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TRANSITIVE SUBSETS FOR SEMIGROUP ACTIONS

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Abstract. In this paper, we study transitive subsets of semigroup actions on topological spaces. Some basic concepts are introduced for dynamical systems of semigroup actions. We discuss some properties of transitive subsets for semigroup actions.

Keywords: Semigroup actions; Weakly mixing subset; Transitive subset.

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1. Introduction

Throughout this paper a dynamical system in the present article is a triple (S,X,π), where S is topological semigroup, X is at least a topological space and

$$\pi : S\times X \to X,\ (s,x)\mapsto sx$$

is a continuous action on X. Thus, $s_1(s_2x) = (s_1s_2)x$ holds for every triple (s_1,s_2,x) in $S\times S\times X$. Sometimes we write the dynamical system as a pair (S,X) (abbreviated by $S-$system) or even as X, when S is understood.

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In the present paper, let $\mathbb{N}$ denote the set of all positive integers and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

If $S = \{f^n\}_{n \in \mathbb{Z}_+}$ and $f : X \to X$ is a continuous map, then dynamical system (S,X) is the classical dynamical system. We use the standard notation: (X, f).

Topological transitivity and weak mixing (see [3, 10, 15, 17]) are global characteristics of classical dynamical systems. Let (X, f) be a classical dynamical system, (X, f) is topologically transitive if for any nonempty open subsets U and V of X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. (X, f) is weakly mixing if for any nonempty open subsets U_1, U_2, V_1 and V_2 of X, there exists $n \in \mathbb{N}$ such that $f^n(U_1) \cap V_1 \neq \emptyset$ and $f^n(U_2) \cap V_2 \neq \emptyset$. It follows from these definitions that mixing weak mixing implies transitivity.

Blanchard introduced overall properties and partial properties in [1]. For example, sensitive dependence on initial conditions, Devaney chaos (see [6]), weak mixing, mixing and more belong to overall properties; Li-Yorke chaos (see [12]) and positive entropy (see [3, 16])belong to partial properties. Weak mixing is an overall property, it is stable under semi-conjugate maps and implies Li-Yorke chaos. We have a weakly mixing system always contains a dense uncountable scrambled set from [9]. In [2], Blanchard and Huang introduced the concepts of weakly mixing subset and partial weak mixing, derived from a result given by Xiong and Yang [18] and showed "partial weak mixing implies Li-Yorke chaos "and "Li-Yorke chaos does not implies partial weak mixing". Let (X, f) be a classical dynamical system and A be a closed subset of X but not a singleton. Then A is weakly mixing subset of X if and only if for any $k \in \mathbb{N}$, any choice of nonempty open subsets $V_1, V_2, \cdots, V_k$ of A and nonempty open subsets $U_1, U_2, \cdots, U_k$ of X with $A \cap U_i \neq \emptyset$, $i = 1, 2, \cdots, k$, there exists $m \in \mathbb{N}$ such that $f^m(V_i) \cap U_i \neq \emptyset$ for $1 \leq i \leq k$. (X, f) is called partial weak mixing if X contains a weakly mixing subset. Motivated by the idea of Blanchard and Huang's notion of "weakly mixing subset", Oprocha and Zhang [13] extended the notion of weakly mixing subset and gave the concept of "transitive subset" and discussed its basic properties. Let A be a nonempty subset of X. A is called a transitive subset of (X, f) if for any choice of nonempty open subset V^A of A and nonempty open subset U of X with $A \cap U \neq \emptyset$, there exists $n \in \mathbb{N}$ such that $f^n(V^A) \cap U \neq \emptyset$.

In recent years, a large number of papers have been devoted to dynamical properties of semigroup and group actions on topological space. Kontorovich and Megrelishvili [11] studied sensitivity of semigroup actions, Polo [14] discussed sensitivity and chaos of group actions, Cairns, Kolganova and Nielsen [5] investigated topological transitivity and mixing for group actions; Braga and Souza [4] studied attractors and chain recurrence for semigroup actions, D.B. Ellis, R. Ellis and Nerurkar [8] discussed the topological dynamics of semigroup actions, Yan and He [19] studied topological complexity of semigroup actions.

In this paper we extend the notions of weakly mixing subset and transitive subset. We give the concepts of transitive subset, weakly mixing subset for semigroup actions and investigate the relationship between transitive subsets, weakly mixing subsets for semigroup actions.

2. Preliminaries

Let X be an $S-$system. The orbit of x is the set $Sx := \{sx : s \in S\}$ for every $x \in X$. If $Sx = \{sx : s \in S\}$ is finite for some $x \in X$, then the orbit of x is periodic. By $\overline{A}$ we will denote the closure of a subset $A \subseteq X$. If (S,X) is a system and Y is a closed $S-$invariant subset, i.e., $SY \subseteq Y$, then we say that (S,Y), the restricted action, is a subsystem of (S,X). Let $\emptyset \neq A \subseteq X$. Put $SA = \bigcup_{s\in S} sA$, the orbit of A under S. Clearly, $(S,\overline{SA})$ forms a subsystem of (S,X). For $U \subseteq X$ and $s \in S$ denote $s^{-1}U := \{x \in X : sx \in U\}$. In fact, for any $U \subseteq X$ and $s \in S$, we have $s(s^{-1}U) \subseteq U$.

Definition 2.1. [11] Let S be a topological semigroup. S is a $F-$semigroup if for every $s_0 \in S$ the subset $S \setminus Ss_0 = \{s \in S : s \notin Ss_0\}$ is finite.

Clearly, every topological group is an $F-$semigroup.

Definition 2.2. [11] The dynamical system (S,X) is called:

(1): topologically transitive if for every pair (U,V) of nonempty open sets U,V in X there exists $s \in S$ such that $U \cap sV \neq \emptyset$;

(2): point transitive if there exists point $x_0 \in X$ with dense orbit $Sx_0 = \{sx_0 : s \in S\}$. Such a point is called transitive point.

We will give the concepts of transitive subset, weakly mixing subset and sensitive subset for semigroup actions.

Definition 2.3. Let (S,X) be a dynamical system and A be a nonempty subset of X. A is called a transitive subset of (S,X) if for any choice of nonempty open subset V^A of A and nonempty open subset U of X with $A\cap U\neq\emptyset$, there exists $s\in S$ such that $sV^A\cap U\neq\emptyset$.

Remark 2.4. X is a transitive subset of (S,X) if and only if (S,X) is topologically transitive.

Definition 2.5. Let (S,X) be a dynamical system and A be a nonempty closed subset of X but not a singleton. A is called a weakly mixing subset of (S,X) if for any $k\in\mathbb{N}$, any choice of nonempty open subsets $V_1^A,V_2^A,\cdots,V_k^A$ of A and nonempty open subsets $U_1,U_2,\cdots,U_k$ of X with $A\cap U_i\neq\emptyset$, $i=1,2,\cdots,k$, there exists $s\in S$ such that $sV_i^A\cap U_i\neq\emptyset$ for $1\leq i\leq k$.

According to the definitions of transitive subset and weakly mixing subset for semigroup action, we have the following results.

Result 1.: If A is a weakly mixing subset of (S,X), then A is a transitive subset of (S,X);

Result 2.: If $a\in X$ is a transitive point of (S,X), then $\{a\}$ is a transitive subset of (S,X);

Result 3.: If $A=Sx$ is a periodic orbit of (S,X) for some $x\in X$, then A is a transitive subset of (S,X).

Example 2.6. *Let $(\sum_2,\sigma)$ be a one-sided symbolic dynamics, $\sum_2=\{x=(x_n)_{n=0}^{\infty}:x_n\in\{0,1\}$ for every $n\}$, $\sigma(x_0,x_1,x_2,\cdots)=(x_1,x_2,\cdots)$. Let $S=\{\sigma^n:n\in\mathbb{Z}_+\}$. Then S is a semigroup on composite operation of mapping. By Robinson* [15] *and Zhou* [20]*, $(\sum_2,\sigma)$ is topologically transitive and weakly mixing. Therefore, $\sum_2$ is a transitive subset and a weakly mixing subset of $(\sum_2,\sigma)$.*

Example 2.7. *Let $\mathbb{R}$ denotes the one-dimensional Euclidean space and let $S=\mathbb{R}$ is a semigroup on general multiplicative operation. Let d is a general metric in $\mathbb{R}$, i.e., $d(x,y)=|x-y|$ for any $x,y\in\mathbb{R}$ and let map*

$$\pi:S\times\mathbb{R}\to\mathbb{R},$$

such that $\pi(r_1,r_2)=r_1\times r_2$ for any $r_1\in S,\ r_2\in\mathbb{R}$. Then $(0,1)$ is a transitive subset of $(S,\mathbb{R})$.

In fact, π is a continuous action on $\mathbb{R}$. We will prove that $(0,1)$ is a transitive subset of $(S,\mathbb{R})$. Let $A=(0,1)$. For any nonempty open set V^A of A and any nonempty open set U of $\mathbb{R}$ with

$A \cap U \neq \emptyset$, then there exists $a \in V^A$ such that $a \neq 0$. Take $b \in U$. There exists $r = \frac{b}{a} \in S$ such that $b \in \frac{b}{a}V^A \cap U$. Therefore, $A = (0,1)$ is a transitive subset of $(S,\mathbb{R})$.

Definition 2.8. [7] Let (X,τ) be a topological space and A be a nonempty set of X. A is a regular closed set of X if $A = \overline{int(A)}$, where $int(A)$ denotes the interior of A.

We easily prove that A is a regular closed set if and only if $int(V^A) \neq \emptyset$ for any nonempty set V^A of A.

Definition 2.9. Let (X,τ) be a topological space. A and B are two nonempty subsets of X. B is dense in A if $A \subseteq \overline{A \cap B}$.

Clearly, B is dense in A if and only if $V^A \cap B \neq \emptyset$ for any nonempty open set V^A of A.

Definition 2.10. Let (S,X) and (S,Y) be two $S-$systems and $h : X \to Y$ be a continuous map.

- **(1):** (S,X) and (S,Y) are called to be topologically semi-conjugate, if h is surjective and $sh(x) = h(sx)$ for every $s \in S$, $x \in X$. Moreover, h is called semi-conjugate map, (S,Y) is called a factor of (S,X) and (S,X) is called an extension of (S,Y).
- **(2):** (S,X) and (S,Y) are called to be topologically conjugate, if h is a homeomorphism and $sh(x) = h(sx)$ for every $s \in S$, $x \in X$. Moreover, h is called a conjugate map.

3. Characterizing transitive subsets for semigroup actions

It is known that a classical dynamical system (X,f) is topologically transitive if and only if for any nonempty open subset U of X, $\bigcup\limits_{n=1}^{\infty} f^n(U)$ is a dense subset of X (see [3]). For transitive subsets of semigroup actions, we have the following proposition.

Proposition 3.1. *Let (S,X) be a dynamical system and A be a nonempty set of X. Then the following conditions are equivalent:*

- **(1):** *A is a transitive subset of (S,X).*
- **(2):** *Let V^A be a nonempty open subset of A and U be a nonempty open subset of X with $A \cap U \neq \emptyset$. Then there exists $s \in S$ such that $V^A \cap s^{-1}U \neq \emptyset$.*
- **(3):** *Let U be a nonempty open set of X with $A \cap U \neq \emptyset$. Then $S^{-1}U = \bigcup\limits_{s\in S} s^{-1}U$ is dense in A.*

Proof. $(1) \Longrightarrow (2)$ Let A be a transitive subset of (S,X). Then for any choice of nonempty open subset V^A of A and nonempty open subset U of X with $A \cap U \neq \emptyset$, there exists $s \in S$ such that $sV^A \cap U \neq \emptyset$. Since $s(V^A \cap s^{-1}U) = sV^A \cap U$, it follows that $V^A \cap s^{-1}U \neq \emptyset$.

$(2) \Longrightarrow (3)$ Let V^A be a nonempty open subset of A and U be a nonempty open subset of X with $A \cap U \neq \emptyset$. By the assumption of (2), there exists $s \in S$ such that $V^A \cap s^{-1}U \neq \emptyset$. Furthermore, we have

$$V^A \cap S^{-1}U = V^A \cap \bigcup_{s \in S} s^{-1}U = \bigcup_{s \in S} (V^A \cap s^{-1}U) \neq \emptyset.$$

Hence, $S^{-1}U$ is dense in A.

$(3) \Longrightarrow (1)$ Let V^A be a nonempty open subset of A and U be a nonempty open subset of X with $A \cap U \neq \emptyset$. Since $S^{-1}U = \bigcup_{s \in S} s^{-1}U$ is dense in A, it follows that $V^A \cap S^{-1}U \neq \emptyset$. Furthermore, there exists $s \in S$ such that $V^A \cap s^{-1}U \neq \emptyset$. Note that $s(V^A \cap s^{-1}U) = sV^A \cap U$, we have $sV^A \cap U \neq \emptyset$. Therefore, A is a transitive subset of (S,X).

Proposition 3.2. *Let (X,d) be a metric $S-$system and A be a nonempty subset of X. Then the following conditions are equivalent:*

- **(1):** *A is a transitive subset of (S,X).*
- **(2):** *Let $a,x \in A$ and $\varepsilon, \delta > 0$. Then there exists $s \in S$ such that $(A \cap B(a,\varepsilon)) \cap s^{-1}B(x,\delta) \neq \emptyset$.*
- **(3):** *Let $a,x \in A$ and $\varepsilon > 0$. Then there exists $s \in S$ such that $(A \cap B(a,\varepsilon)) \cap s^{-1}B(x,\varepsilon) \neq \emptyset$.*

Proof. $(1) \Longrightarrow (2)$ By the definition of transitive subset, (2) is obtained easily.

$(2) \Longrightarrow (3)$ is trivial.

$(3) \Longrightarrow (1)$ Let V^A be a nonempty open subset of A and U be a nonempty open subset of X with $A \cap U \neq \emptyset$. Then there exist $a,x \in A$ and $\varepsilon > 0$ such that $A \cap B(a,\varepsilon) \subseteq V^A$ and $B(x,\varepsilon) \subseteq U$. By the assumption of (3), there exists $s \in S$ such that $(A \cap B(a,\varepsilon)) \cap s^{-1}B(x,\varepsilon) \neq \emptyset$, which implies $V^A \cap s^{-1}U \neq \emptyset$. Therefore, A is a transitive subset of (S,X).

Proposition 3.3. *Let X be an $S-$system and A be a nonempty subset of X. Then A is a transitive subset if and only if $\overline{A}$ is a transitive subset. Thus if A is an invariant (i.e., $SA \subseteq A$) transitive subset, then $(S,\overline{A})$ is topologically transitive.*

Proof. First, assume that $\overline{A}$ is a transitive subset. Let V^A be a nonempty open subset of A and U be a nonempty open subset of X with $A\cap U\neq\emptyset$. Furthermore, there exists an open set V of X such that $V^A=V\cap A$. By the assumptions, there exists $s\in S$ with $(V\cap\overline{A})\cap s^{-1}U\neq\emptyset$, which implies that $(V\cap A)\cap s^{-1}U\neq\emptyset$ (as $V\cap s^{-1}U$ is an open set of X), i.e., $V^A\cap s^{-1}U\neq\emptyset$. Thus, A is a transitive subset. A similar reasoning shows that if A is a transitive subset then so is $\overline{A}$.

Proposition 3.4. *Let A is a transitive subset of (S,X). Then*

- **(1):** *if U is a nonempty open set of X satisfying $A\cap U\neq\emptyset$ and $s^{-1}U\subseteq U$ for every $s\in S$, then U is dense in A;*
- **(2):** *if E is a closed invariant subset of X and $E\subseteq A$, then $E=A$ or E is nowhere dense in A;*
- **(3):** *if A is a regular closed set of X, then $SA=\bigcup\limits_{s\in S}sA$ is dense A.*

Proof. (1) Since $s^{-1}U\subseteq U$ for every $s\in S$, then $\bigcup\limits_{s\in S}s^{-1}U\subseteq U$, By Proposition , we have U is dense in A.

(2) Assume $E\neq A$, we show E is nowhere dense in A. Since E is a closed set of X and $E\subseteq A$, it follows that $U=X\setminus E$ is an open set of X and $U\cap A\neq\emptyset$. As $SE\subseteq E$, we have $S^{-1}U=S^{-1}(X\setminus E)=S^{-1}X\setminus S^{-1}E\subseteq X\setminus E=U$. By the result of (1), U is dense in A. Therefore, E is nowhere dense in A.

(3) Let V^A is a nonempty open subset of A. Since A is a regular closed set of X, then $int(V^A)\neq\emptyset$ and $int(A)\neq\emptyset$. As A is a transitive subset of (S,X), there exists $s\in S$ such that $s(int(A))\cap int(V^A)\neq\emptyset$, which implies $sA\cap V^A\neq\emptyset$. Therefore, $SA=\bigcup\limits_{s\in S}sA$ is dense in A.

Theorem 3.5. *Let (S,X) be a dynamical system and A be a nonempty closed invariant set of X. Then A is a transitive subset of (S,X) if and only if (S,A) is topologically transitive.*

Proof. Let x be a point of X such that $\overline{orb(S,x)}$ is dense in itself. Denote $A=\overline{orb(S,x)}=\overline{Sx}$. Now, let V^A is a nonempty open subset of A and U is a nonempty subset of X with $A\cap U\neq\emptyset$. Since $A=\overline{Sx}$, there exists $s_1\in S$ such that $s_1x\in V^A$. The subset $S\setminus Ss_1$ is finite because S is an $F-$semigroup. Moreover, A is dense in itself, removing the finite subset $(S\setminus Ss_1)x$ from the dense subset Sx of A we get again a dense subset of A. Therefore, Ss_1x is a dense subset of A.

Then there exists $s_2 \in S$ such that $s_2 s_1 x \in A \cap U$. Furthermore, we have $s_2 V^A \cap U \neq \emptyset$. Hence, A is a transitive subset of (S,X), i.e., $\overline{orb(S,x)} = \overline{Sx}$ is a transitive subset of (S,X).

Conflict of Interests

The authors declare that there is no conflict of interests.

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