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COMMON FIXED POINT THEOREMS FOR THREE MAPS UNDER NONLINEAR CONTRACTION OF CYCLIC FORM IN PARTIALLY ORDERED *G*-METRIC SPACES

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Abstract. In this paper, we introduced the notion of a cyclic (ψ, A, B, C)-contraction for the pair (f, g, h) of self-mappings on the set X. We utilize our definition to introduce some common fixed point theorems for the three mappings f, g and h under a set of conditions. As application of our results, we derive some fixed point theorems of integral type.

Keywords: G-metric spaces; Common fixed point; Altering distance function; Weakly increasing mappings; Cyclic map.

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1. Introduction

One of the most important problems in mathematical analysis is to establish existence and uniqueness theorems for some integral and differential equations. Fixed point theorems in (ordered) metric spaces are of great use in it. Mustafa and Sims [1] generalized the concept of a metric space, which called G-metric space. In recent years many authors established important

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results in fixed point theory based on the notion of G-metric spaces (see [5, 6, 7, 8, 9, 11, 16]). One of the popular topics in the fixed point theory is the cyclic contraction. Kirk *et al.* [3] established the first result in this interesting area. Very recently, several authors proved many important results in fixed point theory for cyclic mappings satisfying various nonlinear contractive conditions (see [5-11]) in G-metric space. Some of contractive conditions are based on functions which alter the distance between two points in a G-metric spaces. Such functions were introduced by Khan *et al.* [12], Altun *et al.* [13, 14] introduced the notion of weakly increasing mappings and proved some existing theorems. For some works in the theory of weakly increasing mappings, we refer readers to [15, 16]. In 2013, Wasfi and Mihai [17] introduce the notion of a cyclic (ψ,A,B)-contraction based on the notion of cyclic map, altering distance function and weakly increasing map for the pair (f,T) on the set X. And obtained some common fixed point theorem for the two mappings f and T in ordered metric spaces. In this paper, we generalized the concept of (ψ,A,B)-contraction for the pair (f,T) to (ψ,A,B,C)-contraction for the pair (f,g,h), and obtained some unique common fixed point theorem for three maps f, g and h in partially ordered G-metric spaces. The purpose of this paper is to obtain common fixed point results for three maps satisfying nonlinear contractive conditions of a cyclic form based on the notion of an altering distance function in partially ordered G-metric spaces.

2. Preliminaries

We begin with the definition of the G-metric space.

Definition 2.1. [1] Let X be a nonempty set. A function G: $X\times X\times X\to[0,\infty)$ is called G-metric on X if it satisfy the following properties:

(G1) $G(x,y,z)=0$ if $x=y=z$;

(G2) $0<G(x,x,y)$ for all $x,y\in X$ with $x\neq y$;

(G3) $G(x,x,y)\leq G(x,y,z)$ for all $x,y,z\in X$ with $y\neq z$;

(G4) $G(x,y,z)=G(x,z,y)=G(y,z,x)=\ldots$(:symmetry in all three variables);

(G5) $G(x,y,z)\leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a\in X$.

Then the function G is called a *generalized metric* or, more specifically, a $G-metric$ on X and the pair (X,G) is called a $G-metric\ space$.

This notion of $G-metric$ was introduced by Mustafa and Sims [1]. It can be shown that if (X,d) is a metric space one can define G-metric on X by

$$G(x,y,z) = \max d(x,y), d(y,z), d(z,x) \text{ or } G(x,y,z) = d(x,y) + d(y,z) + d(z,x).$$

Let X be a nonempty set. Then $(X,G,\preceq)$ is called an *partially ordered* $G-metric\ space$ if and only if (X,G) is a G-metric space and $(X,\preceq)$ is a partially ordered set. Two elements $x,y \in X$ are called *comparable* if $x \preceq y$ or $y \preceq x$.

Definition 2.2. [1] Let(X,G) be a G-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is $G-convergent$ to a point $x \in X$ or x_n $G-converges$ to x if, for any $\varepsilon > 0$, there exists $k \in N$ such that $G(x,x_n,x_m) < \varepsilon$ for all $m,n \geq k$, that is, $\lim_{n,m\to+\infty} G(x,x_n,x_m)$. In this case, we write $x_n \to x$ or $\lim_{n\to+\infty} x_n = x$.

Proposition 2.1. [1] Let (X,G) be a G-metric space. Then the following are equivalent:

(1) $\{x_n\}$ is $G-convergent$ to x.

(2) $G(x_n,x_n,x) \to 0$ as $n \to +\infty$.

(3) $G(x_n,x,x) \to 0$ as $n \to +\infty$.

(4) $G(x_n,x_m,x) \to 0$ as $n,m \to +\infty$.

Definition 2.3. [1] Let (X,G) be a G-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a $G-Cauchy\ sequence$ if, for any $\varepsilon > 0$, there exists $k \in N$ such that $G(x_n,x_m,x_l)$ for all $m,n,l \geq k$, that is, $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to +\infty$.

Proposition 2.2 [1] Let (X,G) be a G-metric space. Then the following are equivalent:

(1) The sequence x_n is a $G-Cauchy\ sequence$.

(2) For any $\varepsilon > 0$, there exists $k \in N$ such that $G(x_n,x_m,x_m) < \varepsilon)$ for all $m,n \geq k$.

Proposition 2.3. [1] Let (X,G) be a G-metric space. Then, f: $X \to X$ is $G-continuous$ at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $f(x_n)$ is G-convergent to $f(x)$.

Definition 2.4. [1] A G-metric space (X,G) is called $G-complete$ if every G-cauchy sequence is G-convergent in (X,G).

Based on the notion of G-metric space, Some very recent fixed point theorems regarding cyclic maps in G-metric spaces are given in [5, 6]. The notion of cyclic map has been first introduced by Kir-srinavasan-veevamani [3].

Definition 2.5. [3] Let X be a nonempty set and let $Y=\bigcup_{j=1}^{m} A_j$ where $\{A_j\}_{j=1}^{m}$ is a family of nonempty subsets of X. A map $T: Y \to Y$ is called $cyclicmap$ if

$$T(A_j) \subseteq A_{j+1}, j=1,\ldots,m, where A_{m+1}=A_1.$$

In [4], Karapinar gave the following interesting theorem regarding cyclic maps in the G-metric space.

Theorem 2.1. [4] Let (X,G) be a G-complete G-metric space and $\{A_j\}_{j=1}^{m}$ be a family of nonempty G-closed subsets of X. Let $Y=\bigcup_{j=1}^{m} A_j$ and $T: Y \to Y$ be a map satisfying

$$T(A_j) \subseteq A_{j+1}, j=1,\ldots,m, where\ A_{m+1}=A_1.$$

If there exists $k \in (0,1)$ such that

$$G(Tx,Ty,Tz) \leq kG(x,y,z)$$

hold for all $x \in A_j$ and $y,z \in A_{j+1}$, $j=1,\ldots m$, then T has a unique fixed point in $\bigcap_{j=1}^{m} A_j$.

Definition 2.6. [11] The function ψ: $[0,+\infty) \to [0,+\infty)$ is called an $altering\ distance\ function$ if the following properties are satisfied:

(1) ψ is continuous and nondecreasing;

(2) $\psi(t)=0$ if and only if $t=0$.

Remark 2.7. [13] Let $(X,\preceq)$ be a partially ordered set. Two mappings $F,G: X \to X$ are said to be weakly increasing if $Fx \preceq GFx$ and $Gx \preceq FGx$ for all $x \in X$.

Remark 2.8. [17] Let $(X,\preceq)$ be a partially ordered set and A, B be closed subsets of X with $X=A\cup B$. Let f, T: $X \to X$ be two mappings. The pair (f,T) is said to be $(A,B)-$*weakly increasing* if $fx \preceq Tfx$ for all $x \in A$ and $Tfx \preceq fTx$ for all $x \in B$.

3. Main results

We start with the following definition.

Definition 3.1. Let $(X,\preceq)$ be a partially ordered set and A, B, C be closed ordered subsets of X with $X=A\cup B\cup C$. Let f, g, h: $X\to X$ be three mappings. The pair (f,g,h) is said to be $(A,B,C)-$*weakly increasing* if $fx\preceq gfx$ for all $x\in A$, $gx\preceq hgx$ for all $x\in B$ and $hx\preceq fhx$ for all $x\in C$.

Definition 3.2. Let $(X,G,\preceq)$ be an partially ordered G-metric space and A, B, C be nonempty closed ordered subsets of X. Let f, g, h: $X\to X$ be three mappings. The pair (f,g,h) is called a *cyclic* $(\psi,A,B,C)-$*contraction* if

(1) ψ is an altering distance function;

(2) $A\cup B\cup C$ has a cyclic representation w.r.t the pair (f,g,h); that is, $fA\subseteq B$,$gB\subseteq C$,$hC\subseteq A$ and $X=A\cup B\cup C$;

(3) There exists $0<\delta<1$ such that for three comparable elements $x,y,z\in X$ with $x\in A$, $y\in B$ and $z\in C$, we have

$$\psi(2G(fx,gy,hz))\leq\delta(M(x,y,z)),$$

where $M(x,y,z)=max\{G(x,y,z),G(x,fx,gy),G(y,gy,hz),G(z,hz,fx),$
$\frac{1}{4}(G(fx,y,z)+G(x,gy,z)+G(x,y,hz))\})$.

From now on, by ψ we mean altering distance functions unless otherwise stated. In the rest of this paper, $\mathbb{N}$ stands for the set of nonnegative integer numbers and $M(x,y,z)=max\{G(x,y,z),G(x,fx,gy),G(y,gy,$
$G(x,gy,z)+G(x,y,hz))\})$.

Theorem 3.1. *Let $(X,G,\preceq)$ be an partly ordered complete G-metric space and A, B, C be nonempty closed ordered subsets of X. Let f, g, h: $X\to X$ be three mappings such that the pair (f,g,h) is (A,B,C)-weakly increasing. Assume the following:*

(1) The pair (f,g,h) is a cyclic (ψ,A,B,C)-contraction;

(2) Two of f, g and h are continuous.

Then, at least, one of the mappings of f, g, or h has a fixed point, or, the maps f, g and h have a unique common fixed point in $A\cap B\cap C$.

Proof. Choose $x_0 \in A$. Let $x_1 = fx_0$. Since $fA \subseteq B$, we have $x_1 \subseteq B$. Also, let $x_2 = gx_1$. Since $gB \subseteq C$, we have $x_2 \subseteq C$. Let $x_3 = hx_2$. Since $hC \subseteq A$, we have $x_3 \subseteq A$. Continuing this process, we can construct a sequence $\{x_n\}$ in X such $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$, $x_{3n+3} = hx_{3n+2}$, $x_{3n} \in A$, $x_{3n+1} \in B$, $x_{3n+2} \in C$. Since f, g and h are (A,B,C)-weakly increasing, we have

$$x_1 = fx_0 \preceq gfx_0 = x_2 = gx_1 \preceq Tgx_1 = x_3 \preceq fTx_2 = x_4 \preceq \dots$$

We divide our proof into the following cases.

Case 1: Suppose that $x_n = x_{n+1}$for some $n \in N$.

If $x_{3n} = x_{3n+1}$, then x_{3n} is a fixed point of f. If $x_{3n+1} = x_{3n+2}$, then x_{3n+1} is a fixed point of g. If $x_{3n+2} = x_{3n+3}$, then x_{3n+2} is a fixed point of h. Thus, at least, one of the mappings of f, g, or h has a fixed point.

Case 2: $x_n \neq x_{n+1}$ for all n.

We divide our proof into the following steps.

Step 1. We will show that x_n is a Cauchy sequence in (X,G). Since x_{3n}, x_{3n+1}, x_{3n+2} are comparable elements in X with $x_{3n} \in A$, $x_{3n+1} \in B$, $x_{3n+2} \in C$, let $d_n = G(x_n, x_{n+1}, x_{n+2})$, we obtain that

$$\begin{aligned}
\psi(2d_{3n+1}) &= \psi(2G(fx_{3n}, gx_{3n+1}, hx_{3n+2})) \\
&\leq \delta\psi(max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, fx_{3n}, gx_{3n+1}) \\
&\quad G(x_{3n+1}, gx_{3n+1}, hx_{3n+2}), G(x_{3n+2}, hx_{3n+2}, fx_{3n}) \\
&\quad \frac{1}{4}(G(fx_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n}, gx_{3n+1}, x_{3n+2}) \\
&\quad +G(x_{3n}, x_{3n+1}, hx_{3n+2})\}) \\
&= \delta\psi(max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \\
&\quad \frac{1}{4}(G(x_{3n+1}, x_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+2}, x_{3n+2}) \\
&\quad +G(x_{3n}, x_{3n+1}, x_{3n+3})\}).
\end{aligned}$$

Let

$$\omega = \frac{1}{4}(G(x_{3n+1}, x_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+2}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, x_{3n+3})).$$

Then we have

$$G(x_{3n+1},x_{3n+1},x_{3n+2}) \le G(x_{3n+1},x_{3n+2},x_{3n+3}) = d_{3n+1},$$

$$G(x_{3n},x_{3n+2},x_{3n+2}) \le G(x_{3n},x_{3n+1},x_{3n+2}) = d_{3n} = d_{3n},$$

$$\begin{aligned} G(x_{3n},x_{3n+1},x_{3n+3}) &\le G(x_{3n},x_{3n+1},x_{3n+2}) + G(x_{3n+2},x_{3n+2},x_{3n+3}) \\ &< d_{3n} + d_{3n+1} \\ &< 2\max(d_{3n},d_{3n+1}). \end{aligned}$$

Thus, we have

$$4\omega < d_{3n} + d_{3n+1} + 22\max(d_{3n},d_{3n+1}) \Rightarrow \omega < \max(d_{3n},d_{3n+1}).$$

Then we have

$$\psi(2d_{3n+1}) \le \delta\psi(\max(d_{3n},d_{3n+1})).$$

If $\max(d_{3n},d_{3n+1}) = d_{3n+1}$, then $\psi(2d_{3n+1}) \le \delta\psi(d_{3n+1})$, which is a contradiction. Thus $\max(d_{3n},d_{3n+1}) = d_{3n}$. Therefore, we have

$$\psi(d_{3n+1}) \le \psi(2d_{3n+1}) \le \delta\psi(d_{3n}). \tag{3.1}$$

Also, we have

$$\begin{aligned} \psi(2d_{3n+2}) &= \psi(2G(fx_{3n+3},gx_{3n+1},hx_{3n+2})) \\ &\le \delta\psi(max\{G(x_{3n+3},x_{3n+1},x_{3n+2}),G(x_{3n+3},fx_{3n+3},gx_{3n+1}) \\ &\quad G(x_{3n+1},gx_{3n+1},hx_{3n+2}),G(x_{3n+2},hx_{3n+2},fx_{3n+3},) \\ &\quad \frac{1}{4}(G(fx_{3n+3},x_{3n+1},x_{3n+2}) + G(x_{3n+3},gx_{3n+1},x_{3n+2}) \\ &\quad +G(x_{3n+3},x_{3n+1},hx_{3n+2})\}) \\ &= \delta\psi(max\{G(x_{3n+3},x_{3n+1},x_{3n+2}),G(x_{3n+3},x_{3n+4},x_{3n+2}), \\ &\quad \frac{1}{4}(G(x_{3n+4},x_{3n+1},x_{3n+2}) + G(x_{3n+3},x_{3n+2},x_{3n+2}) \\ &\quad +G(x_{3n+3},x_{3n+1},x_{3n+3})\}) \\ &= \delta\psi(max(d_{3n+1},d_{3n+2})). \end{aligned}$$

If $\max(d_{3n+1}, d_{3n+2}) = d_{3n+2}$, then $\psi(d_{3n+2}) \leq \delta\psi(d_{3n+2})$, which is a contradiction. Thus $\max(d_{3n+1}, d_{3n+2}) = d_{3n+1}$. Therefore, we have

$$\psi(d_{3n+2}) \leq \delta\psi(d_{3n+1}). \tag{3.2}$$

Similarly, we can obtain

$$\psi(d_{3n+3}) \leq \delta\psi(d_{3n+2}). \tag{3.3}$$

From (3.1), (3.2) and (3.3), we have

$$\psi(d_{n+1}) \leq \delta\psi(d_n). \tag{3.4}$$

Since ψ is an altering distance function, we have $\{d_n\}$ is a bounded nonincreasing sequence. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d_n = r$. Letting $n \to \infty$ in (3.4), we have $\psi(r) \leq \delta\psi(r)$. Since $0 < \delta < 1$, we have $\psi(r) = 0$ and hence $r = 0$. Thus

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+2}) = 0. \tag{3.5}$$

Since $x_{n+1} \neq x_{n+2}$ for every n, so by property $(G3)$, we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}).$$

Hence,

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0 \tag{3.6}$$

Also, by proposition 2.1, we have

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0 \tag{3.7}$$

Now, we prove that $\{x_n\}$ is a G-Cauchy sequence. For this aim, it is sufficient to show that the subsequence $\{x_{3n}\}$ is G-Cauchy in X. Assume on contrary that $\{x_{3n}\}$ is not G-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{3m_k}\}$ and $\{x_{3n_k}\}$ of $\{x_{3n}\}$ such that m_k is the smallest index for which $3m_k > 3n_k > k$ and

$$G(x_{3n_k}, x_{3m_k}, x_{3m_k}) \geq \varepsilon \tag{3.8}$$

This means that

$$G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) < \varepsilon. \tag{3.9}$$

Since x_{3n_k}, x_{3m_k-2} and x_{3m_k-1} are comparable elements in X with $x_{3n_k} \in A$, $x_{3m_k-2} \in B$ and $x_{3m_k-1} \in C$, we have

$$\begin{aligned}
& \psi(2G(x_{3n_k+1}, x_{3m_k-1}, x_{3m_k})) \\
= \; & \psi(2G(fx_{3n_k}, gx_{3m_k-2}, hx_{3m_k-1})) \\
\leq \; & \delta\psi(max\{G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}), G(x_{3n_k}, fx_{3n_k}, gx_{3m_k-2}), \\
& G(x_{3m_k-2}, gx_{3m_k-2}, hx_{3m_k-1}), G(x_{3m_k-1}, hx_{3m_k-1}, fx_{3n_k}), \\
& \frac{1}{4}(G(fx_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) + G(x_{3n_k}, gx_{3m_k-2}, x_{3m_k-1}) \\
& + G(x_{3n_k}, x_{3m_k-2}, hx_{3m_k-1}\}) \\
= \; & \delta\psi(max\{G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}), G(x_{3n_k}, x_{3n_k+1}, x_{3m_k-1}, \\
& G(x_{3m_k-2}, x_{3m_k-1}, x_{3m_k}), G(x_{3m_k-1}, x_{3m_k}, x_{3n_k+1}, \\
& \frac{1}{4}(G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1}) + G(x_{3n_k}, x_{3m_k-1}, x_{3m_k-1}) \\
& + G(x_{3n_k}, x_{3m_k-2}, x_{3m_k})\})
\end{aligned}$$

Note that

$$\begin{aligned}
2\varepsilon \; & \leq \; 2G(x_{3n_k}, x_{3m_k}, x_{3m_k}) \\
& \leq \; 2G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+1}) + 2G(x_{3n_k+1}, x_{3m_k}, x_{3m_k}) \\
& \leq \; 2G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+1}) + 2G(x_{3n_k+1}, x_{3m_k}, x_{3m_k-1}).
\end{aligned}$$

Taking the upper limit as $k \to \infty$ and from (3.6), we have

$$2\varepsilon \leq \limsup_{k\to\infty} 2G(x_{3n_k+1}, x_{3m_k}, x_{3m_k-1}) \tag{3.10}$$

Using ($G5$), we obtain that

$$G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) \leq G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-3}, x_{3m_k-2}, x_{3m_k-1}).$$

Taking the upper limit as $k \to \infty$ and using (3.5) and (3.9) we obtain that

$$\limsup_{k\to\infty} G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) \leq \varepsilon. \tag{3.11}$$

Also, from $(G5)$ we have

$$\begin{aligned}
& G(x_{3n_k}, x_{3n_k+1}, x_{3m_k-1}) \\
\leq\ & G(x_{3m_k-1}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-3}, x_{3n_k}, x_{3n_k+1}) \\
\leq\ & G(x_{3m_k-1}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-3}, x_{3n_k}, x_{3n_k}) \\
& + G(x_{3n_k}, x_{3n_k}, x_{3n_k+1}) \\
\leq\ & G(x_{3m_k-1}, x_{3m_k-3}, x_{3m_k-3}) + 2G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) \\
& + G(x_{3n_k}, x_{3n_k}, x_{3n_k+1}).
\end{aligned}$$

Taking the upper limit as $k \to \infty$ and from (3.5)-(3.7) and (3.9) we obtain that

$$\limsup_{k\to\infty} G(x_{3n_k}, x_{3n_k+1}, x_{3m_k-1}) \leq 2\varepsilon. \tag{3.12}$$

On the other hand, from $(G5)$ we have

$$\begin{aligned}
& G(x_{3m_k-1}, x_{3m_k}, x_{3n_k+1}) \\
\leq\ & G(x_{3n_k+1}, x_{3n_k}, x_{3n_k}) + G(x_{3n_k}, x_{3m_k}, x_{3m_k-1}) \\
\leq\ & G(x_{3n_k+1}, x_{3n_k}, x_{3n_k}) + G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) \\
& + G(x_{3m_k-3}, x_{3m_k}, x_{3m_k-1}).
\end{aligned}$$

By taking the upper limit as $k \to \infty$ and using (3.5)-(3.7) and (3.9) we have

$$\limsup_{k\to\infty} G(x_{3m_k-1}, x_{3m_k}, x_{3m_k+1}) \leq \varepsilon. \tag{3.13}$$

Using (G5) we have

$$\begin{aligned}
& G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1}) \\
\leq\ & G(x_{3n_k+1}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-3}, x_{3m_k-2}, x_{3m_k-1}) \\
\leq\ & G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3n_k}, x_{3n_k}, x_{3n_k+1}) \\
& + G(x_{3m_k-3}, x_{3m_k-2}, x_{3m_k-1})
\end{aligned}$$

Taking the upper limit as $k \to \infty$ and using (3.5)-(3.7) and (3.9), we have

$$\limsup_{k\to\infty} G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1}) \leq \varepsilon \tag{3.14}$$

Similarly, we obtain

$$\limsup_{k\to\infty} G(x_{3n_k}, x_{3m_k-2}, x_{3m_k}) \leq \varepsilon \tag{3.15}$$

On the other hand, we obtain from (G3) and (3.11)

$$\limsup_{k\to\infty} G(x_{3n_k}, x_{3m_k-1}, x_{3m_k-1}) \leq \limsup_{k\to\infty} G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) \leq \varepsilon. \tag{3.16}$$

From (3.14), (3.15) and (3.16) we have

$$\limsup_{k\to\infty}\{\frac{1}{4}G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1}) + G(x_{3n_k}, x_{3m_k-1}, x_{3m_k-1}) + G(x_{3n_k}, x_{3m_k-2}, x_{3m_k})\} \leq \frac{3}{4}\varepsilon < \varepsilon. \tag{3.17}$$

Taking the upper limit as $k \to \infty$ and using (3.10)-(3.13) and (3.17), we have

$$\begin{aligned}
\psi(2\varepsilon) &\leq \psi(\limsup_{k\to\infty} 2G(x_{3n_k+1}, x_{3m_k-1}, x_{3m_k})) \\
&= \psi(\limsup_{k\to\infty} 2G(fx_{3n_k}, gx_{3m_k-2}, hx_{3m_k-1})) \\
&\leq \delta\psi(\limsup_{k\to\infty} \max\{G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}), G(x_{3n_k}, x_{3n_k+1}, x_{3m_k-1}), \\
&\quad G(x_{3m_k-2}, x_{3m_k-1}, x_{3m_k}), G(x_{3m_k-1}, x_{3m_k}, x_{3n_k+1}), \\
&\quad \frac{1}{4}(G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1}) + G(x_{3n_k}, x_{3m_k-1}, x_{3m_k-1}) \\
&\quad + G(x_{3n_k}, x_{3m_k-2}, x_{3m_k}) \\
&\leq \delta\psi(2\varepsilon))\}).
\end{aligned}$$

Since $0 < \delta < 1$ and ψ is an altering distance function, we have $\psi(2\varepsilon) = 0$ and hence $\varepsilon = 0$, a contradiction.Thus $\{x_{3n}\}$ is a G-Cauchy sequence in (X, G).

Step 2. Existence of a common fixed point.

Since (X, G) is complete and $\{x_n\}$ is a G-Cauchy sequence in X, we have $\{x_n\}$ G-converges to some $u \in X$, Therefore

$$\lim_{n\to\infty} x_{3n+1} = \lim_{n\to\infty} fx_{3n} = \lim_{n\to\infty} x_{3n+2}$$

$$= \lim_{n\to\infty} gx_{3n+1} = \lim_{n\to\infty} x_{3n+3} = \lim_{n\to\infty} hx_{3n+2} = u.$$

Since x_{3n} is a sequence in A, A is closed and $x_{3n} \to u$, we have $u \in A$. Since x_{3n+1} is a sequence in B, B is closed and $x_{3n+1} \to u$, we have $u \in B$. Also, since x_{3n+2} is a sequence in C, C is closed and $x_{3n+2} \to u$, we have $u \in C$. Thus, $u \in A \cap B \cap C$. Now, we show that u is a common fixed point of f, g and h. Without loss of generality, we may assume f and g are continuous, since $x_{3n} \to u$, we get $x_{3n+1} = fx_{3n} \to fu$. By the uniqueness of limit, we have $u = fu$. Similarly, we obtain $u = gu$. Now, we show that $u = hu$. Since $u \preceq u \preceq u$ with $u \in A$, $u \in B$ and $u \in C$, we have

$$\begin{aligned}
\psi(2G(u,u,hu)) &= \psi(G(fu,gu,hu)) \\
&\leq \delta\psi(\max\{G(u,u,u), G(u,fu,gu), \\
&\quad G(u,gu,hu), G(u,hu,fu)), \\
&\quad \frac{1}{4}(G(fu,u,u) + G(u,gu,u) + G(u,u,hu))\}) \\
&= \delta\psi(G(u,u,hu)).
\end{aligned}$$

Since $0 < \delta < 1$ and ψ is an altering distance function, we get that $G(u,u,hu) = 0$ and hence $u = hu$. Now, we show that the common fixed point of f, g and h is unique. Assume on contrary that v is another fixed point of f, g and h i.e., fv=gv=hv=v, since $X = A \cup B \cup C$, it is easy to see $v \in A \cap B \cap C$. And from the proof above we know $u \in A \cap B \cap C$, since A, B and C are ordered subsets of X, we have, u and v are comparable with $u \in A$, $u \in B$ and $v \in C$, then we have

$$\begin{aligned}
\psi(2G(u,u,v)) &= \psi(2G(fu,gu,hv)) \\
&\leq \delta\psi(\max\{G(u,u,v), G(u,fu,gu), G(u,gu,hv), \\
&\quad G(v,hv,fu), \frac{1}{4}(G(fu,u,v) + G(u,gu,v) + G(u,u,hv))\}) \\
&= \delta\psi(\max\{G(u,u,v), G(v,v,u)\}).
\end{aligned}$$

Thus, we have

$$\psi(2G(u,u,v)) \leq \delta\psi(\max\{G(u,u,v), G(v,v,u)\}). \tag{3.18}$$

Similarly, we obtain

$$\psi(2G(v,v,u)) \leq \delta\psi(\max\{G(u,u,v), G(v,v,u)\}). \tag{3.19}$$

If $\max\{G(u,u,v),G(v,v,u)\} = G(u,u,v)$, we have from (3.18) $\psi(2G(u,u,v)) \leq \delta\psi(G(u,u,v))$. Since $0 < \delta < 1$, we get that $G(u,u,v) = 0$, and hence $u = v$. Similarly, if $\max\{G(u,u,v),G(v,v,u)\} = G(v,v,u)$, we also can obtain $u = v$. Thus, we have f, g and h have a unique common fixed in $A\cap B\cap C$.

Theorem 3.1 can be proved without assuming the continuity of f, g or h. For this instance, we assume that X satisfies the following property.

(P) [17] If $\{x_n\}$ is a nondecreasing sequence in X with $x_n \to x$, then $x_n \preceq x$.

Now, we state and prove the following result.

Theorem 3.2. *Let $(X,G,\preceq)$ be an partly ordered complete G-metric space and A, B, C be nonempty closed ordered subsets of X. Let f, g, h: $X \to X$ be three mappings such that the pair (f,g,h) is (A,B,C)-weakly increasing. Assume the following:*

(1) The pair (f,g,h) is a cyclic (ψ,A,B,C)-contraction;

(2) X satisfies the property (P).

Then, at least, one of the mappings of f, g and h has a fixed point, or, the mappings f, g and h have a unique common fixed point in $A\cap B\cap C$.

Proof. We follow the proof of Theorem 3.1 step by step to construct a nondecreasing sequence $\{x_n\}$ in X with $x_{3n} \in A$, $x_{3n+1} \in B$, $x_{3n+2} \in C$ and $x_n \to u$ for some $u \in A\cap B\cap C$. Using property (P), we get $x_n \preceq u$ for all $n \in \mathbb{N}$. Since $x_{3n} \in A$, $x_{3n+1} \in B$, $u \in C$, we have

$$\begin{aligned}
\psi(2G(x_{3n+1},x_{3n+2},hu)) &\leq \delta\psi(\max\{G(x_{3n},x_{3n+1},u),G(x_{3n},fx_{3n},gx_{3n+1}),\\
&\quad G(x_{3n+1},gx_{3n+1},hu),G(u,hu,fx_{3n}),\\
&\quad \frac{1}{4}(G(fx_{3n},x_{3n+1},u)+G(x_{3n},gx_{3n+1},u)\\
&\quad +G(x_{3n},x_{3n+1},hu))\})\\
&= \delta\psi(\max\{G(x_{3n},x_{3n+1},u),G(x_{3n},x_{3n+1},x_{3n+2}),\\
&\quad G(x_{3n+1},gx_{3n+1},hu),G(u,hu,x_{3n+1}),\\
&\quad \frac{1}{4}(G(x_{3n+1},x_{3n+1},u)+G(x_{3n},x_{3n+2},u)\\
&\quad +G(x_{3n},x_{3n+1},hu))\}).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get $\psi(2G(u,u,hu)) \leq \delta \psi(G(u,u,hu))$. Since $0 < \delta < 1$ and ψ is an altering distance function, we get $(G(u,u,hu)) = 0$, hence $u = hu$. Similarly, we may show that $u = fu$ and $u = gu$. Thus u is a common fixed point of f, g and h. And follow the proof of Theorem 3.1, we have, $u \in A \cap B \cap C$ is the unique fixed point of f, g and h.

Taking $\psi = I_{[0,+\infty)}$ (the identity function) in Theorem 3.1, we have the following result.

Corollary 3.1. *Let $(X,G,\preceq)$ be an partly ordered complete G-metric space and A, B, C be nonempty closed ordered subsets of X. Let f, g, h: $X \to X$ be three mappings such that the pair (f,g,h) is (A,B,C)-weakly increasing and $A \cap B \cap C$ has a cyclic representation with respect to the pair (f,g,h). Suppose that there exists $0 < \delta < 1$ such that for any three comparable elements $x,y,z \in X$ with $x \in A$, $y \in B$ and $z \in C$, we have $2G(fx,gy,hz) \leq \delta(M(x,y,z))$. If two of f, g and h are continuous, then, at least one of the mappings f, g and h has a fixed point, or, the mappings f, g and h have a unique common fixed point in $A \cap B \cap C$.*

By taking $f = g = h$ in Theorem 3.1, we have the following result.

Corollary 3.2. *Let $(X,G,\preceq)$ be an partly ordered complete G-metric space and A, B, C be nonempty closed ordered subsets of X. Let f: $X \to X$ be a map such that $fx \preceq f(fx)$ for all $x \in X$. Suppose that there exists $0 < \delta < 1$ such that for any three comparable elements $x,y,z \in X$ with $x \in A$, $y \in B$ and $z \in C$, we have*

$$\psi(2G(fx,fy,fz)) \leq \delta\psi(max\{G(x,y,z),G(x,fx,fy),G(y,fy,fz),G(z,fz,fx),$$
$$\frac{1}{4}(G(fx,y,z)+G(x,fy,z)+G(x,y,fz))\})$$

Assume the following:

(1) f is a cyclic map;

(2) f is continuous.

Then f has a unique fixed point in $A \cap B \cap C$.

Taking $A = B = C = X$ in Theorem 3.1, we have the following result.

Corollary 3.3. *Let $(X,G,\preceq)$ be an partially ordered complete G-metric space. Let f, g, h: $X \to X$ be three weakly increasing mappings. Suppose that there exists $0 < \delta < 1$ such that for any three comparable elements $x,y,z \in X$, we have $\psi(2G(fx,gy,hz)) \leq \delta\psi(M(x,y,z))$. If two* *of f, g and T are continuous, then at least, one of the mappings f, g or h has a fixed point, or, the mappings f, g and h have a unique common fixed point.*

4. Applications

Denote by Λ the set of functions $\mu : [0,+\infty) \to [0,+\infty)$ satisfying the following hypotheses:

(h1) μ is a Lebesgue-integrable mapping on each compact of $[0,+\infty)$;

(h2) For every $\varepsilon > 0$,we have $\int_0^{\varepsilon} \mu(t)dt > 0$.

Theorem 4.1. *Let $(X,G,\preceq)$ be an partly ordered complete G-metric space and A, B, C be nonempty closed ordered subsets of X. Let $f,g,h : X \to X$ be three mappings such that the pair (f,g,h) is (A,B,C)-weakly increasing and $A\cap B\cap C$ has a cyclic representation with respect to the pair (f,g,h). Suppose that there exists $0 < \delta < 1$ such that for any three comparable elements $x,y,z \in X$ with $x \in A$, $y \in B$ and $z \in C$, we have $\int_0^{2G(fx,gy,hz)} \mu(s)ds \leq \delta \int_0^{M(x,y,z)} \mu(s)ds$. If two of f, g and h are continuous, then, at least, one of the mappings f, g or h has a fixed point, or, the mappings f, g and h have a unique common fixed point in $A\cap B\cap C$.*

By taking $f = g = h$ in Theorems 4.1, we have the following results.

Corollary 4.1. *Let $(X,G,\preceq)$ be an partly ordered complete G-metric space and A, B, C be nonempty closed ordered subsets of X. Let $f : X \to X$ be a map such that $fx \preceq f(fx)$ for all $x \in X$. Suppose that there exists $0 < \delta < 1$ such that for any three comparable elements $x,y,z \in X$ with $x \in A$, $y \in B$ and $z \in C$, we have $\int_0^{2G(fx,fy,fz)} \mu(s)ds \leq \delta \int_0^{M(x,y,z)} \mu(s)ds$. Assume the following:*

(1) f is a cyclic map;

(2) f is continuous.

Then f has a unique fixed point in $A\cap B\cap C$.

Taking $A = B = C = X$ in Theorem 4.1, we have the following result.

Corollary 4.2. *Let $(X,G,\preceq)$ be an partially ordered complete G-metric space. Let $f,g,h : X \to X$ be three weakly increasing mappings. Suppose that there exists $0 < \delta < 1$ such that for any three comparable elements $x,y,z \in X$, we have $\int_0^{2G(fx,gy,hz)} \mu(s)ds \leq \delta \int_0^{M(x,y,z)} \mu(s)ds$. If two* *of f, g and h are continuous,then, at least, one of the mappings f, g or h has a fixed point, or, the mappings f, g and h have a common fixed point.*

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex. Anal. 7 (2006), 289-297.

[2] B.S. Choudhury, P. Maity, Coupled fixed point results in generalized metric spaces, Math. Comput. Modelling 54 (2011), 73-79 .

[3] W.A. Krirk, P.S. Srinavasan, P. Veeramani, Fixed points for mappings satisfying cyclic contractive conditions, Fixed point Theory Appl. 4 (2003) 79-89.

[4] Z. Mustafa, A new structure for generalized metric spaces with applications to fixed point theory. PhD thesis. The University of Newcastle, Callaghan, Australia,(2005).

[5] E. Karapinar, I.M. Erhan, A. Yildiz-Uhus, Abst. Appl. Anal. 2012 (2012), Article ID 12947.)

[6] E. Karapinar, I.M. Erhan, Filomat, 26 (2012), 777-782.

[7] S. Bose, S.M. Hossein, Fixed point theorems for weak contraction in partially ordered G-metric space, eprint arxiv:1403.1935 (2014).

[8] T. Karapinar, R. Agarwal, Further fixed point results on G-metric spaces, Fixed Point Theory Appl. 2013 (2013), Article ID 154.

[9] M. Jleli, B. Samet, Remarks on G-metric spaces and fixed point theorems, Fixed point Theory Appl. 2012 (2012), Article ID 210.

[10] M.J, C.V,N.G,S.K, Mathematica Moravica 18 (2014), 45-62

[11] W. Shatanawi, Fixed point theory for contractive mappings satisfying ϕ-maps in G-metric spaces, Fixed Point Theory Appl. 9 (2010), Article ID 134.

[12] M.S. Khan, M. Sessa, Fixed point theorems by altering distances between the points. Bull. Aust. Math. Soc. 30 (1984), 1-9.

[13] I. Altun, B. Damjanoviv, D. Djoric, Fixed point and common fixed point theorems on ordered cone metric spaces, Appl. Math. Lett. 23 (2010), 310-316.

[14] I. Altun, H. Simsek, Some fixed point theorems on ordered metric spaces and application, Fixed Point Theory Appl. 11 (2010), 395-400.

[15] W. Shatanawi, Some coincidence point results in cone metric spaces, Math. Comput. Model. 55 (2012) 2023-2028.

[16] W. Shatanawi, Some fixed point theorems in ordered G-metric spaces and applications, Abstr. Appl. Anal. 2011 (2011), 1-11.

[17] W. Shatanawi, M. Postolache, Common fixed point results for mappings under nonlinear contraction of cyclic form in ordered metric spaces, Fixed Point Theory APP. 2013 (2013), Article ID 60.

[18] R.R. Jamal, S. Nabiollah, S. Shaban, P. Vahid, R. Stojan, Common fixed point theorems for three maps in discontinuous G_b-metric spaces, Acta Math. Sci. 34 (2014), 1643-1654.