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F_w-CONTRACTIONS IN A COMPLETE G-METRIC SPACE

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Abstract. In this paper, we define a w-distance on a complete G-metric space. Also, we extend and generalize the concept of the F-contraction to the F_w-contraction and prove a fixed point theorem for F_w-contractions in a complete G-metric space.

Keywords: Fixed point; w-distance; F-contraction; F_w-contraction; Complete G-metric spaces.

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1. Introduction and preliminaries

The Banach fixed point theorem for contraction mappings has been generalized and extended in many directions; see ([1], [2], [4], [5], [11], [12] and [13]) and the reference therein. In [6], Dhage introduced the D-metric space as a generalization of the metric space and proved some results in this setting. In 2005, Mustafa and Sims [10] proved that these results are not true in topological structure and hence they introduced the G-metric space as a generalized form of the metric space. Since then, many authors have been studying fixed points of nonlinear operators in the framework of the G-metric space. In 2012, Wardowski [17] introduced a new concept of the F-contraction and proved a fixed point theorem for such a map on a complete

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metric space which generalizes the Banach Contraction Principle in a different direction. In [3], Batra and Vashistha generalized the concept of the F-contraction to the F_w-contraction and proved a fixed point theorem for the F_w-contraction in a complete metric space. Recently, Gupta [7] introduced the notion of the F-contraction in the G-metric space and proved a fixed point theorem concerning the F-contraction.

In this paper, using the concept of the G-metric, we define a w-distance on a complete G-metric space, which is a generalization of the concept of the w-distance due to Kada, Suzuki and Takahashi [8]. Also, we introduce the concept of the F_w-contraction in a complete G-metric space and extend the fixed point theorem due to Gupta.

Now, we recall the following definitions.

Definition 1.1. [10] Let X be a nonempty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x,y,z) = 0$ if $x = y = z$,

(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x = y$,

(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $x \neq y$,

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$(symmetry in all three variables),

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or a G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 1.2. [10] Let (X,G) be a G-metric space.

(1) A sequence $\{x_n\}$ in X, is said to be G-convergent to a point $x \in X$ if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $m,n \geq n_0$,

$G(x_m,x_n,x) < \varepsilon$;

(2) A sequence $\{x_n\}$ in X, is said to be G-Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $m,n,l \geq n_0$,

$G(x_m,x_n,x_l) < \varepsilon$.

Proposition 1.3. [10] *Let (X,G) be a G-metric space. Then the following are equivalent:*

(1) The sequence $\{x_n\}$ is G-Cauchy.

(2) For every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$, for all $n,m \geq k$.

Proposition 1.4. [10] Let (X,G) be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.

Definition 1.5. [17] Let $F:\mathbb{R}^+\to\mathbb{R}$ be a mapping satisfying:

(F1) F is strictly increasing. That is, $\alpha<\beta\Rightarrow F(\alpha)<F(\beta)$ for all $\alpha,\beta\in\mathbb{R}^+$.

(F2) For every sequence $\{\alpha_n\}$ in $\mathbb{R}^+$, we have $\lim_{n\to\infty}\alpha_n=0$ if and only if $\lim_{n\to\infty}F(\alpha_n)=-\infty$.

(F3) There exists a number $k\in(0,1)$ such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha)=0$.

Definition 1.6. [7] Let (X,G) be a G-metric space. A mapping $T:X\to X$ is said to be a F-contraction if there exists a number $\tau>0$ such that

$$G(Tx,Ty,Tz)>0\Rightarrow \tau+F(G(Tx,Ty,Tz))\le F(G(x,y,z)) \text{ for all } x,y,z\in X.$$

Remark 1.7. Clearly Definition 1.6 and (F1) implies that $G(Tx,Ty,Tz)<G(x,y,z)$ for all $x,y,z\in X$ with $Tx\neq Ty\neq Tz$. Hence every F- contraction mapping is continuous.

Next we give the notion of w-distance with some properties and examples.

Definition 1.8. Let (X,G) be a G-metric space. A function $p:X\times X\times X\to[0,\infty)$ is called a w-distance on X if the following conditions hold:

(w1) $p(x,y,z)\le p(x,a,a)+p(a,y,z)$ for all $x,y,z,a\in X$;

(w2) for any $x,y\in X$, $p(x,y,.),p(x,.,y):X\to[0,\infty)$ are lower semicontinuous;

(w3) for each $\varepsilon>0$, there exists $\delta>0$ such that $p(a,x,x)\le\delta$ and $p(a,y,z)\le\delta$ imply $G(x,y,z)\le$ ε.

Example 1.9. *Let (X,G) be a G-metric space. Then $p=G$ is a w-distance on X.*

Proof. (w1) and (w2) are obvious. We show (w3). Let $\varepsilon>0$ be given and put $\delta=\varepsilon/2$. If $G(a,x,x)\le\delta$ and $G(a,y,z)\le\delta$, we have, $G(a,a,x)\le\delta$, which imply that $G(x,y,z)\le 2\delta=\varepsilon$.

Example 1.10. *Let $X=[0,\infty)$ and $G:X^3\to[0,\infty)$ be defined by $G(x,y,z)=\frac{1}{3}(\mid x-y\mid+\mid y-z\mid$ $+\mid x-z\mid)$ for all $x,y,z\in X$. Then (X,G) is a G-metric space and the function $p:X^3\to[0,\infty)$ defined by $p(x,y,z)=max\{y,z\}$ for all $x,y,z\in X$ is a w-distance on X.*

Proof. The proofs of $(w1)$ and $(w2)$are immediate. To show $(w3)$, for any $\varepsilon>0$, put $\delta=\varepsilon/2$. Then if $p(a,x,x)\le\delta$ and $p(a,y,z)\le\delta$, we have $\mid x-y\mid\le 2\delta,\mid y-z\mid\le 2\delta$ and $\mid x-z\mid\le 2\delta$, which imply $G(x,y,z)\le\varepsilon$.

Example 1.11. *In $X = \mathbb{R}$, we consider the G-metric G defined by*
$G(x,y,z) = \frac{1}{3}(\mid x-y \mid + \mid y-z \mid + \mid x-z \mid)$ *for all* $x,y,z \in X$.

Then the function $p : \mathbb{R}^3 \to [0,\infty)$ *defined by* $p(x,y,z) = \frac{1}{3}(|z-x| + |x-y|)$ *for all* $x,y,z \in \mathbb{R}$ *is a w-distance on* $\mathbb{R}$.

Proof. The proofs of $(w1)$ and $(w2)$ are immediate. We show $(w3)$. Let $\varepsilon > 0$ be given and put $\delta = \varepsilon/5$. If $p(a,x,x) \le \delta$ and $p(a,y,z) \le \delta$, we have, respectively, $|x-a| \le 3\delta, |a-z| \le 3\delta$ and $|y-a| \le 3\delta$, which imply that $G(x,y,z) \le 5\delta = \varepsilon$.

Lemma 1.12. *Let X be a G-metric space with metric G and p be a w-distance on X. Let* $\{x_n\}, \{y_n\}$ *be sequences in X,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0,\infty)$ *converging to zero and let* $x,y,z,a \in X$ *. Then we have the following:*

(1) If $p(x_n,y,y) \le \alpha_n$ *and* $p(x_n,y,z) \le \beta_n$ *for* $n \in N$*, then* $G(y,y,z) < \varepsilon$ *and hence* $y = z$*;*

(2) If $p(x_n,y_n,y_n) \le \alpha_n$ *and* $p(x_n,y_m,z) \le \beta_n$ *for any* $m > n \in N$*, then* $G(y_n,y_m,z) \to 0$ *and hence* $y_n \to z$*;*

(3) If $p(x_n,x_m,x_l) \le \alpha_n$ *for any* $l,n,m \in N$ *with* $n \le m \le l$*, then* $\{x_n\}$ *is a G-Cauchy sequence;*

(4) If $p(a,x_n,x_n) \le \alpha_n$ *for any* $n \in N$*, then* $\{x_n\}$ *is a G-Cauchy sequence.*

Proof. We first prove (2). Let $\varepsilon > 0$ be given. From the definition of w-distance, there exists a $\delta > 0$ such that $p(a,u,u) \le \delta$ and $p(a,v,w) \le \delta$ imply $G(u,v,w) \le \varepsilon$. Choose $n_0 \in N$ such that $\alpha_n \le \delta$ and $\beta_n \le \delta$ for every $n \ge n_0$.Then we have, for any $m > n \ge n_0$, $p(x_n,y_n,y_n) \le \alpha_n \le \delta, p(x_n,y_m,z) \le \beta_n \le \delta$, and hence $G(y_n,y_m,z) \le \varepsilon$, so that $\{y_n\}$ converges to z. It follows from (2) that (1) holds. Let us now prove (3). Let $\varepsilon > 0$ be given. As in the proof of (2), choose $\delta > 0$ and then $n_0 \in N$.Then, for any $l \ge m \ge n \ge n_0$, $p(x_{n-1},x_n,x_n) \le \alpha_{n-1} \le \delta$, $p(x_{n-1},x_m,x_l) \le \alpha_{n-1} \le \delta$, and hence $G(x_n,x_m,x_l) \le \varepsilon$. This implies that $\{x_n\}$ is a G-Cauchy sequence. Condition (4) is a special case of (3).

We now define the notion of the F_w-contraction in a G-metric space and give some examples.

Definition1.13. Let (X,G) be a G-metric space and p be a w-distance on X. Let F be a mapping as defined in Definition 1.5. A mapping $T : X \to X$ is said to be a F_w-contraction if

(i) $p(x,y,z) = 0 \Rightarrow p(Tx,Ty,Tz) = 0$;

(ii) There exists a number $\tau > 0$ such that

$\tau + F(p(Tx,Ty,Tz)) \leq F(p(x,y,z))$ for all $x,y,z \in X$

with $p(Tx,Ty,Tz) > 0$.

Remark 1.14. *Clearly, (ii) of Definition 1.13 implies that*

$p(Tx,Ty,Tz) < p(x,y,z)$ *for all* $x,y,z \in X$ *with* $p(Tx,Ty,Tz) > 0$.

Example 1.15. Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\alpha) = ln\alpha$. Then F satisfies (F1), (F2) and (F3) (for all $k \in (0,1)$) of Definition 1.5. A mapping $T : X \to X$ satisfies

$$p(Tx,Ty,Tz) \leq \lambda p(x,y,z), \tag{1.1}$$

for all $x,y,z \in X$ and some $\lambda \in [0,1)$ if and only if T is a F_w-contraction. Let us start with a mapping $T : X \to X$ satisfying (1.1). If $\lambda = 0$ then (i) and (ii) in Definition 1.13 are vacuously satisfied. For $0 < \lambda < 1$, (i) is obvious and (ii) is satisfied for $\tau = ln\frac{1}{\lambda}$. Thus T is a F_w-contraction.

Conversely, if $T : X \to X$ is a F_w-contraction then (ii) of Definition 1.13 implies that

$p(Tx,Ty,Tz) \leq e^{-\tau}p(x,y,z)$ for all $x,y,z \in X$ with $p(Tx,Ty,Tz) > 0$. Clearly it is satisfied even for $p(Tx,Ty,Tz) = 0$. Thus $p(Tx,Ty,Tz) \leq \lambda p(x,y,z)$ for all $x,y,z \in X$, where $\lambda = e^{-\tau} \in [0,1)$.

Example 1.16. *Consider* $H(\alpha) = ln\alpha + \alpha$ *for all* $\alpha > 0$. *Then* H *satisfies (F1), (F2) and (F3) of Definition 1.5. A mapping* $T : X \to X$ *is a* H_w*-contraction if and only if*

$$p(Tx,Ty,Tz)e^{\{p(Tx,Ty,Tz)-p(x,y,z)\}} \leq \lambda p(x,y,z), \tag{1.2}$$

for all $x,y,z \in X$ *and* $\lambda = e^{-\tau} \in [0,1)$. *(Reason is similar to above example)*

Example 1.17. *Consider* $K(\alpha) = ln(\alpha^2 + \alpha)$ *for all* $\alpha > 0$. *Then* K *satisfies (F1), (F2) and (F3) of Definition 1.5. A mapping* $T : X \to X$ *is a* K_w*-contraction if and only if*

$$\frac{p(Tx,Ty,Tz)(p(Tx,Ty,Tz)+1)}{p(x,y,z)(p(x,y,z)+1)} \leq \lambda, \tag{1.3}$$

for all $x,y,z \in X$ *and* $\lambda = e^{-\tau} \in [0,1)$.

Remark 1.18. *Let* $F,H : \mathbb{R}^+ \to \mathbb{R}$ *be mappings satisfying (F1), (F2) and (F3) of Definition 1.5 together with* $F(\alpha) \leq H(\alpha)$ *for all* $\alpha > 0$. *Let* $K = H - F$ *be nondecreasing. Then every*

F_w-contraction $T : X \to X$ is a H_w-contraction. Indeed for any $x,y,z \in X$ with $p(Tx,Ty,Tz) > 0$, we have,

$$\begin{aligned}\tau + H(p(Tx,Ty,Tz)) =& \tau + F(p(Tx,Ty,Tz)) + K(p(Tx,Ty,Tz)) \\ &\leq F(p(x,y,z)) + K(p(x,y,z)) = H(p(x,y,z)).\end{aligned}$$

Example 1.19. *Let $X = [0,\infty)$ and $G(x,y,z) = \frac{1}{3}(\mid x-y \mid + \mid y-z \mid + \mid x-z \mid)$ for all $x,y,z \in X$. Then (X,G) is a complete G-metric space. Define $p : X \times X \times X \to \mathbb{R}^+$ by $p(x,y,z) = max\{y,z\}$ for all $x,y,z \in X$. Then p is a w-distance on X. Define a mapping $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x^2}{2}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Since T is not continuous, therefore it is not a F-contraction for any mapping F as described in Definition 1.5. Now consider the mapping F as described in Example 1.15. We note that $p(Tx,Ty,Tz) = max\{Ty,Tz\} > 0$ if and only if $0 \leq y \leq 1$ or $0 \leq z \leq 1$.

Now we have the following cases:

For $x,y,z \in X$ with $0 < y \leq 1 < z$, we have $\frac{p(Tx,Ty,Tz)}{p(x,y,z)} = \frac{y^2/2}{z} \leq \frac{1}{2}$.
For $x,y,z \in X$ with $0 < z \leq 1 < y$, we have $\frac{p(Tx,Ty,Tz)}{p(x,y,z)} = \frac{z^2/2}{y} \leq \frac{1}{2}$.
For $x,y,z \in X$ with $0 \leq y < z \leq 1$, we have $\frac{p(Tx,Ty,Tz)}{p(x,y,z)} = \frac{z^2/2}{z} = \frac{z}{2} \leq \frac{1}{2}$.
For $x,y,z \in X$ with $0 \leq z < y \leq 1$, we have $\frac{p(Tx,Ty,Tz)}{p(x,y,z)} = \frac{y^2/2}{y} = \frac{y}{2} \leq \frac{1}{2}$.
So p satisfies (1.1) for all $x,y,z \in X$ and for $\lambda = \frac{1}{2}$. Thus T is a F_w -contraction which is not a F-contraction for any F.

2. Main results

Theorem 2.1. *Let (X,G) be a complete G-metric space and p be a w-distance on X. Let $T : X \to X$ be a F_w-contraction. Then T has a unique fixed point x^* in X and for every $x_0 \in X$, there is a sequence $\{T^n x_0\}$ in X that converges to x^*. Further $p(x^*,x^*,x^*) = 0$.*

Proof. For any two fixed points x^* and y^* of T in X with $p(Tx^*,Ty^*,Ty^*) > 0$, we have

$$\tau \leq F(p(x^*,y^*,y^*)) - F(p(Tx^*,Ty^*,Ty^*)) = 0.$$

Thus $p(Tx^*,Ty^*,Ty^*) = p(x^*,y^*,y^*) = 0$ for any two fixed points x^* and y^* of T in X. In particular, $p(Tx^*,Tx^*,Tx^*) = p(x^*,x^*,x^*) = 0$. By Lemma 1.12 (1), we obtain $x^* = y^*$ for any two fixed points x^* and y^* of T in X. Hence fixed point x^* of T if exists is unique and satisfies $p(x^*,x^*,x^*) = 0$.

Now we show the existence of a fixed point of T. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Let $p_n = p(x_{n-1},x_n,x_n)$ for all $n \in \mathbb{N}$. If there exists $k \in \mathbb{N}$ with $p(x_{k-1},x_k,x_k) = 0$ then, by (i) of Definition 1.13, $p(Tx_{k-1},Tx_k,Tx_k) = 0$, that is, $p(x_k,x_{k+1},x_{k+1}) = 0$. Therefore $p(x_{k-1},x_{k+1},x_{k+1}) \leq p(x_{k-1},x_k,x_k) + p(x_k,x_{k+1},x_{k+1}) = 0$. By Lemma 1.12 (1) we have $x_k = x_{k+1}$. Inductively, we have $x_k = x_{k+i}$ for all $i \in \mathbb{N}$. This implies $T^i(x_k) = x_k$ for all $i \in \mathbb{N}$ and in particular, for $i = 1, T(x_k) = x_k$. Also $\lim\limits_{n\to\infty} T^n(x_0) = \lim\limits_{i\to\infty} T^{k+i}(x_0) = \lim\limits_{i\to\infty} T^i(x_k) = x_k$. Thus we can take $x^* = x_k$ in this case and settle the proof.

Now assume that $p_n = p(x_{n-1},x_n,x_n) > 0$ for all $n \in \mathbb{N}$. Then by (ii) of Definition 1.13 we get

$$F(p_n) \leq F(p_{n-1}) - \tau \leq F(p_{n-2}) - 2\tau \leq ... \leq F(p_0) - n\tau. \tag{2.1}$$

From (2.1), we get $\lim\limits_{n\to\infty} F(p_n) = -\infty$. By (F2) of Definition 1.5, we have

$$\lim_{n\to\infty} p_n = 0. \tag{2.2}$$

Now, by (F3) of Definition 1.5, we find that there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} p_n^k F(p_n) = 0. \tag{2.3}$$

By (2.1), we find that following holds for all $n \in \mathbb{N}$.

$$p_n^k F(p_n) - p_n^k F(p_0) = p_n^k (F(p_n) - F(p_0)) \leq ?n p_n^k \tau. \tag{2.4}$$

Letting $n \to \infty$ in (2.4) and using (2.2) and (2.3), we have

$$\lim_{n\to\infty} n p_n^k = 0. \tag{2.5}$$

By (2.5), there exists a positive integer n_0 such that $np_n^k < 1$ for all $n \geq n_0$. Consequently, we have

$$p_n < \frac{1}{n^{\frac{1}{k}}} \ \ \forall n \geq n_0. \tag{2.6}$$

Since the series $\Sigma_{n=1}^{\infty}\frac{1}{n^{\frac{1}{k}}}$ is convergent, therefore, by (2.6), the series $\Sigma_{n=1}^{\infty}p_n$ is also convergent. Now for any $m > n$ we have

$$p(x_n, x_m, x_m) \leq p_{n+1} + p_{n+2} + \cdots + p_m < \alpha_n, \tag{2.7}$$

where $\alpha_n = \Sigma_{i=n+1}^{\infty} p_i \to 0$ as $n \to \infty$. By Lemma 1.12 (3), $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. From (2.7) and (ii) of Definition 1.8, we get

$$p(x_n, x^*, x^*) \leq \alpha_n. \tag{2.8}$$

Now for $p(Tx_{n-1}, Tx^*, Tx^*) > 0$, we find from Remark 1.14 and (2.8) that

$$p(x_n, Tx^*, Tx^*) = p(Tx_{n-1}, Tx^*, Tx^*) < p(x_{n-1}, x^*, x^*) \leq \alpha_{n-1}. \tag{2.9}$$

Clearly (2.9) is satisfied even for $p(Tx_{n-1}, Tx^*, Tx^*) = 0$. Thus

$$p(x_n, Tx^*, Tx^*) \leq \alpha_{n-1} \ \ \forall n \in \mathbb{N}. \tag{2.10}$$

From (2.8), (2.10) and Lemma 1.12 (1), we get $Tx^* = x^*$. Also we have seen above that $x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n(x_0)$.

Example 2.2. *Consider the F_w-contraction T defined in Example 1.19. We note that $x = 0$ is the unique fixed point of T and $p(0,0,0) = 0$.*

Remark 2.3. *From Example 1.9 it is clear that Theorem 2.9 of* [7] *is a particular case of our Theorem 2.1.*

Now, since every contraction $T : X \to X$ satisfying (1.1) is a F_w-contraction for $F(\alpha) = ln\alpha, \alpha > 0, F(\alpha) < ln\alpha + \alpha = H(\alpha)$ for all $\alpha > 0$ and $H - F$ is non decreasing, therefore, by Remark 1.18, T is a H_w-contraction and hence satisfies (1.2). In the following example we shall present a mapping $T : X \to X$ which is a H_w-contraction but not a F_w-contraction and hence satisfies (1.2) but not (1.1). Thus our theorem deals with the fixed points of a more general class of contractions.

Example 2.4. *Consider the sequence $a_n = \frac{n(n-1)}{2}$ for $n \in \mathbb{N}$. Let $X = \{a_n : n \in \mathbb{N}\}$ and $G(x,y,z) = \frac{1}{3}(\mid x-y \mid + \mid y-z \mid + \mid x-z \mid)$ for all $x,y,z \in X$. Then (X,G) is a complete G-metric space. Define $p : X \times X \times X \to \mathbb{R}^+$ by $p(x,y,z) = max\{y,z\}$ for all $x,y,z \in X$. Then p is* *a w-distance on X. Define a mapping* $T: X \to X$ *by* $Ta_1 = a_1$, $Ta_n = a_{n-1}$ *for* $n > 1$. *Take F as in Example 1.15 and H as in Example 1.16. T is not a* F_w*-contraction as* $\lim_{n\to\infty} \frac{p(Ta_1,Ta_n,Ta_n)}{p(a_1,a_n,a_n)} = \lim_{n\to\infty} \frac{a_{n-1}}{a_n} = 1$. *But T is a* H_w*-contraction. We first observe that* $p(Ta_m, Ta_n, Ta_r) > 0 \Leftrightarrow Ta_n > 0$ *or* $Ta_r > 0 \Leftrightarrow n > 2$ *or* $r > 2$. *Now we have the following cases:*

For $n > 2 > r$, *we have*

$$\frac{p(Ta_m,Ta_n,Ta_r)}{p(a_m,a_n,a_r)} e^{p(Ta_m,Ta_n,Ta_r)-p(a_m,a_n,a_r)} = \frac{a_{n-1}}{a_n} e^{a_{n-1}-a_n} = (1-\frac{2}{n})e^{1-n} < e^{1-n} < e^{-1}.$$

For $r > 2 > n$, *we have*

$$\frac{p(Ta_m,Ta_n,Ta_r)}{p(a_m,a_n,a_r)} e^{p(Ta_m,Ta_n,Ta_r)-p(a_m,a_n,a_r)} = \frac{a_{r-1}}{a_r} e^{a_{r-1}-a_r} = (1-\frac{2}{r})e^{1-r} < e^{1-r} < e^{-1}.$$

For $n > r > 2$, *we have*

$$\frac{p(Ta_m,Ta_n,Ta_r)}{p(a_m,a_n,a_r)} e^{p(Ta_m,Ta_n,Ta_r)-p(a_m,a_n,a_r)} = \frac{a_{n-1}}{a_n} e^{a_{n-1}-a_n} = (1-\frac{2}{n})e^{1-n} < e^{1-n} < e^{-1}.$$

For $r > n > 2$, *we have*

$$\frac{p(Ta_m,Ta_n,Ta_r)}{p(a_m,a_n,a_r)} e^{p(Ta_m,Ta_n,Ta_r)-p(a_m,a_n,a_r)} = \frac{a_{r-1}}{a_r} e^{a_{r-1}-a_r} = (1-\frac{2}{r})e^{1-r} < e^{1-r} < e^{-1}.$$

Thus T is an H_w*-contraction for* $\tau = 1$. *Clearly* $a_1 = 0$ *is a fixed point of T,* $p(a_1,a_1,a_1) = a_1 = 0$ *and for any* $a_m \in X$, $\lim_{n\to\infty} T^n a_m = \lim_{n\to\infty} T^{n+m} a_m = \lim_{n\to\infty} T^n(T^m a_m) = \lim_{n\to\infty} T^n a_1 = a_1$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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