Available online at http://scik.org
Adv. Fixed Point Theory, 5 (2015), No. 3, 329-341
ISSN: 1927-6303

SPLIT EQUALITY FIXED POINT PROBLEMS FOR LIPSCHITZ HEMI-CONTRACTIVE MAPPINGS

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Abstract. A very important general class of split feasiblity problem was introduced by Moudafi and Al-Shamas [13], in the case when the mappings are firmly nonexpansive defined on real Hilbert spaces. We propose in this paper a new Krasnoselskii's-type algorithm to solve the problem in the more general case when the mappings are Lipschitz hemicontractive. We show that the proposed algorithm converges weakly to a solution of the problem. We also show that the iterative sequence obtained converges strongly to a solution of the problem under suitable compactness assumptions.

Keywords: Split equality problem; Lipschitz pseudocontraction; Split feasibility problem; Inverse problem.

2010 AMS Subject Classification: 47H09, 47H10.

1. Introduction

Let H be a real Hilbert space and let K be a closed convex and bounded subset of H. Let $T : K \to K$ be a mapping. A fixed point of T is simply a point $x \in K$ such that $Tx = x$. The collection of all fixed points of T is denoted by $F(T)$. The mapping T is said to be

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Received April 8, 2015

- *demi-contractive* if

$$\|Tx - Tp\|^2 \leq \|x - p\|^2 + k\|x - Tx\|$$

  for some $k \in (0,1)$ and all $(x,p) \in K \times F(T)$,

- *hemicontractive* if

$$\|Tx - Tp\|^2 \leq \|x - p\|^2 + \|x - Tx\|$$

  for all $(x,p) \in K \times F(T)$.

- *Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|.$$

The split equality problem was introduced by Moudafi and Al-Shemas[13] in (2013) as a generalization of the split feasibilty problem which appear as inverse problems in phase retrivial, medical image recontruction, intensity modulated radiation therapy (IMRT) and so on (see e.g., Byrne [3], Censor *et al.* [4], Censor *et al.* [5], and Censor and Elfving [6]). It serves as a model for inverse problems in the case where constraints are imposed on the solutions in the domain of a linear transformation an also in its range.

The split equality problem of Moudafi is stated as follows:

$$\text{Find } x \in C = F(S) \text{ and } y \in Q = F(T) \text{ such that } Ax = By, \tag{1}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, H_1, H_2, and H_3 are real Hilbert spaces, while $S : H_1 \to H_1$ and $T : H_2 \to H_2$ firmly quasi-nonexpansive mappings, respectively.

They studied the convergence of a weakly coupled iterative algorithm given by

$$(SEP)\begin{cases} x_{n+1} = S(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = T(y_n + \gamma_n B^*(Ax_n - By_n)), n \geq 1, \end{cases} \tag{2}$$

where A^* and B^* are the adjoints of A and B, respectively, while λ is the sum of the spectral radii of A^*A and $\gamma_n \in (0, \frac{2}{\lambda})$.

The iterative algorithm of Moudafi was for firmly quasi-nonexpansive mapping which has very attractive properties that makes the use of this simple iterative algorithm introduced suitable.

The algorithm of Moudafi and Al-shamas has great merits because it is implementable without the use of projections and yet it is a generalization of the split equality problem if we set $H_3 = H_2$ and $B = I$. The algorithm was extended by Yuan-Fang *et al*. [17] who introduced the following algorithm for solving problem (2):

$$(3) \qquad \begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n T(y_n + \gamma_n B^*(Ax_n - By_n)), \ \forall n \geq 1, \end{cases}$$

where $S : H_1 \to H_1$, $T : H_2 \to H_2$ are still two *firmly quasi-nonexpansive mappings*, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are bounded linear operators, A^* and B^* are the adjoints of A and B, respectively, $\gamma_n \in (0, \frac{2}{\lambda})$, where λ is the sum of the spectral radii of A^*A and B^*B, respectively, and $\{\alpha_n\} \subset [\alpha, 1]$ (for some $\alpha > 0$). Under suitable conditions, the authors obtained strong and weak convergence results, respectively. It was therefore natural to investigate if the split equality problem can be extended to a more general class of mappings apart from the class of firmly quasi-nonexpansive mappings studied by Moudafi and Al-Shamas [13], and Yuan-Fang *et al*. [17].

Motivated by the work of Moudafi and Al-Shamas, Chidume *et al.* [10] studied convergence theorems for split equality problem involving two *demi-contractive* mappings. They introduced the following Krasnoselskii-type iterative algorithm

$$(4) \qquad \begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2, \\ x_{n+1} = (1-\alpha)\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) + \alpha U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big), \\ y_{n+1} = (1-\alpha)\Big(y_n + \gamma B^*(Ax_n - By_n)\Big) + \alpha T\Big(y_n + \gamma B^*(Ax_n - By_n)\Big), \ \forall n \geq 1, \end{cases}$$

where $U : H_1 \to H_1$, $T : H_2 \to H_2$ are two *demi-contractive mappings* defined on Hilbert spaces. The class of demi-contractive mappings properly contains the class of firmly quasi-nonexpansive mappings which was studied by Moudafi and Al-Shemas [13].

The aim of the present study is to extend the split equality problem of Moudafi and Al-Shamas [13], and Chidume *et al.* [10], to Lipschitz hemicontractive mappings. The very important class of hemicontractive mapping contains pseudocontractive mappings with nonempty fixed point sets. The later has been studied extensively, for example, by Browder and Petryshn [1], Browder [2], Chidume [8], Chidume and Zegeye [9], Kirk [11], Maruster[12], Xu [15] and a host of other authors, and is known to properly contain the important class of demicontractive mappings studied by Chidume *et al.* [10]. We will discuss some weak and strong convergence theorem for a mean value sequence introduced.

Our theorems and corollaries extend and generalize the results of Censor and Segal [7], Chidume *et al.* [10], Maruster *et al*. [12], Moudafi and Al-Shemas [13], Xu [16], Yuan-Fang *et al*. [17], and a host of other results.

2. Preliminaries

We introduce in this section some definitions, notations and results which will be needed in proving our main theorem. In the sequel, strong convergence is denoted by " $\to$ " and weak convergence by " $\rightharpoonup$ ". We recall the following useful definitions and lemmas.

Definition 2.1. [Demiclosedness principle] Let $T: K \to K$ be a mapping. Then $I-T$ is called *demiclosed* at zero if for any sequence $\{x_n\}$ in H such that $x_n \rightharpoonup x$, and $\|x_n - Tx_n\| \to 0$, then $Tx = x$.

Definition 2.2. A mapping $T: K \to K$ is called *hemicompact* if, for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence, say, $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.

Trivial examples of hemicompact mappings are mappings with compact domains.

Lemma 2.3. *Let H be a Hilbert space. Then the following identity holds:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{5}$$

where $\lambda \in (0,1)$ and $x, y \in H$.

Lemma 2.4. (Xu [15]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation $a_{n+1} \leq a_n + \sigma_n$, $n \geq 0$, such that $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then, $\lim a_n$ exists. If, in addition, $\{a_n\}$ has a subsequence that converges to 0, then a_n converges to 0 as $n \to \infty$.*

Lemma 2.5. ([Opial's Lemma [14]) *Let H be a real Hilbert space and x_n be a sequence in H for which there exists a nonempty set $\Gamma \subseteq H$ such that for every $x \in \Gamma$, $\lim_{n\to\infty} \|x_n - x\|$ exists and any weak-cluster point of the sequence belongs to Γ. Then, there exists $x^* \in \Gamma$ such that $\{x_n\}$ converges weakly to x^*.*

Lemma 2.6. *Let H_1 and H_2 be two real Hilbert spaces. Then, the product $H_1 \times H_2$ is a Hilbert with inner product $\langle (x_1,x_2),(y_1,y_2)\rangle_* := \langle x_1,y_1\rangle_1 + \langle x_2,y_2\rangle_2$ where $\langle .,.\rangle_1$, $\langle .,.\rangle_2$ are the inner products on H_1 and H_2 respectively.*

3. Main results

In this section, we propose a coupled iterative algorithm for solving the split equality fixed point problem, involving hemicontractive mappings, as stated below:

$$\text{Find } \; x \in C = F(S) \; \text{ and } \; y \in Q = F(T) \text{ such that } \; Ax = By, \tag{6}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, H_1, H_2, and H_3 are real Hilbert spaces, while $S : H_1 \to H_1$ and $T : H_2 \to H_2$ hemicontractive mappings, respectively. Henceforth, given two Lipschitz hemicontractive mappings S and T, we define the set

$$\Gamma := \{(p,q) \in H_1 \times H_2 : Sp = p, Tq = q\}, \tag{7}$$

and a mapping $G : H_1 \times H_2 \to H_1 \times H_2$ by

$$G(x,y) := (S(x - \lambda A^*(Ax - By)), T(y + \lambda B^*(Ax - By))). \tag{8}$$

It is easily to seen that G is Lipschitz. Moreover, for $(p,q) \in \Gamma$, $G(p,q) = (p,q)$. Now consider the coupled iterative algorithm given below

$$(9) \quad \begin{cases} (x_1, y_1) \in H_1 \times H_2, \text{ chosen arbitarily,} \\ (x_{n+1}, y_{n+1}) = (1-\alpha)((x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)) + \alpha G(u_n, v_n), \\ (u_n, v_n) = (1-\alpha)((x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)) + \alpha G(x_n, y_n), \\ \alpha \in (0, L^{-2}(\sqrt{L^2+1} - 1)) \\ \lambda \in (0, \frac{2\alpha}{\bar{\lambda}(A,B)}), \end{cases}$$

where $\bar{\lambda}(A,B)$ is the sum of the spectral radii of A^*A and B^*B and L the Lipschitz constant of G. We show in what follows that the iterative sequence generated by the algorithm above converges weakly to a solution of split equalty problem (6).

Theorem 3.1. *Let H_1, H_2, H_3 be real Hilbert spaces, $S : H_1 \to H_1$ and $T : H_2 \to H_2$ two Lipschitz hemicontractive mappings, and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear mappings. Then the coupled sequence (x_n, y_n) generated by the algorithm (3.4) converges weakly to a solution (x^*, y^*) of problem (6).*

Proof. Define $\|(x,y)\|_*^2 = \|x\|_1^2 + \|y\|_2^2$. Taking $(p,q) \in \Gamma$ and using Lemma 2.3, we obtain

$$\begin{aligned} \|(x_{n+1}, y_{n+1}) - (p,q)\|_*^2 &= \|(1-\alpha)((x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)) - (p,q)) \\ &\quad + \alpha(G(u_n, v_n) - (p,q))\|_*^2 \\ &\leq (1-\alpha)\Big[\|(x_n, y_n) - (p,q)\|_*^2 - 2\lambda\|Ax_n - By_n\|_*^2 + \lambda^2(\bar{\lambda}(A,B))\|Ax_n - By_n\|^2\Big] \\ &\quad + \alpha\|G(u_n, v_n) - (p,q)\|_*^2 \\ &\quad - \alpha(1-\alpha)\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)) - G(u_n, v_n)\|_*^2. \end{aligned}$$

It follows from the definition of the mapping G and the hemicontractive properties of S and T we get

$$\begin{aligned}
\|G(u_n,v_n)-(p,q)\|_*^2 &= \|G(u_n,v_n)-G(p,q)\|_*^2 \\
&\leq \|(u_n-\lambda A^*(Au_n-Bv_n), v_n+\lambda B^*(Au_n-Bv_n))-(p,q)\|_*^2 \\
&+ \|(u_n-\lambda A^*(Au_n-Bv_n), v_n+\lambda B^*(Au_n-Bv_n))-G(u_n,v_n)\|_*^2 \\
&\leq \|(u_n,v_n)-(p,q)\|_*^2-\lambda(2-\lambda(\bar{\lambda}(A,B)))\|Au_n-Bv_n\|^2 \\
&+ \|(u_n-\lambda A^*(Au_n-Bv_n), v_n+\lambda B^*(Au_n-Bv_n))-G(u_n,v_n)\|_*^2.
\end{aligned}$$

In view of the inequalities above, we obtain

(10)
$$\|(x_{n+1},y_{n+1})-(p,q)\|_*^2 \leq (1-\alpha)\Big[\|(x_n,y_n)-(p,q)\|_*^2-\lambda(2-\lambda(\bar{\lambda}(A,B))\|Ax_n-By_n\|^2\Big]$$

(11) $$+\alpha\Big[\|(u_n,v_n)-(p,q)\|_*^2-\lambda(2-\lambda(\bar{\lambda}(A,B)))\|Au_n-Bv_n\|^2$$

(12) $$+\|(u_n-\lambda A^*(Au_n-Bv_n), v_n+\lambda B^*(Au_n-Bv_n))-G(u_n,v_n)\|_*^2.\Big]$$

(13) $$-\alpha(1-\alpha)\|(x_n-\lambda A^*(Ax_n-By_n), y_n+\lambda B^*(Ax_n-By_n)-G(u_n,v_n)\|_*^2.$$

Using the definition of u_n and v_n, we have the folowing chain of inequalities:

$$\begin{aligned}
\|(u_n,v_n)-(p,q)\|_*^2 &= \|(1-\alpha)[(x_n-\lambda A^*(Ax_n-By_n), y_n+\lambda B^*(Ax_n-By_n))-(p,q) \\
&+\alpha[G(x_n,y_n)-(p,q)]\|_*^2 \\
&\leq (1-\alpha)\Big[\|(x_n,y_n)-(p,q)\|_*^2-\lambda(2-\lambda(\bar{\lambda}(A,B)))\|Ax_n-By_n\|^2\Big] \\
&+\alpha\Big[\|(x_n,y_n)-(p,q)\|_*^2-\lambda(2-\lambda(\bar{\lambda}(A,B)))\|Ax_n-By_n\|^2 \\
&+\|(x_n-\lambda A^*(Ax_n-By_n), y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2\Big] \\
&-\alpha(1-\alpha)\|(x_n-\lambda A^*(Ax_n-By_n), y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|(u_n-\lambda A^*(Au_n-Bv_n),v_n+\lambda B^*(Au_n-Bv_n))-G(u_n,v_n)\|_*^2.\\
&\quad\le(1-\alpha)\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n))-G(u_n,v_n)\|^2\\
&\quad+\alpha\|G(x_n,y_n)-G(u_n,v_n)\|^2\\
&\quad-\alpha(1-\alpha)\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n))-G(x_n,y_n)\|^2.
\end{aligned}$$

If we substitute these inequalities into their rightful positions in inequality (10), we get the following:

$$\begin{aligned}
\|(x_{n+1},y_{n+1})-(p,q)\|_*^2&\le(1-\alpha)\Big[\|(x_n,y_n)-(p,q)\|_*^2-\lambda(2-\lambda(\bar\lambda(A,B))\|Ax_n-By_n\|^2\Big]\\
&+\alpha\Big[(1-\alpha)\Big[\|(x_n,y_n)-(p,q)\|_*^2-\lambda(2-\lambda(\bar\lambda(A,B)))\|Ax_n-By_n\|^2\Big]\\
&+\alpha\Big[\|(x_n,y_n)-(p,q)\|_*^2-\lambda(2-\lambda(\bar\lambda(A,B)))\|Ax_n-By_n\|^2\\
&+\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2\Big]\\
&-\alpha(1-\alpha)\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2.\\
&-\lambda(2-\lambda(\bar\lambda(A,B)))\|Au_n-Bv_n\|^2\Big]\\
&+(1-\alpha)\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n))-G(u_n,v_n)\|^2\\
&+\alpha\|G(x_n,y_n)-G(u_n,v_n)\|^2\\
&-\alpha(1-\alpha)\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n))-G(x_n,y_n)\|^2\Big]\\
&-\alpha(1-\alpha)\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)-G(u_n,v_n)\|_*^2.
\end{aligned}$$

Gathering all the similar terms together, we obtain

$$\begin{aligned}
\|(x_{n+1},y_{n+1})-(p,q)\|_*^2&\le\|(x_n,y_n)-(p,q)\|_*^2-\lambda(2-\lambda(\bar\lambda(A,B)))\|Ax_n-By_n\|^2\\
&-(\alpha^2-2\alpha^3)\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2\\
&-\alpha\lambda(2-\lambda(\bar\lambda(A,B)))\|Au_n-Bv_n\|^2\Big]\\
&+\alpha^2\|G(x_n,y_n)-G(u_n,v_n)\|_*^2.
\end{aligned}$$

Again since S and T are Lipschiptz with Lipschitz constant, say, L_s and L_t respectively. Set $L=\max\{L_s,L_t\}$. Then,

$$\begin{aligned}
\|G(x_n,y_n)-G(u_n,v_n)\|^2 &= \|S(x_n-\lambda A^*(Ax_n-By_n)-S(u_n-\lambda A^*(Au_n-Bv_n)\|_1^2\\
&+\|T(y_n+\lambda B^*(Ax_n-By_n)-T(v_n+\lambda B^*(Au_n-Bv_n)\|_2^2\\
&\le L^2\Big[\|(x_n-\lambda A^*(Ax_n-By_n))-u_n\|_1^2\\
&+\|(y_n-\lambda B^*(Ax_n-By_n))-v_n\|_2^2+2\lambda\langle Ax_n-Au_n-\lambda(Ax_n-By_n),Au_n-Bv_n\rangle,\\
&-2\lambda\langle By_n-Bv_n-\lambda(Ax_n-By_n),Au_n-Bv_n\rangle+\lambda^2(\bar{\lambda}(A,B))\|Au_n-Bv_n\|^2\Big].\\
&\le L^2\Big[\alpha^2\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2\\
&+2\lambda\langle Ax_n-By_n,Au_n-Bv_n\rangle-\lambda(2-\lambda\bar{\lambda}(A,B))\|Au_n-Bv_n\|^2\Big].
\end{aligned}$$

Since $2\lambda\langle Ax_n-By_n,Au_n-Bv_n\rangle\le 2\lambda\|Ax_n-By_n\|^2+2\lambda\|Au_n-Bv_n\|^2$, we conclude that

$$\begin{aligned}
\|G(x_n,y_n)-G(u_n,v_n)\|^2 \le L^2\Big[&\alpha^2\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)\\
&-G(x_n,y_n)\|_*^2+2\lambda\|Ax_n-By_n\|^2+\lambda^2\bar{\lambda}(A,B))\|Au_n-Bv_n\|^2\Big].
\end{aligned}$$

Substituting this in its rightful place gives

$$\begin{aligned}
\|(x_{n+1},y_{n+1})-(p,q)\|_*^2 &\le \|(x_n,y_n)-(p,q)\|_*^2-\lambda(2-\lambda(\bar{\lambda}(A,B)))\|Ax_n-By_n\|^2\\
&-(\alpha^2-2\alpha^3)\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2\\
&-\alpha\lambda(2-\lambda(\bar{\lambda}(A,B)))\|Au_n-Bv_n\|^2\Big]\\
&+\alpha^2L^2\Big[\alpha^2\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2\\
&2\lambda\|Ax_n-By_n\|^2+\lambda^2\bar{\lambda}(A,B))\|Au_n-Bv_n\|^2\Big]\\
&=\|(x_n,y_n)-(p,q)\|_*^2\\
&+[-2\lambda+2\lambda\alpha^2L^2+\lambda^2(\bar{\lambda}(A,B))]\|Ax_n-By_n\|^2\\
&-\alpha^2(1-2\alpha-\alpha^2L^2)\times\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2\\
&+[-2\alpha\lambda+\alpha\lambda^2\bar{\lambda}(A,B))+\alpha^2L^2\lambda^2\bar{\lambda}(A,B))]\|Au_n-Bv_n\|^2.
\end{aligned}$$

Finally, if we observe that $1-2\alpha-\alpha^2L^2>0$ is the same as $|\alpha+\frac{1}{L^2}|<L^{-2}\sqrt{L^2+1}$, then, since $\alpha\in(0,L^{-2}[\sqrt{1+L^2}-1])$, we have $1-2\alpha-\alpha^2L^2>0$. Therefore, we have $\alpha^2L^2<1-2\alpha$ and $2-2\alpha^2L^2>0$. Certainly, $\alpha<\min\{\frac{1}{2},\frac{1}{L}\}$, and $-2\lambda+2\lambda\alpha^2L^2+\lambda^2(\bar{\lambda}(A,B))<-2\lambda+2\lambda(1-2\alpha)+\lambda^2(\bar{\lambda}(A,B))=-4\alpha\lambda+\lambda^2(\bar{\lambda}(A,B))<0$ since $\lambda<\frac{2\alpha}{\bar{\lambda}(A,B)}$. Finally, we have

$$\begin{aligned}&-2\alpha\lambda+\alpha\lambda^2\bar{\lambda}(A,B))+(\alpha L\lambda)^2\bar{\lambda}(A,B)\\&<-2\alpha\lambda+\alpha\lambda^2\bar{\lambda}(A,B))+\lambda^2\bar{\lambda}(A,B)(1-2\alpha)\\&<-2\alpha\lambda+\lambda^2\bar{\lambda}(A,B)<0.\end{aligned}$$

From the previous chain of inequalities we may now conclude the following,

$$\|(x_{n+1},y_{n+1})-(p,q)\|_*^2\le\|(x_n,y_n)-(p,q)\|_*^2, \tag{14}$$

$$[2\lambda-2\lambda\alpha^2L^2-\lambda^2(\bar{\lambda}(A,B))]\|Ax_n-By_n\|^2\le\|(x_n,y_n)-(p,q)\|_*^2-\|(x_{n+1},y_{n+1})-(p,q)\|_*^2 \tag{15}$$

and

$$[\alpha^2(1-2\alpha-\alpha^2L^2)\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2$$

$$\le\|(x_n,y_n)-(p,q)\|_*^2-\|(x_{n+1},y_{n+1})-(p,q)\|_*^2. \tag{16}$$

Using Lemma 2.4 we have by (14) that $\|(x_n,y_n)-(p,q)\|_*^2$ has a limit. Therefore, taking limits on both sides of (15), and (16) respectively, we have that

$$\lim_{n\to\infty}\|Ax_n-By_n\|^2=0, \tag{17}$$

$$\lim_{n\to\infty}\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2=0. \tag{18}$$

Next, we show that $\lim_{n\to\infty}\|x_n-S(x_n)\|_1=0$ and $\lim_{n\to\infty}\|y_n-S(y_n)\|_2=0$. The fact that $\|(x_n,y_n)-(p,q)\|_*^2$ has a limit shows that both $\{x_n\}$ and $\{y_n\}$ are bounded. Suppose that x^* and y^* are weak cluster points of the sequences $\{x_n\}$ and $\{y_n\}$ such that $x_{n_k}\rightharpoonup x^*$ and $y_{n_k}\rightharpoonup y^*$ repectively. Then

$$\lim_{k\to\infty}\|S(x_{n_k}-\lambda A^*(Ax_{n_k}-By_{n_k}))-Sx_{n_k}\|\le L_s\bar{\lambda}(A,B)\lim_{k\to\infty}\|Ax_{n_k}-By_{n_k}\|=0,$$

and similarly,

$$\lim_{k\to\infty} \|T(y_{n_k} + \lambda B^*(Ax_{n_k} - By_{n_k})) - Ty_{n_k}\| \leq L_t \bar{\lambda}(A,B) \lim_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0.$$

Therefore we have

$$\begin{aligned} \|x_{n_k} - S(x_{n_k})\| &\leq \|x_{n_k} - (x_{n_k} - \lambda A^*(Ax_{n_k} - By_{n_k})\| \\ &\quad + \|(x_{n_k} - \lambda A^*(Ax_{n_k} - By_{n_k})) - S(x_{n_k} - \lambda A^*(Ax_{n_k} - By_{n_k}))\| \\ &\quad L_s \bar{\lambda}(A,B) \|Ax_{n_k} - By_{n_k}\| \to 0 \ as \ k \to \infty. \end{aligned}$$

A similar computation gives that $\lim_{k\to\infty} \|y_{n_k} - T(y_{n_k})\| = 0$. Since S and T are demiclosed at zero, we conclude that $x^* = S(x^*)$ and $y^* = T(y^*)$. Again, since $x_{n_k} \rightharpoonup x^*$ and $y_{n_k} \rightharpoonup y^*$, we have that

$$Ax_{n_k} - By_{n_k} \rightharpoonup Ax^* - By^*,$$

and by the weak lower semi-continuity of norm square.

$$\|Ax^* - By^*\| \leq \liminf_{n\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0.$$

So, $Ax^* = By^*$ and thus $(x^*, y^*) \in \Gamma$. In conclusion, we have obtain thus far that for each $(p,q) \in \Gamma$, the sequence $\|(x_n, y_n) - (p,q)\|_*^2$ has a limit. Moreover, each weak cluster point of the sequence (x_n, y_n) is an element of Γ. We may now invoke the celebrated Opial's Lemma 2.5 to conclude that there exist $(x^*, y^*) \in \Gamma$ such that (x_n, y_n) converges weakly to (x^*, y^*). Hence the iterative sequence (x_n, y_n) converges weakly to a solution of the spit equality problem (6). The proof is complete.

We may strengthen the conditions of the theorem and obtain the strong convergence of the sequence as follows.

Theorem 3.2. *Suppose that the assupmtions of Theorem 3.1 are fulfilled. Assume, in addition, that the mappings S and T are also hemicompact. Then, for any initial point (x_1, y_1), the coupled iterative sequence (x_n, y_n) derived from the algorithm converges strongly to a solution of problem (SEP).*

Proof. We have obtained from Theorem 3.1 that (x_n, y_n) is bounded, and that $\lim_{n\to\infty} \|x_n - S(x_n)\| = 0$, and $\lim_{n\to\infty} \|y_n - T(y_n)\| = 0$. On the other hand, since S and T are hemicompact, we have some subsequence $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that $\{x_{n_k}\} \to x^*$ and $\{y_{n_k}\} \to y^*$. The subsequence also converge weakly and therefore $Ax_{n_k} - By_{n_k} \rightharpoonup Ax^* - By^*$. As we have shown above, this yields $Ax^* = By^*$ and $(x^*, y^*) \in \Gamma$. Going back to the proof of Theorem 3.1, we have that $\lim_{n\to\infty} \|(x_n, y_n) - (x^*, y^*)\|_*^2$ exists and $\lim_{n\to\infty} \|(x_{n_k}, y_{n_k}) - (x^*, y^*)\|_*^2$. We may conclude by Lemma 2.4 that $(x_n, y_n) \to (x^*, y^*) \in \Gamma$. So our iterative algorith converges to a solution of (SEP) and the proof is complete.

Corollary 3.3. *Suppose that the mappings S and T in Theorem 3.2 are hemicompact and demicontractive. Then, for any initial point (x_1, y_1), the coupled iterative sequence (x_n, y_n) derived from the algorithm converges strongly to a solution of problem (SEP).*

In conclusion, our theorems extend and complement the results of Chidume *et al.* [10], Xu [16], Moudafi and Al-Shamas [13] and many other authors to the more general class of Lipschitz hemicontractive mappings.

Conflict of Interests

The authors declare that there is no conflict of interests.

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