Available online at http://scik.org
Adv. Fixed Point Theory, 6 (2016), No. 2, 123-135
ISSN: 1927-6303

FIXED POINT THEOREMS FOR CONTRACTIVE AND EXPANSIVE MAPPINGS OF GERAGHTY TYPE ON 2-METRIC SPACES

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Abstract. In this paper, we discuss the unique existence of fixed points for mappings with contractive functions or expansive functions on 2-metric spaces, and give some new versions of the fixed point theorems on real metric spaces. Our results generalize and improve the Banach's contraction principle and some fixed point theorems for expansive mappings on real metric spaces.

Keywords: 2-metric space; Fixed point; Contractive function; Expansive function.

2010 AMS Subject Classification: 47H05, 47H10, 54E40.

1. Introduction and preliminaries

Banach's contraction principle is as follows:

Theorem A. [1] *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. If for each $x, y \in X$,*

$$d(fx, fy) \leq k d(x,y),$$

where $k \in [0,1)$. Then f has a unique fixed point.

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Received March 24, 2015

The next result is a unique fixed point theorem for I-expansion mappings on a a metric space:

Theorem B. [2] *Let (X,d) be a complete metric space and $f : X \to X$ be a onto mapping. If*

$$d(fx, fy) \geq h\,d(x,y),\ \forall x,y \in X,$$

where $h > 1$. Then f has a unique fixed point.

In 1962, Rakotch generalized the Banach's contraction principle. He obtained the following theorem by replacing the constant k by a contraction function γ.

Theorem C. [3] *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. If for each $x,y \in X$,*

$$d(fx, fy) \leq \gamma\big(d(x,y)\big)\,d(x,y),$$

where $\gamma : [0,\infty) \to [0,1)$ is non-increasing and continuous function. Then f has a unique fixed point $x_0 \in X$ satisfying $\lim_{n\to\infty} f^n x = x_0$ for any $x \in X$.

In 1973, Geraghty gave another generalization of Banach's contraction principle as follows:

Theorem D. [4-5] *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. If*

$$d(fx, fy) \leq \beta\left(d(x,y)\right)d(x,y),\ \forall x,y \in X,$$

where $\beta : [0,\infty) \to [0,1)$ is a function satisfying the following condition: $\beta(t_n) \to 1 \Longrightarrow t_n \to 0$. Then f has a unique fixed point.

The aim of this paper is to give some new versions of Geraghty' theorem on 2-metric spaces and investigate Geraghty type fixed point theorems for expansive mappings on real metric s-paces and 2-metric spaces respectively.

Definition 1.1. [6-8] A 2-metric space (X,d) consists of a nonempty set X and a function $d : X \times X \times X \to [0,+\infty)$ such that

(i) for distant elements $x,y \in X$, there exists an $u \in X$ such that $d(x,y,u) \neq 0$;

(ii) $d(x,y,z) = 0$ if and only if at least two elements in $\{x,y,z\}$ are equal;

(iii) $d(x,y,z) = d(u,v,w)$, where $\{u,v,w\}$ is any permutation of $\{x,y,z\}$;

(iv) $d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all $x,y,z,u \in X$.

Definition 1.2. [6-8] A sequence $\{x_n\}_{n\in\mathbb{N}}$ in 2-metric space (X,d) is said to be a Cauchy sequence, if for each $\varepsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n,x_m,a) < \varepsilon$ for all $a \in X$ and $n,m > N$.

Definition 1.3. [6-8] A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to $x \in X$, if for each $a \in X$, $\lim_{n\to+\infty} d(x_n,x,a) = 0$. And we write that $x_n \to x$ and call x the limit of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 1.4. [6-8] A 2-metric space (X,d) is said to be complete, if every cauchy sequence in X is convergent.

Lemma 1.5. [9] *Let $\{x_n\}$ be a sequence in 2-metric space (X,d) such that $\lim_{n\to\infty} d(x_n,x_{n+1},a) = 0$ for all $a \in X$. If $\{x_n\}$ is not a Cauchy sequence, then there exist $a \in X$ and $\varepsilon > 0$ such that for each $i \in \mathbb{N}$ there exist $m(i),n(i) \in \mathbb{N}$ with $m(i),n(i) > i$ satisfying the following conditions*

(i) $m(i) > n(i)$ and $n(i) \to \infty$ as $i \to \infty$;

(ii) $d(x_{m(i)},x_{n(i)},a) > \varepsilon$, but $d(x_{m(i)-1},x_{n(i)},a) \le \varepsilon$.

Lemma 1.6. [6-8] *If a sequence $\{x_n\}$ in a 2-metric space (X,d) converges to $x \in X$. Then*

$$\lim_{n\to\infty} d(x_n,b,c) = d(x,b,c), \forall b,c \in X.$$

2. Unique fixed point theorems

First, we give some Geraghty type fixed point theorems for mappings with a contractive function on 2-metric spaces as follows:

Theorem 2.1. *Let (X,d) be a complete 2-metric space and $f : X \to X$ be a mapping. If*

$$d(fx, fy,a) \le \beta\big(d(x,y,a)\big)\, d(x,y,a),\ \forall x,y,a \in X \tag{2.1}$$

where $\beta : [0,\infty) \to [0,1)$ is a function satisfying the following condition: $\beta(t_n) \to 1$ as $n \to \infty$ which implies $t_n \to 0$ as $n \to \infty$. Then f has a unique fixed point $u \in X$ and $\lim_{n\to\infty} f^n x_0 = u$ for all $x_0 \in X$.

Proof. We take an $x_0 \in X$ and construct a sequence $\{x_n\}$ as follows

$$x_{n+1} = fx_n = f^{n+1}x_0,\ n = 0, 1, 2, \cdots.$$

For any fixed $a \in X$ and $n = 1, 2, \cdots$, by (2.1), we have

$$d(x_{n+1}, x_n, a) = d(fx_n, fx_{n-1}, a) \le \beta\big(d(x_n, x_{n-1}, a)\big)d(x_n, x_{n-1}, a) < d(x_n, x_{n-1}, a). \tag{2.2}$$

Hence $\{d(x_{n+1}, x_n, a)\}$ is a non-increasing sequence for any fixed $a \in X$. Therefore there is a $r(a) \ge 0$ such that $d(x_{n+1}, x_n, a) \to r(a)$ as $n \to \infty$. If $r(a) > 0$, then $d(x_{n+1}, x_n, a) > 0$ for all n. Hence by (2.2), we ahve

$$0 < \frac{d(x_{n+1}, x_n, a)}{d(x_n, x_{n-1}, a)} \le \beta(d(x_n, x_{n-1}, a)) < 1.$$

Let $n \to \infty$, then using the above inequality, we obtain

$$\lim_{n\to\infty} \beta(d(x_n, x_{n-1}, a)) = 1.$$

Hence

$$\lim_{n\to\infty} d(x_{n+1}, x_n, a) = 0, \forall a \in X. \tag{2.3}$$

Taking $a = x_{n-1}$ in (2.2), we obtain

$$d(x_{n+1}, x_n, x_{n-1}) \le \beta\big(d(x_n, x_{n-1}, x_{n-1})\big)d(x_n, x_{n-1}, x_{n-1}).$$

Hence

$$d(x_{n-1}, x_n, x_{n+1}) = 0, \forall\ n = 1, 2, \cdots. \tag{2.4}$$

Fix $k \in \mathbb{N}$ and suppose that $d(x_k, x_n, x_{n+1}) = 0$, where $n > k+1$. Using (2.1), we obtain

$$d(x_k, x_{n+1}, x_{n+2}) = (fx_n, fx_{n+1}, x_k) \le \beta\big(d(x_n, x_{n+1}, x_k)\big)d(x_n, x_{n+1}, x_k) = 0.$$

Combining (2.4), we obtain

$$d(x_k, x_n, x_{n+1}) = 0, \forall n \ge k \ge 1. \tag{2.5}$$

So, by (2.5), for all $m > n > k$, we have

$$d(x_k, x_n, x_m) \le d(x_k, x_n, x_{m-1}) + d(x_k, x_{m-1}, x_m) + d(x_n, x_{m-1}, x_m) = d(x_k, x_n, x_{m-1}).$$

Repeating this process, we obtain

$$d(x_k,x_n,x_m) \le d(x_k,x_n,x_{m-1}) \le \cdots \le d(x_k,x_n,x_{n+1}) = 0.$$

Hence, we have the following fact

$$d(x_m,x_n,x_k) = 0, \forall m,n,k \in \mathbb{N}. \tag{2.6}$$

Suppose that $\{x_n\}$ is not a Cauchy sequence, then by Lemma 1.5 and (2.3), there exist $a \in X$ and $\varepsilon > 0$ such that for any $i \in \mathbb{N}$ there exist $m(i), n(i) \in \mathbb{N}$ with $m(i), n(i) > i$ satisfying

(i) $m(i) > n(i)+1$ and $n(i) \to \infty$ as $i \to \infty$;

(ii) $d(x_{m(i)},x_{n(i)},a) > \varepsilon$, but $d(x_{m(i)-1},x_{n(i)},a) \le \varepsilon, i = 1,2,\cdots$.

Using (2.3), (2.6), (ii) and

$$d(x_{m(i)},x_{n(i)},a) \le d(x_{m(i)},x_{m(i)-1},a) + d(x_{m(i)-1},x_{n(i)},a) + d(x_{m(i)},x_{n(i)},x_{m(i)-1}),$$

we obtain

$$\lim_{i\to\infty} d(x_{m(i)},x_{n(i)},a) = \lim_{i\to\infty} d(x_{m(i)-1},x_{n(i)},a) = \varepsilon. \tag{2.7}$$

Since the following two inequalities

$$|d(x_{m(i)},x_{n(i)},a) - d(x_{m(i)},x_{n(i)-1},a)| \le d(x_{n(i)-1},x_{n(i)},a) + d(x_{m(i)},x_{n(i)},x_{n(i)-1}),$$

$$|d(x_{m(i)-1},x_{n(i)-1},a) - d(x_{m(i)},x_{n(i)-1},a)| \le d(x_{m(i)-1},x_{m(i)},a) + d(x_{m(i)},x_{m(i)-1},x_{n(i)-1})$$

hold, we see from (2.3), (2.6) and (2.7) that

$$\begin{aligned} \lim_{i\to\infty} d(x_{m(i)},x_{n(i)},a) &= \lim_{i\to\infty} d(x_{m(i)-1},x_{n(i)},a) \\ &= \lim_{i\to\infty} d(x_{m(i)},x_{n(i)-1},a) \\ &= \lim_{i\to\infty} d(x_{m(i)-1},x_{n(i)-1},a) = \varepsilon. \end{aligned} \tag{2.8}$$

By (2.1) and (2.6), one sees that

$$\begin{aligned}
&d(x_{m(i)},x_{n(i)},a)\\
\le& d(x_{m(i)},x_{m(i)+1},a)+d(x_{n(i)},x_{m(i)+1},a)+d(x_{m(i)},x_{n(i)},x_{m(i)+1})\\
\le& d(x_{m(i)},x_{m(i)+1},a)+d(x_{n(i)},x_{n(i)+1},a)+d(x_{m(i)+1},x_{n(i)+1},a)+d(x_{n(i)},x_{m(i)+1},x_{n(i)+1})\\
=& d(x_{m(i)},x_{m(i)+1},a)+d(x_{n(i)},x_{n(i)+1},a)+d(fx_{m(i)},fx_{n(i)},a)\\
\le& d(x_{m(i)},x_{m(i)+1},a)+d(x_{n(i)},x_{n(i)+1},a)+\beta\big(d(x_{m(i)},x_{n(i)},a)\big)\,d(x_{m(i)},x_{n(i)},a).
\end{aligned}$$

It follows that

$$\varepsilon < d(x_{m(i)},x_{n(i)},a) \le \frac{1}{1-\beta\big(d(x_{m(i)},x_{n(i)},a)\big)}\,[d(x_{m(i)},x_{m(i)+1},a)+d(x_{n(i)},x_{n(i)+1},a)].$$

Hence, by (2.3), we must have the following fact

$$\limsup_{i\to\infty}\frac{1}{1-\beta\big(d(x_{m(i)},x_{n(i)},a)\big)}=+\infty,$$

that is

$$\limsup_{i\to\infty}\beta\big(d(x_{m(i)},x_{n(i)},a)\big)=1.$$

Hence, we arrive at

$$\limsup_{i\to\infty} d(x_{m(i)},x_{n(i)},a)=0,$$

which is contradict with the assumption $d(x_{m(i)},x_{n(i)},a)>\varepsilon$ for all i. Therefore, $\{x_n\}$ is a Cauchy sequence and there exists $u\in X$ such that $x_n\to u$ as $n\to\infty$ by the completeness of X. By (2.1), we obtain that for any $a\in X$,

$$\begin{aligned}
&d(u,fu,a)\\
\le& d(fu,x_{n+1},a)+d(u,x_{n+1},a)+d(u,x_{n+1},fu)\\
=& d(fu,fx_n,a)+d(u,x_{n+1},a)+d(u,x_{n+1},fu)\\
\le& \beta\big(d(u,x_n,a)\big)\,d(u,x_n,a)+d(u,x_{n+1},a)+d(u,x_{n+1},fu)\\
<& d(u,x_n,a)+d(u,x_{n+1},a)+d(u,x_{n+1},fu).
\end{aligned}$$

Let $n\to\infty$, then $d(u,fu,a)=0, \forall a\in X$ by Lemma 1.6, hence $fu=u$. Suppose that v is another fixed point of f, then there is $b\in X$ such that $d(u,v,b)>0$. By (2.1),

$$d(u,v,b)=d(fu,fv,b)\le\beta\big(d(u,v,b)\big)\,d(u,v,b).$$

If $\beta\left(d(u,v,b)\right)=0$, then $d(u,v,b)=0$; If $\beta\left(d(u,v,b)\right)\neq 0$, then $d(u,v,b)<d(u,v,b)$. These two results are both contradictions. So u is the unique fixed point of f. ∎

From the proof of Theorem 2.1, we have observed that if $\{x_n\}$ is not a Cauchy sequence, then (2.8) holds. On the other hand, using (2.6), we have

$$\begin{aligned}
&d(x_{m(i)+1},x_{n(i)+1},a)\\
\leq& d(x_{m(i)},x_{n(i)+1},a)+d(x_{m(i)+1},x_{m(i)},a)+d(x_{m(i)+1},x_{n(i)+1},x_{m(i)})\\
\leq& d(x_{m(i)},x_{n(i)},a)+d(x_{n(i)},x_{n(i)+1},a)+d(x_{m(i)},x_{n(i)+1},x_{n(i)})+d(x_{m(i)+1},x_{m(i)},a)\\
=& d(x_{m(i)},x_{n(i)},a)+d(x_{n(i)},x_{n(i)+1},a)+d(x_{m(i)+1},x_{m(i)},a).
\end{aligned}$$

Similarly, we obtain

$$d(x_{m(i)},x_{n(i)},a)\leq d(x_{m(i)+1},x_{n(i)+1},a)+d(x_{n(i)},x_{n(i)+1},a)+d(x_{m(i)+1},x_{m(i)},a).$$

Hence we have

$$|d(x_{m(i)},x_{n(i)},a)-d(x_{m(i)+1},x_{n(i)+1},a)|\leq [d(x_{n(i)},x_{n(i)+1},a)+d(x_{m(i)+1},x_{m(i)},a)],$$

so using (2.3), we obtain

$$\lim_{i\to\infty} d(x_{m(i)+1},x_{n(i)+1},a)=\lim_{i\to\infty} d(x_{m(i)},x_{n(i)},a)=\varepsilon.$$

Using the above fact and modifying the proof of Theorem 2.1, we can obtain a variant form of Theorem 2.1 as follows:

Theorem 2.2. *Let (X,d) be a complete 2-metric space and $f:X\to X$ be a mapping. If*

$$d(fx,fy,a)\leq \beta\left(d(fx,fy,a)\right)d(x,y,a), \forall x,y,a\in X, \tag{2.9}$$

where $\beta:[0,\infty)\to[0,1)$ is a function satisfying the following condition: $\beta(t_n)\to 1$ as $n\to\infty$ which implies $t_n\to 0$ as $n\to\infty$. Then f has a unique fixed point.

Remark 2.3. Theorem D is a version of Theorem 2.1 on real metric spaces.

The next result is another version of Theorem 2.2 on real metric spaces. Here, we omit its proof.

Theorem 2.4. *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. If*

$$d(fx, fy) \leq \beta\big(d(fx, fy)\big)\, d(x,y), \forall x,y \in X, \tag{2.10}$$

where $\beta : [0,\infty) \to [0,1)$ is a function satisfying the following condition: $\beta(t_n) \to 1$ as $n \to \infty$ which implies $t_n \to 0$ as $n \to \infty$. Then f has a unique fixed point.

Remark 2.5. Theorem 2.4 is not only a generalization of Banach' contraction principle but also another version of Theorem D.

Example 2.6. Let $X = \{1,2,3\}$ and $d : X \times X \to [0,\infty)$ as follows:

$$d(1,1) = d(2,2) = d(3,3) = 0, d(1,2) = d(2,1) = 2, d(1,3) = d(3,1) = 3, d(2,3) = d(3,2) = 4.$$

It is known that (X,d) is a complete metric space. Let $\beta : [0,\infty) \to [0,1)$ be a function satisfying $\beta(0) = 0$ and $\beta(t) = \frac{1}{1+\frac{t}{12}}$ for all $t > 0$. Then β is non-continuous and non-monotonous, and obviously, $\beta(t_n) \to 1$ as $n \to \infty$ if and only if $t_n \to 0$ as $n \to \infty$. Let $f : X \to X$ be as follows

$$f1 = 1, f2 = 1, f3 = 2.$$

Since

$$d(f1, f3) = 2 < \frac{18}{7} = \frac{1}{1+\frac{2}{12}} \times 3 = \beta(2) \times d(1,3) = \beta(d(f1, f3)) \times d(1,3),$$

$$d(f2, f3) = 2 < \frac{24}{7} = \frac{1}{1+\frac{2}{12}} \times 4 = \beta(2) \times d(2,3) = \beta(d(f2, f3)) \times d(2,3),$$

we have f and β satisfy all conditions of Theorem 2.4. Consequently f has a unique fixed point 1. ∎

Using Theorem 2.4, we obtain the following fixed point theorem for a mapping with a expansive function on real metric spaces.

Theorem 2.7. *Let (X,d) be a complete metric space and $f : X \to X$ be a surjective mapping. If*

$$d(fx, fy) \geq \gamma\big(d(x,y)\big)\, d(x,y), \forall x,y \in X, \tag{2.11}$$

where $\gamma : [0,\infty) \to (1,\infty)$ is a function satisfying the following condition: $\gamma(t_n) \to 1$ as $n \to \infty$ which implies $t_n \to 0$ as $n \to \infty$. Then f has a unique fixed point.

Proof. If $x,y \in X$ and $fx = fy$, then by (2.11), we obtain that $x = y$. This means that f is injective, hence f has its inverse mapping g. By (2.11) again, for each $x,y \in X$,

$$d(x,y) = d(fgx, fgy) \geq \gamma\big(d(gx,gy)\big)d(gx,gy),$$

hence

$$d(gx,gy) \leq \frac{1}{\gamma\big(d(gx,gy)\big)}d(x,y). \tag{2.12}$$

Let $\beta(t) = \frac{1}{\gamma(t)}$, $\forall t \in [0,\infty)$, then $0 \leq \beta(t) < 1$ for all $t \in [0,\infty)$ and $\beta(t_n) \to 1$ as $n \to \infty$ which implies $t_n \to 0$ as $n \to \infty$. On the other hand, (2.12) becomes

$$d(gx,gy) \leq \beta\big(d(gx,gy)\big)\, d(x,y). \tag{2.13}$$

Hence g has a unique fixed point z by Theorem 2.4. Obviously, z is the unique fixed point of f. ∎

Similarly, using Theorem D, we obtain the following.

Theorem 2.8. *Let (X,d) be a complete metric space and $f : X \to X$ be a surjective mapping. If*

$$d(fx, fy) \geq \gamma\big(d(fx, fy)\big)\, d(x,y), \forall x,y \in X, \tag{2.14}$$

where $\gamma : [0,\infty) \to (1,\infty)$ is a function satisfying the following condition: $\gamma(t_n) \to 1$ as $n \to \infty$ which implies $t_n \to 0$ as $n \to \infty$. Then f has a unique fixed point.

Proof. By using the same method we used in the proof of Theorem 2.7, f has its inverse mapping g, so by (2. 14),

$$d(x,y) = d(fgx, fgy) \geq \gamma\big(d(fgx, fgy)\big)d(gx,gy) = \gamma\big(d(x,y)\big)d(gx,gy),$$

hence

$$d(gx,gy) \leq \frac{1}{\gamma\big(d(x,y)\big)}d(x,y).$$

The rest of the proof follows from Theorem D and the method of proof of Theorem 2.7. ∎

Remark 2.9. Theorems 2.7 and 2.8 are the generalizations of Theorem B.

Next, we give Geraghty type fixed point theorems for mappings with a expansive function on 2-metric spaces.

Theorem 2.10. *Let (X,d) be a complete 2-metric space and $f : X \to X$ be a surjective mapping. If*

$$d(fx, fy, a) \geq \gamma\big(d(x,y,a)\big)\, d(x,y,a), \forall x,y,a \in X, \tag{2.15}$$

where $\gamma : [0,\infty) \to (1,\infty)$ is a function satisfying the following condition: $\gamma(t_n) \to 1$ as $n \to \infty$ which implies $t_n \to 0$ as $n \to \infty$. Then f has a unique fixed point.

Proof. Take any $x_0 \in X$. Since f is onto, we can construct a sequence $\{x_n\}$ in X such that

$$x_n = fx_{n+1},\ n = 0,1,2,\cdots.$$

For any fixed n and any $a \in X$,

$$d(x_n,x_{n+1},a) = d(fx_{n+1}, fx_{n+2},a) \geq \gamma\big(d(x_{n+1},x_{n+2},a)\big) d(x_{n+1},x_{n+2},a) > d(x_{n+1},x_{n+2},a). \tag{2.16}$$

From (2.16), we have $\{d(x_n,x_{n+1},a)\}$ is a non-increasing sequence for any fixed $a \in X$. Therefore there exists $r(a) \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1},x_n,a) = r(a)$. If $r(a) > 0$, then $d(x_{n+1},x_n,a) > 0$ for all n, hence using (2.16), we obtain

$$1 < \gamma\big(d(x_{n+1},x_{n+2},a)\big) \leq \frac{d(x_n,x_{n+1},a)}{d(x_{n+1},x_{n+2},a)}.$$

Let $n \to \infty$, then from the above, we obtain

$$\lim_{n\to\infty} \gamma\big(d(x_{n+1},x_{n+2},a)\big) = 1,$$

hence

$$\lim_{n\to\infty} d(x_{n+1},x_{n+2},a) = 0,\ \forall\, a \in X. \tag{2.17}$$

Let $a = x_n$ in (2.16), then we obtain

$$d(x_n,x_{n+1},x_{n+2}) = 0, \forall\, n = 0,1,2,\cdots. \tag{2.18}$$

Fix $k \in \mathbb{N}$ and suppose that $d(x_k,x_{n+1},x_{n+2}) = 0$, where $(n+1)-k > 1$, then by (2.15) and (2.18),

$$0 = d(x_k,x_{n+1},x_{n+2}) = d(fx_{n+2}, f_{n+3},x_k) \geq \gamma\big(d(x_{n+2},x_{n+3},x_k)\big) d(x_{n+2},x_{n+3},x_k),$$

hence

$$d(x_{n+2},x_{n+3},x_k) = 0.$$

By the induction principle, we have the following fact

$$d(x_k, x_n, x_{n+1}) = 0, \forall n \geq k \geq 1. \tag{2.19}$$

For all $m > n > k$, by (2.19), we obtain that

$$d(x_k, x_n, x_m) \leq d(x_k, x_n, x_{m-1}) + d(x_k, x_{m-1}, x_m) + d(x_n, x_{m-1}, x_m) = d(x_k, x_n, x_{m-1}).$$

Repeating this process, we obtain

$$d(x_k, x_n, x_m) \leq d(x_k, x_n, x_{m-1}) \leq \cdots \leq d(x_k, x_n, x_{n+1}) = 0.$$

From the inequality obtained above, we have the following fact

$$d(x_m, x_n, x_k) = 0, \forall m, n, k \in \mathbb{N}. \tag{2.20}$$

Suppose that $\{x_n\}$ is not a Cauchy sequence, then repeating the process of the proof of Theorem 2.1, we are sure that (2.8) in Theorem 2.1 also holds.

By (2.15) and (2.20), we have

$$\begin{aligned}
&\gamma\big(d(x_{m(i)}, x_{n(i)}, a)\big) d(x_{m(i)}, x_{n(i)}, a) \\
\leq& d(fx_{m(i)}, fx_{n(i)}, a) \\
=& d(x_{m(i)-1}, x_{n(i)-1}, a) \\
\leq& d(x_{m(i)-1}, x_{m(i)}, a) + d(x_{m(i)}, x_{n(i)-1}, a) + d(x_{m(i)-1}, x_{n(i)-1}, x_{m(i)}) \\
\leq& d(x_{m(i)-1}, x_{m(i)}, a) + d(x_{m(i)}, x_{n(i)}, a) + d(x_{n(i)}, x_{n(i)-1}, a) + d(x_{m(i)}, x_{n(i)-1}, x_{n(i)}) \\
=& d(x_{m(i)-1}, x_{m(i)}, a) + d(x_{m(i)}, x_{n(i)}, a) + d(x_{n(i)}, x_{n(i)-1}, a).
\end{aligned}$$

Hence we have

$$\varepsilon < d(x_{m(i)}, x_{n(i)}, a) \leq \frac{1}{\gamma\big(d(x_{m(i)}, x_{n(i)}, a)\big) - 1}[d(x_{m(i)-1}, x_{m(i)}, a) + d(x_{n(i)}, x_{n(i)-1}, a)].$$

So we obtain by (2.17)

$$\liminf_{i\to\infty}[\gamma\big(d(x_{m(i)}, x_{n(i)}, a)\big) - 1] = 0,$$

namely

$$\liminf_{i\to\infty}[\gamma\big(d(x_{m(i)}, x_{n(i)}, a)\big)] = 1.$$

Hence we have

$$\liminf_{i\to\infty} d(x_{m(i)}, x_{n(i)}, a) = 0,$$

which is a contradict with the assumption $d(x_{m(i)}, x_{n(i)}, a) \geq \varepsilon$ for all i, so $\{x_n\}$ is Cauchy. Therefore there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$ by the completeness of X. Since f is onto, there exists $z \in X$ such that $u = fz$. By (2.15), for each $a \in X$ and n, we have

$$d(x_n, u, a) = d(fx_{n+1}, fz, a) \geq \gamma\big(d(x_{n+1}, z, a)\big) d(x_{n+1}, z, a) > d(x_{n+1}, z, a).$$

Hence, we have

$$d(u, z, a) = \lim_{n\to\infty} d(x_{n+1}, z, a) \leq \lim_{n\to\infty} d(x_n, u, a) = 0, \forall a \in X.$$

Consequently $z = u = fz$. Namely, z is a fixed point of f.

Suppose that $w \in X$ is another fixed point of f, then there exists $b \in X$ such that $d(z, w, b) > 0$. By (2.15), we have

$$d(z, w, b) = d(fz, fw, b) \geq \gamma\big(d(z, w, b)\big) d(z, w, b) > d(z, w, b),$$

which is a contradiction. Hence u is the unique fixed point of f. ∎

Using the idea of Theorem 2.2 and modifying the proof of Theorem 2.10, we obtain another form of Theorem 2.10 as follows:

Theorem 2.11 *Let (X, d) be a complete 2-metric space and $f : X \to X$ be a surjective mapping. If*

$$d(fx, fy, a) \geq \gamma\big(d(fx, fy, a)\big)\, d(x, y, a), \forall x, y, a \in X, \tag{2.21}$$

where $\gamma : [0, \infty) \to (1, \infty)$ is a function satisfying the following condition : $\gamma(t_n) \to 1$ as $n \to \infty$ which implies $t_n \to 0$ as $n \to \infty$. Then f has an unique fixed point.

Remark 2.12 Theorems 2.7 and 2.8 are some other versions of theorems 2.10 and 2.11 on real metric spaces respectively. Although Theorems 2.7 and 2.8 can follow from Theorem 2.4 and theorem D, Theorems 2.10 and 2.11 can not follow from Theorems 2.1 and 2.2. In fact, f in Theorems 2.10 and 2.11 is not necessarily onto, hence f can not be invertible. Consequently we can not use the method of the proof of Theorem 2.7 and Theorem 2.8.

Conflict of Interests

The author declares that there is no conflict of interests.

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