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COINCIDENCE POINT AND COMMON FIXED POINT THEOREMS IN CONE METRIC TYPE SPACES

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Abstract. In this paper, we prove coincidence point and common fixed point results of two, three and four self mappings in normal cone metric type spaces. The results presented in this paper generalize some recent results announced by many authors.

Keywords: Cone metric type space; Normal cone; Common fixed point; Coincidence point; Weakly compatible mappings.

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1. Introduction

In 1989, The concept of b-metric space was introduced by Bakht, who used it to prove the Banach contraction mapping principle [1-5]. In 2007, Huang and Zhang introduced cone metric spaces and established fixed point theorems of nonlinear operators [8]. Since 2007, fixed point problem in the framework of cone metric spaces have been extensive investigated by many authors; see, for example, [2-12] and the references therein. As a generalization and unification of cone metric spaces and b-metric spaces, Khamsi and Husssain defined a new type of spaces

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which was called cone metric type spaces. For the results in the framework of cone metric type spaces, we refer authors to [13-15] and the references therein.

The aim of this paper is to obtain coincidence points and common fixed points for two, three and four self nonlinear mappings in a normal cone metric type spaces. The results presented in this paper generalize some recent results announced by many authors.

2. Preliminaries

Definition 2.1. [7] A subset P of a real Banach space E is called a cone if it has the following properties:

(1) P is non-empty,closed and $P \neq \{\theta\}$;

(2) $0 \leq a, b \in R$ and $x, y \in P \Rightarrow ax + by \in P$;

(3) $P \cap (-P) = \{\theta\}$.

For a given cone $P \subseteq E$,a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We use $x \ll y$ for $y - x \in intP$,$intP$ stands for the interior of P.

Definition 2.2. [7] A cone P is said to be normal if there exists a constant $\kappa > 0$ such that

$$\| x \| \leq \kappa \| y \|, \quad for\ all\ x, y \in E, \theta \leq x \leq y.$$

The least number κ is called the normal constant of P.

Definition 2.3. [16,17] Let X be a nonempty set,$s \geq 1$ be a real number and E a real Banach space with cone P.Suppose that the mapping $d : X \times X \to E$ satisfies:

(1) $d(x,y) \geq \theta$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,z) \leq s[d(x,y) + d(y,z)]$ for all $x, y, z \in X$.

Then d is called a cone metric type on X and (X, d, s) is called a cone metric type space.

Example 2.4. [13] *Let* $B = \{e_i | i = 1, 2..., n\}$ *be an orthonormal basis of* $\mathbb{R}^n$ *with inner product* $(\cdot, \cdot)$ *and* $p > 0$. *Define*

$$X_p = \{[x] | x : [0,1] \to \mathbb{R}^n, \int_0^1 |(x(t), e_j)|^p dt, \quad j = 1, 2, ..., n\},$$

where $[x]$ represents the class of equivalence of x with respect to relation of functions equal almost everywhere. Let $E=\mathbb{R}^n$ and

$$P_B=\{y\in\mathbb{R}^n|(y,e_i)\geq 0, i=1,2,...,n\}$$

be a solid cone. Define $d: X_P\times X_P\to P_B\subset\mathbb{R}^n$ by

$$d(f,g)=\sum_{i=1}^{n}e_i\int_0^1|((f-g)(t),e_i)|^p dt,\quad f,g\in X_p.$$

Then (X_P,d,s) is a cone metric type space with $s=2^{p-1}$.

Definition 2.5. [16] Let (X,d,s) be a cone metric type space,x_n a sequence in X and $x\in X$.

(1) $\{x_n\}$ converges to x if for $\forall c\in E$ with $0\ll c$ there exists $n_0\in\mathbf{N}$ such that $d(x_n,x)\ll c$ for all $n>n_0$,and we write $\lim_{n\to\infty}d(x_n,x)=\theta$.

(2) $\{x_n\}$ is called a Cauchy sequence if for $\forall c\in E$ with $0\ll c$ there exists $n_0\in\mathbf{N}$ such that $d(x_n,x_m)\ll c$ for all $m,n>n_0$,and we write $\lim_{n\to\infty}d(x_n,x_m)=\theta$.

Lemma 2.6. [10] *Let (X,d,s) be a cone metric type space and P a normal cone,then*

(1) $\{x_n\}$ converges to x if and only if $d(x_n,x)\to\theta$,as $n\to\infty$;

(2) $\{x_n\}$ is called a Cauchy sequence if and only if $d(x_n,x_m)\to\theta$,as $n,m\to\infty$.

Definition 2.7. [18] Let f and g be self-mappings on a set X,if

$$w=fx=gx\quad for\ some\ x\ in\ X,$$

then x is called coincidence point of f and g ,w is called a point of coincidence of f and g.

Definition 2.8. [18] Let f and g be self-mappings on a set X,if $fgw=gfw$ for all coincidence points w, then the pair (f,T) is said to be weakly compatible.

3. Main results

Theorem 3.1. *Let (X,d,s) be a cone metric type space with coefficient $s\geq 1$ and P a normal cone with normal constant κ.Suppose the mappings $f,g: X\to X$ for all $x,y\in X$ satisfy:*

$$d(fx,fy)\leq a_1d(gx,gy)+a_2d(fx,gx)+a_3d(fy,gy)+a_4d(fx,gy)+a_5d(fy,gx)$$

where $a_i \geq 0, i = 1, \cdots, 5$ *with*

$$2sa_1 + (s+1)(a_2+a_3) + (s^2+s)(a_4+a_5) < 2. \tag{3.1}$$

Also, suppose that $f(X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X*. Then* f *and* g *have a unique point of coincidence. Moreover,if* (f,g) *is weakly compatible,then* f *and* g *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \subset g(X)$, we can choose a point x_1 in X such that $fx_0 = gx_1$. Similarly, choose a point x_2 in X such that $fx_1 = gx_2$. Continuing this process, we obtain the sequence $\{x_n\}$ by $fx_n = gx_{n+1}$ for all $n \geq 0$. Then

$$\begin{aligned}
d(gx_{n+1}, gx_n) &\leq a_1 d(gx_n, gx_{n-1}) + a_2 d(fx_n, gx_n) + a_3 d(fx_{n-1}, gx_{n-1}) \\
&+ a_4 d(fx_n, gx_{n-1}) + a_5 d(fx_{n-1}, gx_n) \\
&= a_1 d(gx_n, gx_{n-1}) + a_2 d(gx_{n+1}, gx_n) + a_3 d(gx_n, gx_{n-1}) \\
&+ a_4 d(gx_{n+1}, gx_{n-1}) + a_5 d(gx_n, gx_n) \\
&\leq a_1 d(gx_n, gx_{n-1}) + a_2 d(gx_{n+1}, gx_n) + a_3 d(gx_n, gx_{n-1}) \\
&+ sa_4 d(gx_{n-1}, gx_n) + sa_4 d(gx_n, gx_{n+1}) \\
&= (a_1 + a_3 + sa_4) d(gx_n, gx_{n-1}) + (a_2 + sa_4) d(gx_{n+1}, gx_n),
\end{aligned}$$

and

$$\begin{aligned}
d(gx_n, gx_{n+1}) &= d(fx_{n-1}, fx_n) \\
&\leq a_1 d(gx_{n-1}, gx_n) + a_2 d(fx_{n-1}, gx_{n-1}) + a_3 d(fx_n, gx_n) \\
&+ a_4 d(fx_{n-1}, gx_n) + a_5 d(fx_n, gx_{n-1}) \\
&= a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gx_{n+1}, gx_n) \\
&+ a_4 d(gx_n, gx_n) + a_5 d(gx_{n+1}, gx_{n-1}) \\
&\leq a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gx_{n+1}, gx_n) \\
&+ sa_5 d(gx_{n+1}, gx_n) + sa_5 d(gx_n, gx_{n-1}) \\
&= (a_1 + a_2 + sa_5) d(gx_n, gx_{n-1}) + (a_3 + sa_5) d(gx_{n+1}, gx_n).
\end{aligned}$$

Adding the last two inequalities, we have

$$2d(gx_n,gx_{n+1}) \leq (2a_1+a_2+a_3+sa_4+sa_5)d(gx_n,gx_{n-1})+(a_2+a_3+sa_4+sa_5)d(gx_n,gx_{n+1}).$$

Then

$$d(gx_n,gx_{n+1}) \leq \frac{2a_1+a_2+a_3+sa_4+sa_5}{2-a_2-a_3-sa_4-sa_5}d(gx_n,gx_{n-1}),$$

for all $n \geq 0$. Put

$$\lambda = \frac{2a_1+a_2+a_3+sa_4+sa_5}{2-a_2-a_3-sa_4-sa_5}.$$

It follows that $s\lambda < 1$ and $d(gx_n,gx_{n+1}) \leq \lambda d(gx_n,gx_{n-1}) \leq \lambda^n d(gx_0,gx_1)$. Now for $m > n$, we have

$$\begin{aligned} d(gx_n,gx_m) &\leq sd(gx_n,gx_{n+1})+s^2d(gx_{n+1},gx_{n+2})+\cdots+s^{m-n-1}d(gx_{m-2},gx_{m-1}) \\ &+s^{m-n}d(gx_{m-1},gx_m) \\ &\leq (s\lambda^n+s^2\lambda^{n+1}+\cdots+s^{m-n-1}\lambda^{m-2}+s^{m-n}\lambda^{m-1})d(gx_0,gx_1) \\ &\leq \frac{s\lambda^n}{1-s\lambda}d(gx_0,gx_1). \end{aligned}$$

Since P is a normal cone with normal constant κ, we have

$$\| d(gx_n,gx_m) \| \leq \kappa\frac{s\lambda^n}{1-s\lambda} \| d(gx_0,gx_1) \|,$$

Thus, if $n,m \to \infty$, then $d(gx_n,gx_m) \to \theta$. Hence, $\{gx_n\}$ is a Cauchy sequence. Since $g(X)$ is complete, there exist $u,v \in X$ such that $gx_n \to v = gu$. Since

$$\begin{aligned} d(gx_n,fu) &= d(fx_{n-1},fu) \\ &\leq a_1d(gx_{n-1},gu)+a_2d(fx_{n-1},gx_{n-1})+a_3d(fu,gu) \\ &+a_4d(fx_{n-1},gu)+a_5d(fu,gx_{n-1}) \\ &= a_1d(gx_{n-1},v)+a_2d(gx_n,gx_{n-1})+a_3d(fu,v) \\ &+a_4d(gx_n,v)+a_5d(fu,gx_{n-1}) \\ &\leq a_1d(gx_{n-1},v)+a_2d(gx_n,gx_{n-1})+sa_3d(fu,gx_n)+sa_3d(gx_n,v) \\ &+a_4d(gx_n,v)+sa_5d(fu,gx_n)+sa_5d(gx_n,gx_{n-1}), \end{aligned}$$

we find that

$$d(gx_n, fu) \leq \frac{1}{1 - sa_3 - sa_5}[a_1 d(gx_{n-1}, v) + (a_2 + sa_5) d(gx_n, gx_{n-1}) + (sa_3 + a_4) d(gx_n, v)].$$

Hence, we have

$$\| d(gx_n, fu) \| \leq \frac{\kappa}{1 - sa_3 - sa_5} \| a_1 d(gx_{n-1}, v) + (a_2 + sa_5) d(gx_n, gx_{n-1}) + (a_4 + sa_3) d(gx_n, v) \| .$$

If $n \to \infty$,then we have $d(gx_n, fu) \to \theta$, Also, $d(gx_n, gu) \to \theta$ as $n \to \infty$. The uniqueness of a limit in a cone metric type space implies that $fu = gu = v$. Now we show that f and g have a unique point of coincidence. For this end, assume that there exists another point u^* in X such that $fu^* = gu^* = v^*$. Then

$$\begin{aligned} d(v, v^*) &= d(fu, fu^*) \\ &\leq a_1 d(gu, gu^*) + a_2 d(fu, gu) + a_3 d(fu^*, gu^*) \\ &+ a_4 d(fu, gu^*) + a_5 d(fu^*, gu) \\ &= a_1 d(v, v^*) + a_2 d(v, v) + a_3 d(v^*, v^*) \\ &+ a_4 d(v, v^*) + a_5 d(v^*, v) \\ &\leq (a_1 + a_4 + a_5) d(v, v^*), \end{aligned}$$

which gives a contraction, Hence,w e have $v = v^*$. If (f, g) is weakly compatible, then $fv = fgu = gfu = gv$. So $u = v$ by uniqueness. Thus v is the unique common fixed point of f and g.

Corollary 3.2. *Let (X, d, s) be a cone metric type space with coefficient $s \geq 1$ and P a normal cone with normal constant κ.Suppose the mappings f and g be self-mappings on X,such that $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Suppose that one of the following conditions holds:*

(1) $d(fx, fy) \leq a_1 d(gx, gy) + a_2 d(fx, gx) + a_3 d(fy, gy)$,
for all $x, y \in X$, where $a_1, a_2, a_3 \geq 0$ and $2sa_1 + (s+1)(a_2 + a_3) < 2$.

(2) $d(fx,fy) \leq a_1 d(gx,gy) + a_2 d(fx,gy) + a_3 d(fy,gx)$,

for all $x,y \in X$, *where* $a_1,a_2,a_3 \geq 0$ *and* $2sa_1 + (s^2+s)(a_2+a_3) < 2$.

(3) $d(fx,fy) \leq a_1 d(fx,gx) + a_2 d(fy,gy)$,

for all $x,y \in X$, *where* $a_1,a_2 \geq 0$ *and* $a_1 + a_2 < \frac{2}{s+1}$.

(4) $d(fx,fy) \leq a_1 d(fx,gy) + a_2 d(fy,gx)$,

for all $x,y \in X$, *where* $a_1,a_2 \geq 0$ *and* $a_1 + a_2 < \frac{2}{s^2+s}$.

(5) $d(fx,fy) \leq a_1 d(gx,gy)$,

for all $x,y \in X$, *where* $0 < a_1 < \frac{1}{s}$.

Then f *and* g *have a unique point of coincidence.Moreover,if* (f,g) *is weakly compatible,then* f *and* g *have a unique common fixed point.*

Putting $g = i_X$ in Theorem 3.1 and Corollary 3.2, we get the following results.

Corollary 3.3. *Let* (X,d,s) *be a cone metric type space with coefficient* $s \geq 1$ *and* P *a normal cone with normal constant* κ*.Let* $f : X \to X$ *be a map such that* $f(X)$ *is a complete subspace of* X*. Suppose that one of the following conditions holds:*

(1) $d(fx,fy) \leq a_1 d(x,y) + a_2 d(fx,x) + a_3 d(fy,y) + a_4 d(fx,y) + a_5 d(fy,x)$,

for all $x,y \in X$, *where* $a_1,a_2,a_3,a_4,a_5 \geq 0$ *with* $2sa_1 + (s+1)(a_2+a_3) + (s^2+s)(a_4+a_5) < 2$.

(2) $d(fx,fy) \leq a_1 d(x,y) + a_2 d(fx,x) + a_3 d(fy,y)$,

for all $x,y \in X$, *where* $a_1,a_2,a_3 \geq 0$ *and* $2sa_1 + (s+1)(a_2+a_3) < 2$.

(3) $d(fx,fy) \leq a_1 d(x,y) + a_2 d(fx,y) + a_3 d(fy,x)$,

for all $x,y \in X$, *where* $a_1,a_2,a_3 \geq 0$ *and* $2sa_1 + (s^2+s)(a_2+a_3) < 2$.

(4) $d(fx,fy) \leq a_1 d(fx,x) + a_2 d(fy,y)$,

for all $x,y \in X$, *where* $a_1,a_2 \geq 0$ *and* $a_1 + a_2 < \frac{2}{s+1}$.

(5) $d(fx,fy) \leq a_1 d(fx,y) + a_2 d(fy,x)$,

for all $x,y \in X$, *where* $a_1,a_2 \geq 0$ *and* $a_1 + a_2 < \frac{2}{s^2+s}$.

(6) $d(fx,fy) \leq a_1 d(x,y)$,

for all $x,y \in X$, *where* $0 < a_1 < \frac{1}{s}$.

Then f *has a unique fixed point.*

Theorem 3.4. *Let(X,d,s) be a cone metric type space with coefficient* $s \geq 1$ *and* P *a normal cone with normal constant* κ*. Suppose the mappings* S,T *and* f *are three self-mappings on* X*,* *satisfy:* $d(Sx,Ty) \leq a_1d(fx,fy) + a_2d(Sx,fx) + a_3d(Ty,fy) + a_4d(Sx,fy) + a_5d(Ty,fx)$ *for all* $x,y \in X$, *where* $a_1,a_2,a_3,a_4,a_5 \geq 0$ *with*

$$2sa_1 + (s+1)(a_2+a_3) + (s^2+s)(a_4+a_5) < 2. \tag{3.2}$$

Also, suppose that $S(X)\bigcup T(X) \subseteq f(X)$ *and* $f(X)$ *is a complete subspace of* X. *Then* S,T *and* f *have a unique point of coincidence.Moreover,if* (S,f) *and* (T,f) *are weakly compatible,then* S,T *and* f *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Since $S(X)\bigcup T(X) \subseteq f(X)$, we can choose a point x_1 in X such that $Sx_0 = fx_1$. Similarly, choose a point x_2 in X such that $Tx_1 = fx_2$. Continuing this process, we obtain the sequence $\{x_n\}$ by $fx_{2k+1} = Sx_{2k}, fx_{2k+2} = Tx_{2k+1}$, for all $k \geq 0$. Then

$$\begin{aligned}
d(fx_{2k+1},fx_{2k+2}) &\leq a_1d(fx_{2k},fx_{2k+1}) + a_2d(Sx_{2k},fx_{2k}) + a_3d(Tx_{2k+1},fx_{2k+1}) \\
&+ a_4d(Sx_{2k},fx_{2k+1}) + a_5d(Tx_{2k+1},fx_{2k}) \\
&\leq a_1d(fx_{2k},fx_{2k+1}) + a_2d(fx_{2k+1},fx_{2k}) + a_3d(fx_{2k+2},fx_{2k+1}) \\
&+ a_4d(fx_{2k+1},fx_{2k+1}) + a_5d(fx_{2k+2},fx_{2k}) \\
&\leq a_1d(fx_{2k},fx_{2k+1}) + a_2d(fx_{2k+1},fx_{2k}) + a_3d(fx_{2k+2},fx_{2k+1}) \\
&+ sa_5d(fx_{2k+2},fx_{2k+1}) + sa_5d(fx_{2k+1},fx_{2k}),
\end{aligned}$$

which implies that

$$d(fx_{2k+1},fx_{2k+2}) \leq \frac{a_1+a_2+sa_5}{1-a_3-sa_5}d(fx_{2k},fx_{2k+1}).$$

Similarly, we have

$$\begin{aligned}
d(fx_{2k+3},fx_{2k+2}) &\leq a_1d(fx_{2k+2},fx_{2k+1}) + a_2d(Sx_{2k+2},fx_{2k+2}) + a_3d(Tx_{2k+1},fx_{2k+1}) \\
&+ a_4d(Sx_{2k+2},fx_{2k+1}) + a_5d(Tx_{2k+1},fx_{2k+2}) \\
&\leq a_1d(fx_{2k+2},fx_{2k+1}) + a_2d(fx_{2k+3},fx_{2k+2}) + a_3d(fx_{2k+2},fx_{2k+1}) \\
&+ a_4d(fx_{2k+3},fx_{2k+1}) + a_5d(fx_{2k+2},fx_{2k+2}) \\
&\leq a_1d(fx_{2k+2},fx_{2k+1}) + a_2d(fx_{2k+3},fx_{2k+2}) + a_3d(fx_{2k+2},fx_{2k+1}) \\
&+ sa_4d(fx_{2k+3},fx_{2k+2}) + sa_4d(fx_{2k+2},fx_{2k+1}).
\end{aligned}$$

Hence, we have

$$d(fx_{2k+2},fx_{2k+3}) \leq \frac{a_1+a_3+sa_4}{1-a_2-sa_4}d(fx_{2k+1},fx_{2k+2}).$$

Let

$$\lambda = \frac{a_1+a_2+sa_5}{1-a_3-sa_5}, \mu = \frac{a_1+a_3+sa_4}{1-a_2-sa_4}.$$

By induction, we have

$$\begin{aligned} d(fx_{2k+1},fx_{2k+2}) &\leq \lambda d(fx_{2k},fx_{2k+1}) \\ &\leq \lambda\mu d(fx_{2k-1},fx_{2k}) \\ &\leq \lambda\mu\lambda d(fx_{2k-2},fx_{2k-1}) \\ &\leq \cdots \leq \lambda(\mu\lambda)^k d(fx_0,fx_1) \end{aligned}$$

and

$$\begin{aligned} d(fx_{2k+2},fx_{2k+3}) &\leq \mu d(fx_{2k+1},fx_{2k+2}) \\ &\leq \mu\lambda d(fx_{2k},fx_{2k+1}) \\ &\leq \cdots \leq (\mu\lambda)^{k+1} d(fx_0,fx_1) \end{aligned}$$

for all $k \geq 0$. From the condition (2.2), we have $\lambda\mu < \frac{1}{s^2}$. Now, for $p < q$, we have

$$\begin{aligned} d(fx_{2p},fx_{2q+1}) &\leq sd(fx_{2p},fx_{2p+1}) + s^2 d(fx_{2p+1},fx_{2p+2}) + s^3 d(fx_{2p+2},fx_{2p+3}) \\ &+ \cdots + s^{2q-2p+1} d(fx_{2q},fx_{2q+1}) \\ &\leq s(\lambda\mu)^p d(fx_0,fx_1) + s^2\lambda(\lambda\mu)^p d(fx_0,fx_1) + s^3(\lambda\mu)^{p+1} d(fx_0,fx_1) \\ &+ \cdots + s^{2q-2p+1}(\lambda\mu)^{q+1} d(fx_0,fx_1) \\ &\leq [\frac{s(\lambda\mu)^p}{1-s^2(\lambda\mu)} + \frac{s^2\lambda(\lambda\mu)^p}{1-s^2(\lambda\mu)}] d(fx_0,fx_1) \\ &\leq (1+s\lambda)\frac{s(\lambda\mu)^p}{1-s^2(\lambda\mu)} d(fx_0,fx_1). \end{aligned}$$

Similarly, we can obtain

$$d(fx_{2p},fx_{2q}) \leq (1+s\lambda)\frac{s(\lambda\mu)^p}{1-s^2(\lambda\mu)} d(fx_0,fx_1),$$

$$d(fx_{2p+1}, fx_{2q}) \leq (1+s\mu)\frac{s\lambda(\lambda\mu)^p}{1-s^2(\lambda\mu)}d(fx_0, fx_1),$$

$$d(fx_{2p+1}, fx_{2q+1}) \leq (1+s\mu)\frac{s\lambda(\lambda\mu)^p}{1-s^2(\lambda\mu)}d(fx_0, fx_1).$$

Hence, for $0 < n < m$, there exists $p < n < m$ such that $p \to \infty$ as $n \to \infty$, and

$$d(fx_n, fx_m) \leq \max\{(1+s\lambda)\frac{s(\lambda\mu)^p}{1-s^2(\lambda\mu)}, (1+s\mu)\frac{s\lambda(\lambda\mu)^p}{1-s^2(\lambda\mu)}\}d(fx_0, fx_1).$$

Since P is a normal cone with normal constant κ, we have

$$\parallel d(fx_n, fx_m) \parallel \leq \kappa \max\{(1+s\lambda)\frac{s(\lambda\mu)^p}{1-s^2(\lambda\mu)}, (1+s\mu)\frac{s\lambda(\lambda\mu)^p}{1-s^2(\lambda\mu)}\} \parallel d(fx_0, fx_1) \parallel .$$

Since $\lambda\mu < \frac{1}{s^2}$, we have if $n, m \to \infty$. Then

$$\max\{(1+s\lambda)\frac{s(\lambda\mu)^p}{1-s^2(\lambda\mu)}, (1+s\mu)\frac{s\lambda(\lambda\mu)^p}{1-s^2(\lambda\mu)}\} \to 0.$$

So $d(fx_n, fx_m) \to \theta$. Hence, $\{fx_n\}$ is a Cauchy sequence. Since $f(X)$ is complete, there exist $u, v \in X$ such that $fx_n \to fu = v$. Since

$$\begin{aligned}
d(Su, fu) &\leq sd(fu, fx_{2n}) + sd(fx_{2n}, Su) \\
&= sd(fu, fx_{2n}) + sd(Tx_{2n-1}, Su) \\
&\leq sd(fu, fx_{2n}) + sa_1d(fu, fx_{2n-1}) + sa_2d(Su, fu) + sa_3d(Tx_{2n-1}, fx_{2n-1}) \\
&+ sa_4d(Su, fx_{2n-1}) + sa_5d(Tx_{2n-1}, fu) \\
&\leq sd(fu, fx_{2n}) + sa_1d(fu, fx_{2n-1}) + sa_2d(Su, fu) + sa_3d(fx_{2n}, fx_{2n-1}) \\
&+ sa_4d(Su, fx_{2n-1}) + sa_5d(fx_{2n}, fu) \\
&\leq sd(fu, fx_{2n}) + sa_1d(fu, fx_{2n-1}) + sa_2d(Su, fu) + sa_3d(fx_{2n}, fx_{2n-1}) \\
&+ s^2a_4d(Su, fu) + s^2a_4d(fu, fx_{2n-1}) + sa_5d(fx_{2n}, fu),
\end{aligned}$$

we find that

$$\begin{aligned}
&d(Su, fu) \\
&\leq \frac{1}{1-sa_2-s^2a_4}[sd(fu, fx_{2n}) + sa_1d(fu, fx_{2n-1}) + sa_3d(fx_{2n}, fx_{2n-1}) \\
&+ s^2a_4d(fu, fx_{2n-1}) + sa_5d(fx_{2n}, fu)].
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\| d(Su,fu) \| \\
&\leq \frac{\kappa}{1-sa_2-s^2a_4} \| sd(fu,fx_{2n})+sa_1d(fu,fx_{2n-1})+sa_3d(fx_{2n},fx_{2n-1}) \\
&+s^2a_4d(fu,fx_{2n-1})+sa_5d(fx_{2n},fu) \| .
\end{aligned}$$

If $n\to\infty$, then we have $\| d(fu,Su) \|=0$. Hence, $fu=Su$. Similarly, we can show that $fu=Tu$, that is, $v=fu=Su=Tu$. Now we show that S,T and f have a unique point of coincidence. For this, assume that there exists another point u^* in X such that $fu^*=Su^*=Tu^*=v^*$. Then

$$\begin{aligned}
d(v,v^*) &= d(Su,Tu^*) \\
&\leq a_1d(fu,fu^*)+a_2d(Su,fu)+a_3d(Tu^*,fu^*) \\
&+a_4d(Su,fu^*)+a_5d(Tu^*,fu) \\
&= a_1d(v,v^*)+a_2d(v,v)+a_3d(v^*,v^*) \\
&+a_4d(v,v^*)+a_5d(v^*,v) \\
&\leq (a_1+a_4+a_5)d(v,v^*),
\end{aligned}$$

which gives a contraction, Hence, we have $v=v^*$. If (S,f) and (T,f) are weakly compatible, then $Sv=Sfu=fSu=fv$ and $Tv=Tfu=fTu=fv$ It implies that $Sv=Tv=fv$. So $u=v$ by uniqueness. Thus v is the unique common fixed point of S,T and f.

Corollary 3.5. *Let (X,d,s) be a cone metric type space with coefficient $s\geq 1$ and P a normal cone with normal constant κ. Suppose the mappings S,T and f be self-mappings on X,such that $S(X)\bigcup T(X)\subseteq f(X)$ and $f(X)$ is a complete subspace of X. Suppose that one of the following two conditions holds:*

(1) $d(Sx,Ty)\leq a_1d(fx,fy)+a_2d(Sx,fx)+a_3d(Ty,fy)$

for all $x,y\in X$, where $a_1,a_2,a_3\geq 0$ and $2sa_1+(s+1)(a_2+a_3)<2$.

(2) $d(Sx,Ty)\leq a_1d(fx,fy)+a_2d(Sx,fy)+a_3d(Ty,fx)$

for all $x,y\in X$, where $a_1,a_2,a_3\geq 0$ and $2sa_1+(s^2+s)(a_2+a_3)<2$.

(3) $d(Sx,Ty)\leq a_1d(Sx,fx)+a_2d(Ty,fy)$

for all $x,y\in X$, where $a_1,a_2\geq 0$ and $a_1+a_2<\frac{2}{s+1}$.

(4) $d(Sx,Ty) \leq a_1 d(Sx,fy) + a_2 d(Ty,fx)$

for all $x,y \in X$*, where* $a_1, a_2 \geq 0$ *and* $a_1 + a_2 < \frac{2}{s^2+s}$.

(5) $d(Sx,Ty) \leq a_1 d(fx,fy)$

for all $x,y \in X$*, where* $a_1 \geq 0$ *and* $a_1 < \frac{1}{s}$.

Then S,T *and* f *have a unique point of coincidence.Moreover,if* (S,f) *and* (T,f) *are weakly compatible, then* S,T *and* f *have a unique common fixed point.*

Theorem 3.6. *Let* (X,d,s) *be a cone metric type space with coefficient* $s \geq 1$ *and* P *a normal cone with normal constant* κ*.Suppose the mappings* f,g,S *and* T *be self-mappings on* X*, satisfying:*

$d(fx,gy) \leq a_1 d(Sx,Ty) + a_2 d(fx,Sx) + a_3 d(gy,Ty) + a_4 d(fx,Ty) + a_5 d(gy,Sx)$

for all $x,y \in X$*, where* $a_1,a_2,a_3,a_4,a_5 \geq 0$ *with*

$$2sa_1 + (s+1)(a_2+a_3) + (s^2+s)(a_4+a_5) < 2 \qquad (3.3).$$

Also, suppose that $f(X) \subset T(X), g(X) \subset S(X)$ *and one of* $f(X), g(X), S(X), T(X)$ *is a complete subspace of* X*. Then* (f,S) *and* (g,T) *have a common point of coincidence.Moreover,if* (f,S) *and* (g,T) *are weakly compatible, then* f,g,S *and* T *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \subset T(X), g(X) \subset S(X)$, we can choose a point x_1 in X such that $fx_0 = Tx_1$. Similarly, choose a point x_2 in X such that $gx_1 = Sx_2$. Continuing this process, we obtain the sequence $\{x_n\}$ and $\{y_n\}$ by $y_{2n-1} = fx_{2n-2} = Tx_{2n-1}, y_{2n} = gx_{2n-1} = Sx_{2n}$, for all $n \geq 0$. Then

$$\begin{aligned}
d(y_{2n-1},y_{2n}) &= d(fx_{2n-2},gx_{2n-1}) \\
&\leq a_1 d(Sx_{2n-2},Tx_{2n-1}) + a_2 d(fx_{2n-2},Sx_{2n-2}) + a_3 d(gx_{2n-1},Tx_{2n-1}) \\
&+ a_4 d(fx_{2n-2},Tx_{2n-1}) + a_5 d(gx_{2n-1},Sx_{2n-2}) \\
&\leq a_1 d(y_{2n-2},y_{2n-1}) + a_2 d(y_{2n-1},y_{2n-2}) + a_3 d(y_{2n},y_{2n-1}) \\
&+ a_4 d(y_{2n-1},y_{2n-1}) + a_5 d(y_{2n},y_{2n-2}) \\
&\leq a_1 d(y_{2n-2},y_{2n-1}) + a_2 d(y_{2n-1},y_{2n-2}) + a_3 d(y_{2n},y_{2n-1}) \\
&+ sa_5 d(y_{2n},y_{2n-1}) + sa_5 d(y_{2n-1},y_{2n-2}),
\end{aligned}$$

which implies that $d(y_{2n-1},y_{2n}) \leq \frac{a_1+a_2+sa_5}{1-a_3-sa_5} d(y_{2n-2},y_{2n-1})$. Similarly we can show that

$$d(y_{2n},y_{2n+1}) \leq \frac{a_1+a_3+sa_4}{1-a_2-sa_4} d(y_{2n-1},y_{2n}).$$

Letting $\lambda = \max\{\frac{a_1+a_2+sa_5}{1-a_3-sa_5}, \frac{a_1+a_3+sa_4}{1-a_2-sa_4}\}$, we know that $0 < \lambda < \frac{1}{s}$. Therefore $d(y_n,y_{n+1}) \leq \lambda d(y_{n-1},y_n) \leq \lambda^n d(y_0,y_1)$, for all $n \in \mathbf{N}$. Now, for $m > n$ we have

$$\begin{aligned} d(y_n,y_m) &\leq sd(y_n,y_{n+1}) + s^2 d(y_{n+1},y_{n+2}) + \cdots + s^{m-n-1} d(y_{m-2},y_{m-1}) \\ &+ s^{m-n} d(y_{m-1},y_m) \\ &\leq (s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n}\lambda^{m-1}) d(y_0,y_1) \\ &\leq \frac{s\lambda^n}{1-s\lambda} d(y_0,y_1) \end{aligned}$$

Since P is a normal cone with normal constant κ, we have $\| d(y_n,y_m) \| \leq \kappa \frac{s\lambda^n}{1-s\lambda} \| d(y_0,y_1) \|$ Thus, if $n,m \to \infty$, then $d(y_n,y_m) \to 0$. Hence, $\{y_n\}$ is a Cauchy sequence. Suppose that $S(X)$ is complete. Then there exist $u,v \in X$ such that $Sx_{2n} = y_{2n} \to v = Su$. We claim that $fu = v$. For this end, consider

$$\begin{aligned} d(fu,v) &\leq sd(fu,gx_{2n-1}) + sd(gx_{2n-1},v) \\ &\leq sa_1 d(Su,Tx_{2n-1}) + sa_2 d(fu,Su) + sa_3 d(gx_{2n-1},Tx_{2n-1}) \\ &+ sa_4 d(fu,Tx_{2n-1}) + sa_5 d(gx_{2n-1},Su) + sd(gx_{2n-1},v) \\ &= sa_1 d(v,y_{2n-1}) + sa_2 d(fu,v) + sa_3 d(y_{2n},y_{2n-1}) \\ &+ sa_4 d(fu,y_{2n-1}) + sa_5 d(y_{2n},v) + sd(y_{2n},v) \\ &\leq sa_1 d(v,y_{2n-1}) + sa_2 d(fu,v) + sa_3 d(y_{2n},y_{2n-1}) \\ &+ s^2 a_4 d(fu,v) + s^2 a_4 d(v,y_{2n-1}) + sa_5 d(y_{2n},v) + sd(y_{2n},v). \end{aligned}$$

It implies that

$$\begin{aligned} &d(fu,v) \\ &\leq \frac{1}{1-sa_2-s^2a_4}[sa_1 d(v,y_{2n-1}) + sa_3 d(y_{2n},y_{2n-1}) \\ &+ s^2 a_4 d(v,y_{2n-1}) + (sa_5+s) d(y_{2n},v)]. \end{aligned}$$

Hence, we have

$$\begin{aligned}
&\| d(fu,v) \| \\
&\leq \frac{\kappa}{1-sa_2-s^2a_4} \| sa_1d(v,y_{2n-1})+sa_3d(y_{2n},y_{2n-1}) \\
&+s^2a_4d(v,y_{2n-1})+(sa_5+s)d(y_{2n},v) \| .
\end{aligned}$$

If $n\to\infty$, then we have $\| d(fu,v) \|=0$. Hence, $fu=v=Su$. Since $v\in f(X)\subset T(X)$, there exists a point $w\in X$ such that $Tw=v$. Now we will show that $gw=v$. Since

$$\begin{aligned}
d(gw,v) &\leq sd(fx_{2n},gw)+sd(fx_{2n},v) \\
&\leq sa_1d(Sx_{2n},Tw)+sa_2d(fx_{2n},Sx_{2n})+sa_3d(gw,Tw) \\
&+sa_4d(fx_{2n},Tw)+sa_5d(gw,Sx_{2n})+sd(fx_{2n},v) \\
&=sa_1d(y_{2n},v)+sa_2d(y_{2n+1},y_{2n})+sa_3d(gw,v) \\
&+sa_4d(y_{2n+1},v)+sa_5d(gw,y_{2n})+sd(y_{2n+1},v) \\
&\leq sa_1d(y_{2n},v)+sa_2d(y_{2n+1},y_{2n})+sa_3d(gw,v) \\
&+sa_4d(y_{2n+1},v)+s^2a_5d(y_{2n},v)+s^2a_5d(gw,v)+sd(y_{2n+1},v).
\end{aligned}$$

It implies that

$$\begin{aligned}
&d(gw,v) \\
&\leq \frac{1}{1-sa_3-s^2a_5}[sa_1d(y_{2n},v)+sa_2d(y_{2n},y_{2n+1}) \\
&+(sa_4+s)d(y_{2n+1},v)+s^2a_5d(y_{2n},v)].
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\| d(gw,v) \| \\
&\leq \frac{\kappa}{1-sa_3-s^2a_5} \| sa_1d(v,y_{2n})+sa_2d(y_{2n},y_{2n+1}) \\
&+(sa_4+s)d(v,y_{2n+1})+s^2a_5d(y_{2n},v) \| .
\end{aligned}$$

Letting $n\to\infty$ in the above equality, we get $\| d(gw,v) \|=0$. Hence, $gw=v=Tw$. Thus (f,S) and (g,T) have a common point of coincidence in X. Now if (f,S) and (g,T) are weakly compatible, then $fv = fSu = Sfu = Sv = w_1$ (say) and $gv = gTw = Tgw = Tv = w_2$ (say). Since

$$
\begin{aligned}
d(w_1,w_2) &= d(fv,gv) \\
&\le a_1d(Sv,Tv)+a_2d(fv,Sv)+a_3d(gv,Tv)+a_4d(fv,Tv)+a_5d(gv,Sv) \\
&\le a_1d(w_1,w_2)+a_2d(w_1,w_1)+a_3d(w_2,w_2)+a_4d(w_1,w_2)+a_5d(w_2,w_1) \\
&\le (a_1+a_4+a_5)d(w_1,w_2),
\end{aligned}
$$

we find a contradiction. Thus we have $d(w_1,w_2) = 0$, that is, $w_1 = w_2$. Hence, we have $fv = gv = Sv = Tv$. Now we shall show that $fv = v$. Since

$$
\begin{aligned}
d(fv,v) &= d(fv,gw) \\
&\le a_1d(Sv,Tw)+a_2d(fv,Sv)+a_3d(gw,Tw)+a_4d(fv,Tw)+a_5d(gw,Sv) \\
&\le a_1d(fv,v)+a_2d(fv,fv)+a_3d(v,v)+a_4d(fv,v)+a_5d(v,fv) \\
&\le (a_1+a_4+a_5)d(fv,v),
\end{aligned}
$$

which is a contradiction. Thus we have $fv = v$, and v is a common fixed point of f,g,S and T.

Next we prove the uniqueness. Let v^* be another fixed point. Then

$$
\begin{aligned}
d(v,v^*) &= d(fv,gv^*) \\
&\le a_1d(Sv,Tv^*)+a_2d(fv,Sv)+a_3d(gv^*,Tv^*)+a_4d(fv,Tv^*)+a_5d(gv^*,Sv) \\
&\le a_1d(v,v^*)+a_2d(v,v)+a_3d(v^*,v^*)+a_4d(v,v^*)+a_5d(v^*,v) \\
&\le (a_1+a_4+a_5)d(v,v^*),
\end{aligned}
$$

which is a contradiction. Thus we have $v = v^*$. This completes the proof.

Corollary 3.7. *Let (X,d,s) be a cone metric type space with coefficient $s \ge 1$ and P a normal cone with normal constant κ.Suppose the mappings f,g,S and T be self-mappings on X, such that $f(X) \subset T(X), g(X) \subset S(X)$ and one of $f(X),g(X),S(X),T(X)$ is a complete subspace of X. Suppose that one of the following conditions holds:*

(1) $d(fx,gy) \le a_1d(Sx,Ty)+a_2d(fx,Sx)+a_3d(gy,Ty)$,
where $a_1,a_2,a_3 \ge 0$ and $2sa_1+(s+1)(a_2+a_3) < 2$.

(2) $d(fx,gy) \leq a_1 d(Sx,Ty) + a_2 d(fx,Ty) + a_3 d(gy,Sx)$,

where $a_1, a_2, a_3 \geq 0$ *and* $2sa_1 + (s^2+s)(a_2+a_3) < 2$.

(3) $d(fx,gy) \leq a_1 d(fx,Sx) + a_2 d(gy,Ty)$,

where $a_1, a_2, a_3 \geq 0$ *and* $a_1 + a_2 < \frac{2}{s+1}$.

(4)$d(fx,gy) \leq a_1 d(fx,Ty) + a_2 d(gy,Sx)$,

where $a_1, a_2, a_3 \geq 0$ *and* $a_1 + a_2 < \frac{2}{s^2+s}$.

(5) $d(fx,gy) \leq a_1 d(Sx,Ty)$

for all $x, y \in X$*, where* $a_1 \geq 0$ *and* $a_1 < \frac{1}{s}$.

Then (f,S) *and* (g,T) *have a common point of coincidence. Moreover, if* (f,S) *and* (g,T) *are weakly compatible, then* f, g, S *and* T *have a unique common fixed point.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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