Available online at http://scik.org
Adv. Fixed Point Theory, 5 (2015), No. 4, 407-419
ISSN: 1927-6303

A VERSION OF COUPLED FIXED POINT THEOREMS ON QUASI-PARTIAL b-METRIC SPACES

ANURADHA GUPTA[1], PRAGATI GAUTAM[2*]

[1]Department of Mathematics, Delhi College of Arts and Commerce, University of Delhi, Delhi, India

[2]Department of Mathematics, Kamala Nehru College, August Kranti Marg, University of Delhi, Delhi, India

Abstract. The notion of quasi-partial b-metric spaces was introduced and fixed point and coupled fixed point theorems on this space were studied. The present result is a continuation of the study of coupled fixed point theorems on quasi-partial b-metric spaces and a new version of coupled fixed point theorems on this space.

Keywords: Partial-metric space; Partial b-metric space; Quasi-partial metric space; Quasi-partial b-metric space; Coupled fixed point.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The concept of b-metric spaces was introduced by Czerwik [3] as a generalization of metric spaces. The partial metric space was introduced by Matthews [8] in 1994. Shukla [10] generalized both the concept of b-metric and partial-metric spaces by introducing partial b-metric spaces. Motivated by this a modest attempt has been made to introduce the notion of quasi-partial b-metric space [4] where we have proved fixed point theorems on it. Further, we proved

*Corresponding author
E-mail addresses: dishna2@yahoo.in (A. Gupta), pragati.knc@gmail.com (P. Gautam)
Received July 7, 2015

coupled fixed point theorems on the same space [5]. The present result is a continuation of the study of coupled fixed point theorems on quasi-partial b-metric spaces An example is provided to support the main results.

2. Preliminaries

We begin the section with some basic definitions and concepts.

Definition 2.1. [10] A *partial b-metric* on a non-empty set X is a mapping $p_b : X \times X \to \mathbb{R}^+$ such that for some real numbers $s \geq 1$ and all $x,y,z \in X$,

(P_{b_1}) $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$,

(P_{b_2}) $p_b(x,x) \leq p_b(x,y)$,

(P_{b_3}) $p_b(x,y) = p_b(y,x)$,

(P_{b_4}) $p_b(x,y) \leq s[p_b(x,z) + p_b(z,y)] - p_b(z,z)$.

A *partial b-metric* space is a pair (X, p_b) such that X is a non-empty set and p_b is a partial b-metric on X. The number s is called the coefficient of (X, p_b).

Definition 2.2. [6] A *quasi-partial metric* on a non-empty set X is a function $q : X \times X \to \mathbb{R}^+$ which satisfies:

(QPM_1) If $q(x,x) = q(x,y) = q(y,y)$, the $x = y$,

(QPM_2) $q(x,x) \leq q(x,y)$,

(QPM_3) $q(x,x) \leq q(y,x)$,

(QPM_4) $q(x,y) + q(z,z) \leq q(x,z) + q(z,y)$ for all $x,y,z \in X$.

A *quasi partial metric space* is a pair (X, q) such that X is a non-empty set and q is a *quasi-partial metric* on X.

Let q be a quasi-partial metric on the set X. Then $d_q(x,y) = q(x,y) + q(y,x) - q(x,x) - q(y,y)$ is a metric on X.

Lemma 2.3. [6] *For a quasi-partial metric q on X, $p_q(x,y) = \dfrac{1}{2}[q(x,y) + q(y,x)]$, $x,y \in X$ is a partial metric on X.*

Lemma 2.4. [6] *Let (X,q) be a quasi-partial metric space. Let (X,p_q) be the corresponding partial metric space, and let (X,d_{p_q}) be the corresponding metric space. Then the sequence $\{x_n\}$ is Cauchy in (X,q) iff the sequence $\{x_n\}$ is Cauchy in (X,p_q) iff the sequence $\{x_n\}$ is Cauchy in (X,d_{p_q}).*

Lemma 2.5. [6] *Let (X,q) be a quasi-partial metric space, let (X,p_q) be the corresponding partial metric space, and let (X,d_{p_q}) be the corresponding metric space. Then (X,q) is complete iff (X,p_q) is complete iff (X,d_{p_q}) is complete. Moreover,*

$$\begin{aligned}
\lim_{n\to\infty} dp_q(x,x_n)=0 \Leftrightarrow p_q(x,x) &= \lim_{n\to\infty} p_q(x,x_n) = \lim_{n,m\to\infty} p_q(x_n,x_m) \\
\Leftrightarrow \ q(x,x) &= \lim_{n\to\infty} q(x,x_n) = \lim_{n,m\to\infty} q(x_n,x_m) \\
&= \lim_{n\to\infty} q(x_n,x) = \lim_{n,m\to\infty} q(x_m,x_n)\,.
\end{aligned}$$

The concept of coupled fixed points for a metric space was introduced by Bhaskar and Lakshmikantham [2]. Later, the notion of a coupled coincidence point of mappings on a metric space was given by Lakshmikantham and Ćirić [7].

Definition 2.6. [2] Let X be a nonempty set. An element $(x,y)\in X\times X$, is a *coupled fixed point* of the mapping $F:X\times X\to X$ if $F(x,y)=x$ and $F(y,x)=y$.

Definition 2.7. [7] An element (x,y) in $X\times X$ is called a *coupled coincidence point* of the mapping $F:X\times X\to X$ and $g:X\to X$ if $F(x,y)=gx$ and $F(y,x)=gy$.

The concept of w-compatible mappings was given by Abbas et al. [1] which is defined as:

Definition 2.8. [1] Let X be a nonempty set. The mapping $F:X\times X\to X$ and $g:X\to X$ are *w-compatible* if $gF(x,y)=F(gx,gy)$ whenever

$$gx=F(x,y) \quad \text{and} \quad gy=F(y,x).$$

3. Quasi-partial b-metric spaces

In [4], we introduced the concept of quasi-partial b-metric space and proved fixed point theorem on it.

Definition 3.1. [4] A *quasi-partial b-metric* on a non-empty set X is a mapping $qp_b : X \times X \to \mathbb{R}^+$ such that for some real number $s \geq 1$ and all $x,y,z \in X$.

($\mathrm{QP_{b_1}}$) $qp_b(x,x) = qp_b(x,y) = qp_b(y,y) \Rightarrow x = y$,

($\mathrm{QP_{b_2}}$) $qp_b(x,x) \leq qp_b(x,y)$,

($\mathrm{QP_{b_3}}$) $qp_b(x,x) \leq qp_b(y,x)$, and

($\mathrm{QP_{b_4}}$) $qp_b(x,y) \leq s[qp_b(x,z) + qp_b(z,y)] - qp_b(z,z)$.

A *quasi-partial b-metric space* is a pair (X,qp_b) such that X is a non-empty set and (X,qp_b) is a quasi-partial b-metric on X. The number s is called the coefficient of (X,qp_b).

Let qp_b be a quasi-partial b-metric on the set X. Then $d_{qp_b}(x,y) = qp_b(x,y) + qp_b(y,x) - qp_b(x,x) - qp_b(y,y)$ is a b-metric on X.

Lemma 3.2. [4] *Every quasi-partial metric space is a quasi-partial b-metric space. But the converse may not be true.*

Example 3.3. Let $X = [0,1]$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} (x+y)^2, & x < y \\ 2, & x > y \\ 0, & x = y. \end{cases}$$

First we prove condition (1) of the definition.

Let $qp_b(x,x) = qp_b(x,y) = qp_b(y,y)$. we claim $x = y$.

If $x \neq y$, then we have two cases.

Case 1: $x < y$.

Then $qp_b(x,x) = 0$, $qp_b(x,y) = (x+y)^2$ and $qp_b(y,y) = 0$.

Then the above condition reduces to $0 = (x+y)^2 = 0$.

Since $x,y \geq 0$ therefore $x = y = 0$ which is a contradiction to $x < y$.

Case 2: $x > y$.

Then the above condition reduces to $0 = 2 = 0$ which is absurd.

Hence we must have $x = y$.

Next we prove condition (2) of the definition i.e., $qp_b(x,x) \leq qp_b(x,y)$ for all $x,y \in X$.

Case 1: $x < y$.

$$qp_b(x,x) = 0 \leq (x+y)^2 = qp_b(x,y).$$

Case 2: $x > y$.

$$qp_b(x,x) = 0 < 2 = qp_b(x,y).$$

Case 3: $x = y$.

$$qp_b(x,x) = 0 = qp_b(x,y).$$

Similarly condition (3) of the definition holds.

Finally, we prove condition (4) of definition with $s = 2$. i.e.

$$qp_b(x,y) \leq 2[qp_b(x,z) + qp_b(z,y)] - qp_b(z,z)$$

$$2[qp_b(x,z) + qp_b(z,y)] - qp_b(z,z) - qp_b(x,y) \geq 0.$$

The following cases and subcases arise.

Case 1: $x < y$.

Subcase (i): $z < x < y$.

The above expression reduces to

$$\begin{aligned} 2[2 + (z+y)^2] - 0 - (x+y)^2 &= 4 + 2z^2 + 2y^2 + 4zy - x^2 - y^2 - 2xy \\ &= 4 + (z-x)[z+x+2y] + z^2 + y^2 + 2zy. \end{aligned}$$

Since $x, y, z \in [0,1]$, one has

$$-1 \leq z - x \leq 1 \quad \text{and} \quad 0 \leq z + x + 2y \leq 3\,.$$

Combining the two, we get

$$0 \leq 4 - (z+x+2y) \leq 4 + (z-x)(z+x+2y) \leq 4 + (z+x+2y)\,.$$

Hence

$$\begin{aligned} & 4 + (z-x)(z+x+2y) \geq 0 \\ \Rightarrow \qquad & 4 + (z-x)[z+x+2y] + z^2 + y^2 + 2zy \geq 0\,. \end{aligned}$$

Subcase (ii): $x < z < y$.

The above expression reduces to

$$2[(x+z)^2+(z+y)^2]-0-(x+y)^2 = 2x^2+2z^2+4xz+2z^2+2y^2+4zy-x^2-y^2-2xy$$
$$= (x+2z)^2+y^2+2zy+2y(z-x) \geq 0 \quad \text{since } x<z.$$

Subcase (iii): $x<y<z$.

The above expression reduces to

$$2[(x+z)^2+2]-0-(x+y)^2 = 2(x+z)^2+4-(x+y)^2 \geq 0 \quad \text{since } y<z.$$

Case 2: If $x>y$.

Subcase (i): $z<y<x$.

The above expression reduces to $2[2+(z+y)^2]-0-2 = 2+2(z+y)^2 \geq 0$.

Subcase (ii): $y<z<x$.

The above expression reduces to $2[2+2]-0-2 = 6 \geq 0$.

Subcase (iii): $y<x<z$.

The above expression reduces to $2[(x+z)^2+2]-0-2 = 2(x+z)^2+2 \geq 0$.

Hence all the conditions of definition of quasi-partial b-metric space are satisfied. So, (X,σ) is a quasi-partial b-metric space with coefficient $s=2$.

Definition 3.4. [4] Let (X,qp_b) be a quasi-partial b-metric. Then

(i) a sequence $\{x_n\} \subset X$ *converges* to $x \in X$ if and only if

$$qp_b(x,x) = \lim_{n\to\infty} qp_b(x,x_n) = \lim_{n\to\infty} qp_b(x_n,x).$$

(ii) a sequence $\{x_n\} \subset X$ is called a *Cauchy sequence* if and only if

$$\lim_{n,m\to\infty} qp_b(x_n,x_m) \quad \text{and} \quad \lim_{n,m\to\infty} qp_b(x_m,x_n) \text{ exist (and are finite).}$$

(iii) the quasi-partial b-metric space (X,qp_b) is said to be *complete* if every Cauchy sequence $\{x_n\} \subset X$ converges with respect to τ_{qp_b} to a point $x \in X$ such that

$$qp_b(x,x) = \lim_{n,m\to\infty} qp_b(x_m,x_n) = \lim_{n,m\to\infty} qp_b(x_n,x_m).$$

Lemma 3.5. [4] *Let (X,qp_b) be a quasi-partial b-metric space. Then the following hold.*

(A) *If $qp_b(x,y)=0$, then $x=y$.*

(B) *If $x \neq y$, then $qp_b(x,y) > 0$ and $qp_b(y,x) > 0$.*

Proof. It is similar as for the case of quasi-partial metric space [6].

Shatanawi [9] studied coupled fixed point theorems on quasi-partial metric space. Motivated by this we have studied coupled fixed theorem on quasi-partial b-metric space [5]. Here we prove a different version of coupled fixed point theorem on this space.

4. Main results

Theorem 4.1. *Let (X,qp_b) be a complete quasi-partial b-metric space and let $F : X \times X \to X$, $g : X \to X$ be two mappings. Suppose that there exists a function $\phi : gX \to \mathbb{R}^+$ such that $qp_b(gx,F(x,y)) + qp_b(gy,F(y,x)) \leq \phi(gx) + \phi(gy) - \phi(F(x,y)) - \phi(F(y,x))$ holds for all $(x,y) \in X \times X$. Also, assume that the following hypotheses are satisfied.*

(a) *$F(X \times X) \subseteq g(X)$;*

(b) *if $G : X \times X \to \mathbb{R}$, $G(x,y) = qp_b(F(x,y),gx)$, then for each sequence $(gx_n, gy_n) \to (u,v)$ we have $G(u,v) \leq k \liminf_{n\to\infty} G(x_n,y_n)$ for some $k > 0$. Then F and g have a coupled coincidence point (u,v). In addition, $qp_b(gu,gu) = 0$ and $qp_b(gv,gv) = 0$.*

Proof. Consider $(x_0,y_0) \in X \times X$. As $F(X \times X) \subseteq g(X)$, there are x_1 and y_1 from X such that $gx_1 = F(x_0,y_0)$ and $gy_1 = F(y_0,x_0)$. Again, since $F(X \times X) \subseteq g(X)$, there are x_2 and y_2 from X such that $gx_2 = F(x_1,y_1)$ and $gy_2 = F(y_1,x_1)$. By repeating this process, we construct two sequences, $\{x_n\}$ and $\{y_n\}$ with $gx_{n+1} = F(x_n,y_n)$ and $gy_{n+1} = F(y_n,x_n)$. Let m and n be natural numbers with $m > n$, then using (QPb_4), we get

$$
\begin{aligned}
qp_b(gx_n,gx_{n+2}) &\leq s\{qp_b(gx_n,gx_{n+1}) + qp_b(gx_{n+1},gx_{n+2})\} - qp_b(gx_{n+1},gx_{n+1}) \\
&\leq s\{qp_b(gx_n,gx_{n+1}) + qp_b(gx_{n+1},gx_{n+2})\}. \\
qp_b(gx_n,gx_{n+3}) &\leq s\{qp_b(gx_n,gx_{n+2}) + qp_b(gx_{n+2},gx_{n+3})\} - qp_b(gx_{n+2},gx_{n+2}) \\
&\leq s^2 qp_b(gx_n,gx_{n+1}) + s^2 qp_b(gx_{n+1},gx_{n+2})) + s qp_b(gx_{n+2},gx_{n+3}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
qp_b(gx_n, gx_m) &\le s^{m-n-1}\{qp_b(gx_n, gx_{n+1}) + qp_b(gx_{n+1}, gx_{n+2})\} \\
&\quad + s^{m-n-2}\{qp_b(gx_{n+2}, gx_{n+3})\} + \cdots + s\{qp_b(gx_{m-1}, gx_m)\} \\
&= \sum_{i=n+1}^{m-1} s^{m-i}\{qp_b(gx_i, gx_{i+1})\} + s^{m-n-1}\{qp_b(gx_n, gx_{n+1})\} \\
&= \sum_{i=n}^{m-1} s^{m-i}\{qp_b(gx_i, gx_{i+1})\} + s^{m-n-1}\{qp_b(gx_n, gx_{n+1})\} - s^{m-n}\{qp_b(gx_n, gx_{n+1})\} \\
&= \sum_{i=n}^{m-1} s^{m-i}\{qp_b(gx_i, gx_{i+1})\} - s^{m-n}\{qp_b(gx_n, gx_{n+1})\}\left(1 - \frac{1}{s}\right) \\
&\le \sum_{i=n}^{m-1} s^{m-i}\{qp_b(gx_i, gx_{i+1})\}.
\end{aligned} \tag{4.1}
$$

Similarly,

$$
qp_b(gy_n, gy_m) \le \sum_{i=n}^{m-1} s^{m-1}\{qp_b(gy_i, gy_{i+1})\}. \tag{4.2}
$$

Adding (4.1) and (4.2), we get

$$
\begin{aligned}
qp_b(gx_n, gx_m) + qp_b(gy_n, gy_m) &\le \sum_{i=n}^{m-1} s^{m-i}\{qp_b(gx_i, gx_{i+1}) + qp_b(gy_i, gy_{i+1})\} \\
&= \sum_{i=n}^{m-1} s^{m-i}\{qp_b(gx_i, F(x_i, y_i)) + qp_b(gy_i, F(y_i, x_i))\} \\
&= \sum_{i=n}^{m-1} s^{m-i}\{\phi(gx_i) + \phi(gy_i) - \phi(F(x_i, y_i)) - \phi(F(y_i, x_i))\}
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
&= s^{m-n}\{\phi(gx_n) + \phi(gy_n) - \phi(gx_{n+1}) - \phi(gy_{n+1})\} \\
&\quad + s^{m-n-1}\{\phi(gx_{n+1}) + \phi(gy_{n+1}) - \phi(gx_{n+2}) - \phi(gy_{n+2})\} + \cdots \\
&\quad + s\{\phi(gx_{m-1}) + \phi(gy_{m-1}) - \phi(gx_m) - \phi(gy_m)\} \\
&\le s^{m-n}\phi(gx_n) + s^{m-n}\phi(gy_n) - s^{m-n-1}\phi(gx_{n+1})(s-1) \\
&\quad - s^{m-n-1}\phi(gy_{n+1})(s-1) - \cdots - s\phi(gx_m) - s\phi(gy_m)
\end{aligned}
$$

$$
qp_b(gx_n, gx_m) + qp_b(gy_n, gy_m) \le s^{m-n}[\phi(gx_n) + \phi(gy_n)] - s[\phi(gx_m) + \phi(gy_m)]. \tag{4.4}
$$

Consider $z_n(x) = \sum_{i=0}^{n} [qp_b(gx_i, gx_{i+1}) + qp_b(gy_i, gy_{i+1})]$. Inequality (4.4) implies that

$$\begin{aligned} z_n(x) &\leq \sum_{i=0}^{n} s\{\phi(gx_i) + \phi(gy_i) - \phi(gx_{i+1}) - \phi(gy_{i+1})\} \\ &\leq s\{\phi(gx_0) + \phi(gy_0) - \phi(gx_{n+1}) - \phi(gy_{n+1})\} \\ &\leq s\{\phi(gx_0) + \phi(gy_0)\}. \end{aligned}$$

Hence the non-decreasing sequence $\{z_n\}$ is bounded, so it is convergent. Taking the limit as $n,m \to +\infty$ in (4.3), we conclude

$$\lim_{n,m\to\infty} qp_b(gx_n, gx_m) = \lim_{n,m\to\infty} qp_b(gy_n, gy_m) = 0\,.$$

Using similar arguments, it can be proved that

$$\lim_{n,m\to\infty} qp_b(gx_m, gx_n) = \lim_{n,m\to\infty} qp_b(gy_m, gy_n) = 0\,.$$

As $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the complete quasi-partial b-metric space (X, qp_b), there are u, v in X such that $u = \lim_{n\to\infty} gx_n$ and $v = \lim_{n\to\infty} gy_n$.

Now considering hypotheses (b), the following relations hold true:

$$\begin{aligned} 0 &\leq qp_b(F(u,v), gu) \\ &= G(u,v) \\ &\leq k \liminf_{n\to\infty} G(x_n, y_n) \\ &= k \liminf_{n\to\infty} qp_b(F(x_n,y_n), gx_n) \\ &= k \liminf_{n\to\infty} qp_b(gx_{n+1}, gx_n) = 0\,. \end{aligned}$$

We get $qp_b(F(u,v), gu) = 0$ and by Lemma 3.5, it follows that $F(u,v) = gu$. Similarly, it can be proved that $F(v,u) = gv$. To conclude, (u,v) is a coupled coincidence point of the mappings F and g, and $qp_b(gu,gu) = 0$ and $qp_b(gv,gv) = 0$.

Corollary 4.2. *Let (X, qp_b) be a complete quasi-partial b-metric space and let $F : X \times X \to X$ be a mapping. Suppose that there exists a function $\phi : X \to \mathbb{R}^+$ such that*

$$qp_b(x, F(x,y)) + qp_b(y, F(y,x)) \leq \phi(x) + \phi(y) - \phi(F(x,y)) - \phi(F(y,x))$$

holds for all $(x,y) \in X \times X$. Also assume that the following hypotheses are satisfied:

(i) $F(X \times X) \subseteq X$;

(ii) *if* $G : X \times X \to \mathbb{R}$, $G(x,y) = qp_b(F(x,y),x)$, *then for each sequence* $(x_n,y_n) \to (u,v)$, *we have* $G(u,v) \leq k \liminf_{n\to\infty} G(x_n,y_n)$ *for some* $k > 0$.

Then F *has a coupled coincidence point* (u,v). *In addition,* $qp_b(u,v) = 0$ *and* $qp_b(v,u) = 0$.

Proof. If follows from Theorem 4.1 by taking $g = I_X$ (the identity mapping).

Example 4.3. Let $X = [0,+\infty)$. Define

$$qp_b : X \times X \to \mathbb{R}^+, \quad qp_b(x,y) = |x-y| + y.$$

Also, define

$$F : X \times X \to X,\ F(x,y) = 2x,\ g : X \to X,\ gx = 4x,\ \phi : X \to \mathbb{R}^+,\ \phi(x) = 2x.$$

Then

(i) (X, qp_b) is a complete quasi-partial b-metric space.

(ii) $F(X \times X) \subseteq g(X)$.

(iii) For any $x, y \in X$, we have

$$qp_b(gx, F(x,y)) + qp_b(gy, F(y,x)) \leq \phi(gx) + \phi(gy) - \phi(F(x,y)) - \phi(F(y,x)).$$

(iv) Let $G : X \times X \to \mathbb{R}^+$ be defined by $G(x,y) = qp_b(F(x,y), gx)$. If (gx_n) and (gy_n) are two sequences in X with $(gx_n, gy_n) \to (u,v)$, then

$$G(u,v) \leq 4 \liminf_{n\to\infty} G(x_n,y_n).$$

Proof. To verify (i) we proceed by observing that $qp_b(x,y) = |x-y| + y$ is a quasi-partial b-metric with $s = 1$. Hence a quasi-partial metric. By Lemma 2.5, (X, qp_b) is complete if and only if $(X, d_{p_{qp_b}})$ is complete.

Here,

$$p_{qp_b}(x,y) = \frac{1}{2}[qp_b(x,y)+qp_b(y,x)]$$
$$= |x-y| + \frac{x+y}{2}.$$
$$d_{p_{qp_b}}(x,y) = 2p_{qp_b}(x,y) - p_{qp_b}(x,x) - p_{qp_b}(y,y)$$
$$= 2|x-y| + x + y - x - y$$
$$= 2|x-y|.$$

Clearly, $(X, d_{p_{qp_b}})$ is a complete metric space being a compact space.

Now, we verify (ii).

Let $F(x,y)$ be an arbitrary element of $F(X\times X)$. We need to show

$$F(x,y) \in g(X) = \{g(x) : x \in X\}$$
$$= \{4x : x \in [0,\infty)\}$$
$$= [0,\infty).$$
$$F(x,y) = 2x \in [0,\infty) = g(X).$$

Hence, $F(X\times X) \subseteq g(X)$.

To verify (iii), given $x,y \in X$, $gx = 4x$, $gy = 4y$, $F(x,y) = 2x$, $F(y,x) = 2y$, $\phi(x) = 2x$ and $\phi(y) = 2y$. Thus

$$qp_b(gx,F(x,y)) + qp_b(gy,F(y,x)) = qp_b(4x,2x) + qp_b(4y,2y)$$
$$= 4x + 4y$$
$$= 8x + 8y - 4x - 4y$$

$$= \phi(4x) + \phi(4y) - \phi(2x) - \phi(2y)$$
$$= \phi(gx) + \phi(gy) - \phi(F(x,y)) - \phi(F(y,x)).$$

To verify (iv), let $g(x_n)$ and $g(y_n)$ be two sequences in X such that $(gx_n, gy_n) \to (u,v)$ for some $u,v \in X$. Then $gx_n \to u$ and $gy_n \to v$. Thus,

$$qp_b(gx_n,u) = qp_b(4x_n,u) \to qp_b(u,u)$$

and

$$qp_b(u,gx_n) = qp_b(u,4x_n) \to qp_b(u,u).$$

Therefore, $|4x_n - u| + u \to u$ and $|u - 4x_n| + 4x_n \to u$ Hence $|4x_n - u| \to 0$. It follows that $x_n \to \frac{1}{4}u$ in $\mathbb{R}^+$. Now

$$\begin{aligned}
G(u,v) &= qp_b(F(u,v),u) \\
&= qp_b(2u,u) \\
&\leq 8\left(\frac{1}{4}u\right) \\
&= 8\liminf_{n\to\infty}(x_n) \\
&= 8\liminf_{n\to\infty} G(x_n,x_n) \\
&= 8\liminf_{n\to\infty} G\left(\frac{1}{2}F(x_n,y_n),x_n\right) \\
&= 4\liminf_{n\to\infty} G(F(x_n,y_n),x_n).
\end{aligned}$$

So F and g satisfy all the hypotheses of Theorem 4.1.

Hence, F and g have a coupled coincidence point. Here $(0,0)$ is the coupled coincidence point of F and g.

Competing Interests

The authors declare that they have no competing interests.

References

[1] M. Abbas, M.A. Khan, S. Radenović, Common coupled fixed point theorems in cone-metric spaces for *w*-compatible mapping, Appl. Math. Comput. 217 (2010), 195-202.

[2] T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006), 1379-1393.

[3] S. Czerwik, Contraction mappings in *b*-metric spaces, Acta Math. Inform. Univ. Ostrav 1 (1993), 5-11.

[4] A. Gupta, P. Gautam, Quasi-partial *b*-metric spaces and some related fixed point theorems, Fixed Point Theory Appl. 2015 (2015), Article ID 18.

[5] A. Gupta, P. Gautam, Some coupled fixed point theorems on quasi-partial *b*-metric spaces, Int. J. Math. Anal. 9 (2015), 293-306.

[6] E. Karapinar, I. Erhan, A. Özürk, Fixed point thoerem on quasi-partial metric spaces, Math. Comput. Model, (2012).

[7] V. Lakshmikantham, L. Ćirić, Coupled fixed point theorems for non-linear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009), 4341-4349.

[8] S.G. Matthews, Partial metric topology, in: General Topology and its Applications, Proc. 8th Summer Conf., Queen's College 1992, Ann. N.Y. Acad. Sci., Vol. **728**, pp. 183–197 (1994).

[9] S. Pitea, Some coupled fixed point thoerems in quasi-partial metric spaces, Fixed Point Theory Appl. 2013 (2013), Article ID 153.

[10] S. Shukla, Partial *b*-metric spaces and fixed point theorems, Mediterr. J. Math. http://dx.doi.org/10.1007/S00009-013-0327-4.