \alpha d(x_{2n+2}, x_{2n+3}) + \beta d(x_{2n+2}, x_{2n+2})\},$$
$$\max\{d(x_{2n+2}, x_{2n+3}), \alpha d(x_{2n+1}, x_{2n+3}) + \beta d(x_{2n+1}, x_{2n+2})\}]$$

Now proceeding in the same manner as followed from (4.2) to (4.5) we obtain

$$d(x_{2n+1}, x_{2n+2}) \geq cd(x_{2n+2}, x_{2n+3}) \tag{4.6}$$

Thus is general we have $\quad d(x_{n+1}, x_{n+2}) \leq c^{-1}d(x_n, x_{n+1}) \quad$ (4.7)

Since $c > 1$, and by definition 3.1, $\{x_n\}$ is a Cauchy sequence in X which is complete. Therefore, it has a limit z in X. Also from lemma 3.2, the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have the same limit z. Again since S and T are surjective, there exists two point υ and w in X such that $z = S\upsilon$ and $z = Tw$. Thus, using (4.1) and $z = Tw$, we have

$$d(x_{2n}, z) = d(Sx_{2n+1}, Tw)$$
$$\geq \phi[d(Sx_{2n+1}, x_{2n+1}), d(Sx_{2n+1}, w), d(Tw, x_{2n+1}), d(Tw, w), d(x_{2n+1}, w),$$
$$\max(d(x_{2n+1}, w), \alpha d(Sx_{2n+1}, x_{2n+1}) + \beta d(Sx_{2n+1}, w)),$$
$$\max(d(x_{2n+1}, w), \alpha d(Sw, x_{2n+1}) + \beta d(Sw, w)\}]$$
$$\geq \phi[d(x_{2n}, x_{2n+1}), d(x_{2n}, w), d(z, x_{2n+1}), d(z, w), d(x_{2n+1}, w),$$
$$\max\{d(x_{2n+1}, w), \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n}, w)\},$$
$$\max(d(x_{2n+1}, w), \alpha d(z, x_{2n+1}) + \beta d(z, w)\}]$$

letting $n \to \infty$, we obtain

$$0 = d(z, z) \geq \phi[d(z, z), d(z, w), d(z, z), d(z, w), d(z, w),$$
$$\max(d(z, w), \alpha d(z, z) + \beta d(z, w)), \max\{(d(z, w), \alpha d(z, z) + \beta d(z,w)\}]$$
$$\geq \phi[0, d(z, w), 0, d(z, w), d(z, w), \max\{d(z, w), \alpha.0 + \beta d(z, w)\},$$
$$\max\{d(z, w), \alpha.0+ \beta d(z, w)\}]$$
$$\geq \phi[0, 0 + d(z, w), 0, d(z, w), d(z, w), d(z, w), d(z, w)]$$

which, by (M_3), implies

$$0 \geq hd(z, w)$$

so that z = w. Similarly we can prove z = υ. Therefore z = Sz = Tz and so S and T have a common fixed point in complete cone metric space.

Remark 4.1: Our theorem 4.1 extend the corresponding result of Kang [9], Pathak and Tiwari [10], for S = T, it also extend the result of Khan et al. [8] for expansion type maps on cone metric space.

Corollary 4.1: Let S, T be surjective mappings from a complete cone metric space (X, d) into itself satisfying

$$\begin{aligned} d(Sx, Ty) \geq\ & ad(Sx, x) - ed(Sx, y) - ed(Ty, x) + bd(Ty, y) + cd(x, y) \\ & + f \max\{d(x, y), \alpha d(Sx, x) + \beta d(Sx, y)\} + g \max\{d(x, y), \alpha d(Ty, x) + \beta d(Ty, y)\} \end{aligned}$$

for all $x, y \in X$ with $x \neq y$ where a, b, c, e, f, g are non negative real numbers with $0 \leq a - e < 1$, $0 \leq b - f < 1$, $a - 2e + b + c + f + g > 1$ and $0 \leq \alpha, \beta < 1$ are such that $\alpha + \beta \neq 0$ and $\min\left\{ \frac{(1-\alpha)}{(\alpha+\beta)}, \frac{(1-\beta)}{(\alpha+\beta)} \right\} = k > 1$.

Then S and T have a common fixed in complete cone metric space.

Proof: Let $\phi(t_1, t_2, t_3, t_4, t_5, t_6, t_7) = at_1 - et_2 - et_3 + bt_4 + ct_5 + ft_6 + gt_7$. Then $h = \phi(1, 1, 1, 1, 1, 1, 1) = a - 2e + b + c + f + g > 1$ and it satisfies the following conditions:

(M_1) Obviously

(M_2) Obviously

(M_3) Let $\upsilon, w \in R^+$ such that

$$\upsilon \geq \phi(\upsilon, \upsilon + w, 0, w, w, w, w) = a\upsilon - e(\upsilon + w) + bw + cw + fw + gw,$$

then $\upsilon \geq ((b + c + g + f - e) / (1 - (a - e)) w \geq (a - 2e + b + c + f + g) w \geq hw$.

If $\upsilon \geq \phi(w, 0, \upsilon + w, \upsilon, w, w, w)$ then similarly $\upsilon \geq hw$. Therefore $\phi \in \Phi$ and so the proof of corollary is completed by theorem 4.1.

For our second we have the following implicit relation:

Let $\Phi' \in \{\phi: (R^+)^7 \to R^+\}$ denote the family of all real valued functions satisfying the following conditions:

(N_1) ϕ is non-increasing in 2^{nd} and 3^{rd} coordinate variables.

(N_2) Let $u, w \in R^+$ be such that either $u > \phi(u, u + w, 0, w, w, w, w)$ or $u > \phi(w, 0, u + w, u, w, w, w)$. Then $u > w$.

(N_3) $u \le \phi(0, u, u, 0, u, u, u)$ for each $u > 0$.

Theorem 4.2: Let (X, d) be a cone metric space and let S and T be self surjective mappings satisfying

$$d(Sx, Ty) > \phi[d(Sx, x), d(Sx, y), d(Ty, x), d(Ty, y), d(x, y), \max\{d(x, y), \alpha d(Sx, x) + \beta d(Sx, y)\}, \max\{d(x, y), \alpha d(Ty, x) + \beta d(Ty, y)\}] \tag{4.6}$$

for all $x, y \in X$ with $x \neq y$, where $\phi \in \Phi'$ and $0 \le \alpha < 1$, $0 \le \beta < 1$ are such that $\alpha + \beta \neq 0$ and $\min\left\{\frac{(1-\alpha)}{(\alpha+\beta)}, \frac{(1-\beta)}{(\alpha+\beta)}\right\} > 1$. If, one of S or T is continuous then S and T have a unique common fixed point in compact cone metric space.

Proof: Suppose S is continuous and let $m = \inf\{d(Sx, x): x \in X\}$. Since X is compact, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} x_n = x_0 \tag{4.7}$$

and

$$\lim_{n\to\infty} d(Sx_n, x_n) = m \tag{4.8}$$

Since $d(Sx_0, x_0) \le d(Sx_0, Sx_n) + d(Sx_n, x_n) + d(x_n, x_0)$

then by continuity of S and (4.7) and (4.8), we get

$$d(Sx_0, x_0) \le m, \qquad \text{and thus} \qquad d(Sx_0, x_0) = m.$$

Since T is surjective, there exists a point y_0 in X such that $Ty_0 = x_0$ and thus $d(Sx_0, Ty_0) = m$. Suppose $m > 0$, then by (4.6), we have

$$\begin{aligned} d(Sx_0, Ty_0) &> \phi[d(Sx_0, x_0), d(Sx_0, y_0), d(Ty_0, x_0), d(Ty_0, y_0), d(x_0,y_0), \\ &\qquad \max\{d(x_0, y_0), \alpha d(Sx_0, x_0) + \beta d(Sx_0, y_0)\}, \\ &\qquad \max\{d(x_0, y_0), \alpha d(Ty_0, x_0) + \beta d(Ty_0, y_0)\}] \\ &\ge \phi[d(Sx_0, x_0), d(Sx_0, x_0) + d(x_0, y_0), d(x_0, x_0), d(x_0, y_0), d(x_0, y_0), \end{aligned}$$

$$\max\{d(x_0, y_0), \alpha d(Sx_0, x_0) + \beta d(Sx_0, y_0)\},$$
$$\max\{d(x_0, y_0), \alpha d(x_0, x_0) + \beta d(x_0, y_0)\}]$$

and so $m > \phi[m, m + d(x_0, y_0), 0, d(x_0, y_0), d(x_0, y_0), \max\{d(x_0, y_0), \alpha m + \beta d(Sx_0, y_0)\},$

$$\max\{d(x_0, y_0), \alpha.0 + \beta d(x_0, y_0)\}] \qquad (4.9)$$

Now in the **case I**, when

$$\max \{d(x_0, y_0), \alpha m + \beta d(Sx_0, y_0)\} = \alpha m + \beta d(Sx_0, y_0), \text{ we have}$$
$$\alpha m + \beta d(Sx_0, y_0) \geq d(x_0, y_0)$$

yielding thereby, $m \geq ((1 - \beta) / (\alpha + \beta))\, d(x_0, y_0) > d(x_0, y_0)$

But in the other **case II**, when

$$\max\{d(x_0, y_0), \alpha m + \beta d(Sx_0, y_0)\} = d(x_0, y_0)$$

then (4.9) reduced to

$$m > \phi[m, m + d(x_0, y_0), 0, d(x_0, y_0), d(x_0, y_0), d(x_0, y_0), d(x_0, y_0)]$$

which, by (N_2), implies $m > d(x_0, y_0)$. Thus in general we have

$$m > d(x_0, y_0) \qquad (4.10)$$

Since S is surjective, there exists a point z_0 in X such that $y_0 = Sz_0$ and so

$$d(Sz_0, Ty_0) = d(y_0, x_0) < m.$$

Since $d(Sz_0, z_0) \geq m > 0$, by an application of (4.6), we have

$$d(Sz_0,Ty_0) > \phi[d(Sz_0, z_0), d(Sz_0, y_0), d(Ty_0, z_0), d(Ty_0, y_0), d(z_0, y_0),$$
$$\max\{d(z_0, y_0), \alpha d(Sz_0, z_0) + \beta d(Sz_0, y_0)\},$$
$$\max\{d(z_0, y_0), \alpha d(Ty_0, z_0) + \beta d(Ty_0, y_0)\}]$$
$$\geq \phi[d(Sz_0, z_0), d(y_0, y_0), d(Sz_0, Ty_0) + d(Sz_0, z_0), d(Sz_0, Ty_0), d(Sz_0, z_0),$$
$$\max\{d(Sz_0, z_0), \alpha d(Sz_0, z_0) + \beta d(Sz_0, y_0)\},$$
$$\max\{d(Sz_0, z_0), \alpha d(Ty_0, z_0) + \beta d(Ty_0, y_0)\}]$$

Now proceeding in the same way as followed from (4.9) to (4.10), we obtain

$$d(Sz_0, Ty_0) > d(Tz_0, z_0)$$

so that $m > d(Sz_0, Ty_0) > d(Sz_0, z_0) \geq m$, a contradiction. Hence m = 0 which gives

$$Sx_0 = x_0 = Ty_0 .$$

Again suppose $x_0 \neq y_0$, then by (4.6) we have

$$0 = d(Sx_0, Ty_0) > \phi[d(Sx_0, x_0), d(Sx_0, y_0), d(Ty_0, x_0), d(Ty_0, y_0), d(x_0, y_0),$$

$$\max\{d(x_0, y_0), \alpha d(Sx_0, x_0) + \beta d(Sx_0, y_0)\},$$
$$\max\{d(x_0, y_0), \alpha d(Ty_0, x_0) + \beta d(Ty_0, y_0)\}]$$
$$\geq \phi[0, d(x_0, y_0), 0, d(x_0, y_0), d(x_0, y_0), \max\{d(x_0, y_0), \alpha.0 + \beta d(x_0, y_0)\ ),$$
$$\max\{d(x_0, y_0)\ , \alpha.0 + \beta d(x_0, y_0)\}]$$
$$\geq \phi[0, 0 + d(x_0, y_0), 0, d(x_0, y_0), d(x_0, y_0), d(x_0, y_0), d(x_0, y_0)]$$

which by (N_2) implies $0 > d(x_0, y_0)$ which is a contradiction. Therefore, $x_0 = y_0$ and hence, $Sx_0 = x_0 = Tx_0$. Again suppose z_0 be a point in X such that $Sz_0 = z_0 = Tz_0$, and suppose $z_0 \neq x_0$, then by (4.6) we have

$$d(z_0, x_0) = d(Sz_0, Tx_0) > \phi[d(Sz_0, z_0), d(Sz_0, x_0), d(Tx_0, z_0), d(Tx_0, x_0), d(z_0, x_0),$$
$$\max\{d(z_0, x_0), \alpha d(Sz_0, z_0) + \beta d(Sz_0, x_0)\},$$
$$\max\{d(z_0, x_0), \alpha d(Tx_0, z_0) + \beta d(Tx_0, x_0)\}]$$
$$\geq \phi[0, d(z_0, x_0), d(z_0, x_0), 0, d(z_0, x_0), \max\{d(z_0, x_0), \alpha.0 + \beta d(z_0, x_0)\},$$
$$\max\{d(z_0, x_0), \alpha d(x_0, z_0) + \beta.0)\}]$$

Implying thereby

$$d(z_0, x_0) > \phi[0, d(z_0, x_0), d(z_0, x_0),0, d(z_0, x_0), d(z_0, x_0), d(z_0, x_0)]$$

which, by an application of (N_3), yields a contradiction. Hence, $z_0 = x_0$ and this completes the proof.

Remark 44: Our theorem 4.2 extends and compactifies the corresponding result of Kang [9], Khan et. al. [8], Pathak and Tiwari [10] on compact cone metric space.

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