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FIXED POINT THEOREMS IN CONE RANDOM METRIC SPACES

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Abstract: We define Cone random metric space and find some fixed point results for weak contraction condition.
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1.INTRODUCTION

Random fixed point theorem for contraction mappings in polish spaces and random fixed point theorems are of fundamental importance in probabilistic functional analysis. Their study was initiated by the Prague school of Probabilistics with work of Spacek[28] and Hans[11,12]. For example survey are refer to Bharucha-Ried[8], Itoh[15] proved several random fixed point theorems and gave their applications to random differential equations in Banach spaces. Random coincidence point theorems and random fixed point theorem are stochastic generalization of classical coincidence point theorems and classical fixed point theorems. Sehgal and Singh[26], Papageorgiou[22], Rhoades Sessa Khan[25] and Lin[19] have proved differential stochastic version of well knownSchauder's fixed point theorem.Then Beg and Shahzad[4] studied the structure of common random fixed points and random coincidence points of a pair of compatible random operators.

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In [14] Huang and Zhang generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space, hence they have defined the cone metric spaces. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality of a cone. Subsequently, various authors have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cones. There exist a lot of works involving fixed points used the Banach contraction principle. This principle has been extended kind of contraction mappings by various authors.

2.PRELIMINARY

Definition 2.1: Let (E, τ) be a topological vector space and P a subset of E, P is called a cone if

1. P is non-empty and closed, $P \neq \{0\}$,

2. For $x, y \in P$ and $a, b \in R \Rightarrow ax + by \in P$ where $a,b \geq 0$

1. If $x \in P$ and $-x \in P \Rightarrow x = 0$
   For a given cone $P \subseteq E$, a partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$, $x < y$ if $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int } P$, int P denotes the interior of P.

Definition 2.2 : Measurable function : Let $(\Omega, \sum)$ be a measurable space with $\sum$ a sigma algebra of subsets of Ω and M a non-empty subset of a metric space X = (X, d). Let 2^M be the family of all non-empty subsets of M and C(M) the family of all nonempty closed subsets of M. A mapping $G: \Omega \to 2^M$ is called measurable if, for each open subset U of M,

Definition 2.3: Measurable selector: A mapping $\xi : \Omega \to M$is called a measurable selector of a measurable mapping $G : \Omega \to 2^M$ if $\xi$is measurable and $\xi(w) \in G(w)$ for each $w \in \Omega$.

Definition 2.4: Random operator: Mapping $T : \Omega \times M \to X$ is said to be a random operator if, for each fixed $x \in M$, $T(., x) : \Omega \to X$ is measurable.

Definition 2.5:Continuous Random operator: A random operator $T : \Omega \times M \rightarrow X$ is said to be continuous random operator if, for each fixed $x \in M$, $T(., x) : \Omega \rightarrow X$ is continuous.

Definition 2.6: Randomfixed point: A measurable mapping $\xi : \Omega \rightarrow M$is a random fixed point of a random operator $T : \Omega \times M \rightarrow X$ if $\xi(w) = T(w, \xi(w))$ for each $w \in \Omega$.

Definition 2.7: Let M be a nonempty set and the mapping $d: \Omega \times M \rightarrow X$ and $P \subset X$ be a cone, $w \in \Omega$ be a selector, satisfies the following conditions:

2.7.1) $d(x(w), y(w)) > 0 \, \forall \, x(w), y(w) \in \Omega \times X \Leftrightarrow x(w) = y(w)$
2.7.2) $d(x(w), y(w)) = d(y(w), x(w)) \;\; \forall x, y \in X, w \in \Omega$ *and* $x(w), y(w) \in \Omega \times X$
2.7.3) $d(x(w), y(w)) = d(x(w), z(w)) + d(z(w), y(w)) \forall x, y \in X$ *and* $w \in \Omega$ *be a selector.*
2.7.4) *For any* $x, y \in X, w \in \Omega$, $d(x(w), y(w))$ *is non-increasing and left continuous in* α.

Then d is called cone random metric on M and (M, d) is called a cone random metric space.

Defination 2.8: Implicit Relation

Let Φ be the class of all real-valued continuous functions $\varphi : (R^+)^5 \rightarrow R^+$ non -decreasing in the first argument and satisfying the following conditions:

For $x, y \geq 0$,

$x \leq \varphi(y,0,x,y, x+y)$ or $x \leq \varphi(y,y, x, y,x)$ or $x \leq \varphi(x, y,0,x, y)$

there exists a real number $0< h <1$ such that $x \leq hy$

3. MAIN RESULTS

Theorem 3.1 : Let (X, d) be a complete cone metric space and let M be a nonempty separable closed subset of cone metric space X and let T be continuous random operators defined on M such that for $w \in \Omega, T(w,.) : \Omega \times M \rightarrow M$ satisfying contraction. A

$d(T(x(w), Ty(w)) \leq \phi$ (d(x(w), y(w)) , d(y(w), T(x(w)), d(y(w), T(y(w)) ,

d(x(w), T(x(w)), d(x(w), T(y(w))) ……………………(1)

for all $x, y \in X$, $w \in \Omega$. Then T has a fixed point in X.

Proof: For each $x_0(w) \in \Omega \times X$ and $n \geq 1$, let $x_1 = T x_0$ and $x_{n+1}(w) = T(x_n(w)) = T^{n+1}x_0(w)$. Then

$d(x_n(w), x_{n+1}(w)) = d(T(x_{n-1}(w)), T(x_n(w)))$

$\leq \phi(d(x_{n-1}(w), x_n(w)), d(x_n(w), T(x_{n-1}(w)), d(x_n(w), T(x_n(w)),$

$d(x_{n-1}(w), T(x_{n-1}(w)), d(x_{n-1}(w), T(x_n(w)))$

$\leq \phi(d(x_{n-1}(w), x_n(w)), d(x_n(w), x_n(w)), d(x_n(w), x_{n+1}(w)), d(x_{n-1}(w), x_n(w)), d(x_{n-1}(w), x_{n+1}(w)))$

$\leq \phi(d(x_{n-1}(w), x_n(w)), 0, d(x_n(w), x_{n+1}(w)), d(x_{n-1}(w), x_n(w)),$

$d(x_{n-1}(w), x_n(w)) + d(x_n(w), x_{n+1}(w)))$

Hence from (2.8) we have

$d(x_n(w), x_{n+1}(w)) \leq h(d(x_{n-1}(w), x_n(w)))$

Similarly

$d(x_{n-1}(w), x_n(w)) \leq h\, d(x_{n-2}(w), x_{n-1}(w))$

Hence $d(x_n(w), x_{n+1}(w)) \leq h(d(x_{n-1}(w), x_n(w))) \leq h^2 d(x_{n-2}(w), x_{n-1}(w))$

On continuing this process

$d(x_n(w), x_{n+1}(w)) \leq h^n(d(x_0(w), x_1(w)))$

So for n>m

$d(x_m(w), x_n(w)) \leq (h^m + h^{m+1} + h^{m+2} \ldots\ldots\ldots\ldots + h^{n-1})(d(x_0(w), x_1(w)))$

$\leq \frac{h^m}{1-h}(d(x_0(w), x_1(w)))$

Let $0 \ll c$ be given. Choose a natural number N such that

$\frac{h^m}{1-h}(d(x_0(w), x_1(w))) \ll c$for every $m \geq N$. Thus

$d(x_m(w), x(w)_n) \leq \frac{h^m}{1-h}(d(x(w)_0, x_1(w))) \ll c$for every $n > m \geq N$.

Therefore the sequence $\{x_n(w)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\Omega \times X$. Since (X, d) is complete, there exists $z(\omega) \in \Omega \times X$ such that $x_n(\omega) \rightarrow z(\omega)$. Choose a natural number N_1 such that

Hence we have

$d(z(w), Tz(w)) \leq d(z(w), x_{n+1}(w)) + d(x_{n+1}(w), Tz(w))$

$= d(z(w), x_{n+1}(w)) + d(Tx_n(w), Tz(w))$

$\leq d(z(w), x_{n+1}(w)) + \phi(d(x_n(w), z(w)), d(z(w), Tx_n(w)), d(z(w), Tz(w)),$

$d(x_n(w), Tx_n(w)), d(x_n(w), Tz(w)))$

$\leq d(z(w), x_{n+1}(w)) + \phi(d(x_n(w), z(w)), d(z(w), x_{n+1}(w)), d(z(w),Tz(w)),$

$d(x_n(w),x_{n+1}(w)), d(x_n(w), Tz(w)))$

Taking $n\to\infty$ we have

$d(z(w), Tz(w)) \leq 0 + \phi(0, 0, d(z(w), Tz(w)),0, d(z(w), Tz(w)))$

$d(z(w), Tz(w)) \leq \phi(0, 0, d(z(w), Tz(w)), 0, d(z(w), Tz(w)))$

$d(z(w), Tz(w)) \leq 0$

Thus $-(d(z(w), Tz(w)))\in P$. But $d(z(w), Tz(w)) \in P$.

Therefore $d(z(w), Tz(w))= 0$ and so $Tz(w) = z(w)$.

Example: Let M=R and $P=\{x\in M:x\geq 0\}$, also $\Omega=[0,1]$ and Σ be the sigma algebra of Lebesgue's measurable subset of [0,1]. Let $X = [0,\infty)$ and define mapping as $d:(\Omega xX)x(\Omega xX)\to M$ by $d(x(w),y(w)) = |x(w)-y(w)|$. Then (X,d) is a cone random metric space. Define random operator T from ΩxX to X as $T(x(w)) = x(w)/2$. Also sequence of mapping $x_n:\Omega\to X$ is defined by $x_n(w) = \{(1 -w^2)^{1+1/n}\}$ for every $w\in\Omega$ & $n\in N$. Define measurable mapping $x: \Omega\to X$ as $x(w)=\{1-w^2\}$ for every $w\in\Omega$. T Satisfies all condition of the theorem and hence $(1-w^2)$ is fixed point of the space.

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