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NEW UNIQUE COMMON FIXED POINTS FOR AN INFINITE FAMILY OF MAPPINGS WITH ϕ-CONTRACTIVE OR ψ-φ-CONTRACTIVE CONDITIONS ON 2-METRIC SPACES

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Abstract. In this paper, Some new unique common fixed point results for an infinite family of self-mappings satisfying ϕ-contractive condition or ψ-φ-contractive condition on complete 2-metric spaces are obtained, in which the mappings satisfy some contractive condition determined by semi-continuous functions, but do not satisfy continuity and commutation. The main results generalize and improve many well-known and corresponding conclusions.

Keywords: 2-metric space; Common fixed point; ϕ-contractive condition; ψ-φ-contractive condition; Altering distance function.

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1. Introduction and preliminaries

There have appeared many common fixed point theorems of mappings with some contractive conditions on 2-metric spaces. But most of them held under subsidiary conditions [1-2], for examples; commutativity of mappings or uniform boundness of mappings at some point, and so on. In [3-9], the author obtained some generalized results for infinite or finite family of

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mappings satisfying generalized linear or non-linear contractive or quasi-contractive conditions and expansive conditions under removing the above subsidiary conditions.

In this paper, using real continuous functions, we establish contractive conditions of an infinite family of self-mappings on 2-metric spaces, and discuss the existence problems of common fixed points for the given mappings and obtain unique common fixed point theorems.

Definition 1.1. [2-5] A 2-metric space (X,d) consists of a nonempty set X and a function $d : X \times X \times X \to [0,+\infty)$ such that

(i) for distant elements $x,y \in X$, there exists an $u \in X$ such that $d(x,y,u) \neq 0$;

(ii) $d(x,y,z) = 0$ if and only if at least two elements in $\{x,y,z\}$ are equal;

(iii) $d(x,y,z) = d(u,v,w)$, where $\{u,v,w\}$ is any permutation of $\{x,y,z\}$;

(iv) $d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all $x,y,z,u \in X$.

Definition 1.2. [2-5] A sequence $\{x_n\}_{n\in\mathbb{N}}$ in 2-metric space (X,d) is said to be Cauchy, if for each $\varepsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n,x_m,a) < \varepsilon$ for all $a \in X$ and $n,m > N$.

Definition 1.3. [2-5] A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to $x \in X$, if for each $a \in X$, $\lim_{n\to+\infty} d(x_n,x,a) = 0$. And we write that $x_n \to x$ and call x the limit of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 1.4. [2-5] A 2-metric space (X,d) is said to be complete, if every cauchy sequence in X is convergent.

Lemma 1.5. [10] *Let $\{x_n\}$ be a sequence in 2-metric space (X,d) such that $\lim_{n\to\infty} d(y_n,y_{n+1},a) = 0$ for all $a \in X$. If $\{x_n\}$ is not a Cauchy sequence, then there exist $a \in X$ and $\varepsilon > 0$ such that for each $i \in \mathbb{N}$ there exist $m(i),n(i) \in \mathbb{N}$ with $m(i),n(i) > i$ such that*

(i) $m(i) > n(i)$ and $n(i) \to \infty$ as $i \to \infty$;

(ii) $d(x_{m(i)},x_{n(i)},a) > \varepsilon$, but $d(x_{m(i)-1},x_{n(i)},a) \leq \varepsilon$.

Lemma 1.6. [6-8] *If a sequence $\{x_n\}$ in a 2-metric space (X,d) converges to $x \in X$. Then* $\lim_{n\to\infty} d(x_n,b,c) = d(x,b,c), \forall b,c \in X$.

2. Common fixed point theorems

Theorem 2.1. *Let (X,d) be a complete 2-metric space, $\{f_i\}_{i\in\mathbb{N}}$ a family of self mappings on X. Suppose that for each $i,j\in\mathbb{N}$ with $i\neq j$ and $x,y,a\in X$,*

$$d(f_ix,f_jy,a)\leq\phi\big(\max\{d(x,y,a),d(x,f_ix,a),d(y,f_jy,a),d(x,f_jy,a),d(y,f_ix,a)\}\big),\qquad(2.1)$$

where $\phi:[0,\infty)\to[0,\infty)$ is a upper semi-continuous and non-decreasing real function satisfying $\phi(t)<\frac{t}{2}$ for all $t>0$. Then $\{f_i\}_{i\in\mathbb{N}}$ have a unique common fixed point.

Proof. Take an $x_0\in X$. We construct a sequence $\{x_n\}$ as follows $x_{n+1}=f_{n+1}x_n,\ n=0,1,2,\cdots$.

For fixed n, by (2.1), for any $a\in X$,

$$\begin{aligned}&d(x_{n+1},x_{n+2},a)\\=&d(f_{n+1}x_n,f_{n+2}x_{n+1},a)\\\leq&\phi\big(\max\{d(x_n,x_{n+1},a),d(x_n,x_{n+1},a),d(x_{n+1},x_{n+2},a),d(x_n,x_{n+2},a),0\}\big)\\\leq&\phi\big(\max\{d(x_n,x_{n+1},a),d(x_{n+1},x_{n+2},a),[d(x_n,x_{n+1},x_{n+2})+d(x_n,x_{n+1},a)+d(x_{n+1},x_{n+2},a)]\}\big)\\=&\phi\big(d(x_n,x_{n+1},x_{n+2})+d(x_n,x_{n+1},a)+d(x_{n+1},x_{n+2},a)\big).\end{aligned}\qquad(2.2)$$

Take $a=x_n$ in (2.2), then we obtain

$$d(x_{n+1},x_{n+2},x_n)\leq\phi\big(2d(x_n,x_{n+1},x_{n+2})\big).$$

If $d(x_{n+1},x_{n+2},x_n)>0$, then $d(x_{n+1},x_{n+2},x_n)<\frac{1}{2}2d(x_{n+1},x_{n+2},x_n)=d(x_{n+1},x_{n+2},x_n)$. This is a contradiction. Hence we have the following fact

$$d(x_{n+1},x_{n+2},x_n)=0,\ n=0,1,2,\cdots.\qquad(2.3)$$

Fix $k\in\mathbb{N}$ and suppose that $d(x_k,x_{n-1},x_n)=0$, where $n>k+2$. Then by (2.1) and (2.3),

$$\begin{aligned}&d(x_n,x_{n+1},x_k)\\=&d(f_nx_{n-1},f_{n+1}x_n,x_k)\\\leq&\phi\big(\max\{d(x_{n-1},x_n,x_k),d(x_{n-1},x_n,x_k),d(x_n,x_{n+1},x_k),d(x_{n-1},x_{n+1},x_k),0\}\big)\\\leq&\phi\big(d(x_{n-1},x_n,x_{n+1})+d(x_{n-1},x_n,x_k)+d(x_n,x_{n+1},x_k)\big)\\=&\phi(d(x_n,x_{n+1},x_k)).\end{aligned}$$

Hence using the property of ϕ, we obtain

$$d(x_n, x_{n+1}, x_k) = 0,$$

therefore, combining the above result with (2.3), we have

$$d(x_k, x_n, x_{n+1}) = 0, \forall n \geq k \geq 1. \tag{2.4}$$

For all $k > n > m$,

$$\begin{aligned}
&d(x_m, x_n, x_k)\\
\leq &d(x_m, x_n, x_{k-1}) + d(x_m, x_{k-1}, x_k) + d(x_n, x_{k-1}, x_k) = d(x_m, x_n, x_{k-1})\\
\leq &\cdots \leq d(x_m, x_n, x_{n+1}) = 0.
\end{aligned}$$

Hence, we have the following fact

$$d(x_m, x_n, x_k) = 0, \forall m, n, k \in \mathbb{N}. \tag{2.5}$$

From (2.2) and (2.3), we obtain

$$d(x_{n+1}, x_{n+2}, a) \leq \phi\left(d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+2}, a)\right), \forall\, n = 0, 1, 2, \cdots,\ a \in X. \tag{2.6}$$

If there exists $a \in X$ such that $d(x_n, x_{n+1}, a) < d(x_{n+1}, x_{n+2}, a)$, then

$$d(x_{n+1}, x_{n+2}, a) \leq \phi(2d(x_{n+1}, x_{n+2}, a)) < d(x_{n+1}, x_{n+2}, a),$$

which is a contradiction. Hence

$$d(x_{n+1}, x_{n+2}, a) \leq d(x_n, x_{n+1}, a), \forall\, n = 0, 1, 2 \cdots,\ a \in X. \tag{2.7}$$

So, for any fixed $a \in X$, $\{d(x_n, x_{n+1}, a)\}$ is a decreasing sequence, hence $\lim_{n\to\infty} d(x_n, x_{n+1}, a) = r(a) \geq 0$ for some $r(a) \in \mathbb{R}$. Suppose that $r(a) > 0$. Let $n \to \infty$, then from (2.6), we obtain

$$\begin{aligned}
r(a) = &\lim_{n\to\infty} d(x_{n+1}, x_{n+2}, a)\\
\leq &\limsup_{n\to\infty} \phi\left(d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+2}, a)\right)\\
\leq &\phi\left(\lim_{n\to\infty} [d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+2}, a)]\right)\\
= &\phi(2r(a)) < r(a),
\end{aligned}$$

this ia a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}, a) = 0, \forall\, a \in X. \tag{2.8}$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Otherwise, by Lemma 1.5, there exist $a \in X$ and $\varepsilon > 0$ such that for any $i \in \mathbb{N}$ there exist $m(i), n(i) \in \mathbb{N}$ with $m(i), n(i) > i$ satisfying

(i) $m(i) > n(i)$ and $n(i) \to \infty$ as $i \to \infty$;

(ii) $d(x_{m(i)}, x_{n(i)}, a) > \varepsilon$, but $d(x_{m(i)-1}, x_{n(i)}, a) \leq \varepsilon, i = 1, 2, \cdots$.

Using (2.5) and (2.8) and the following fact

$$d(x_{m(i)}, x_{n(i)}, a) \leq d(x_{m(i)}, x_{m(i)-1}, a) + d(x_{m(i)-1}, x_{n(i)}, a) + d(x_{m(i)}, x_{n(i)}, x_{m(i)-1}),$$

we obtain

$$\lim_{i\to\infty} d(x_{m(i)}, x_{n(i)}, a) = \lim_{i\to\infty} d(x_{m(i)-1}, x_{n(i)}, a) = \varepsilon. \tag{2.9}$$

The following two inequalities hold

$$|d(x_{m(i)}, x_{n(i)}, a) - d(x_{m(i)}, x_{n(i)-1}, a)| \leq d(x_{n(i)-1}, x_{n(i)}, a) + d(x_{m(i)}, x_{n(i)}, x_{n(i)-1}),$$

$$|d(x_{m(i)-1}, x_{n(i)-1}, a) - d(x_{m(i)}, x_{n(i)-1}, a)| \leq d(x_{m(i)-1}, x_{m(i)}, a) + d(x_{m(i)}, x_{m(i)-1}, x_{n(i)-1}),$$

hence using (2.5), (2.8) and (2.9), we obtain

$$\lim_{n\to\infty} d(x_{m(i)}, x_{n(i)}, a) = \lim_{n\to\infty} d(x_{m(i)-1}, x_{n(i)}, a) = \lim_{i\to\infty} d(x_{m(i)}, x_{n(i)-1}, a) = \lim_{i\to\infty} d(x_{m(i)-1}, x_{n(i)-1}, a) = \varepsilon. \tag{2.10}$$

Therefore by (2.1) and (2.10),

$$
\begin{aligned}
0 <& \varepsilon \\
=& \lim_{i\to\infty} d(x_{m(i)}, x_{n(i)}, a) \\
=& \lim_{i\to\infty} d(f_{m(i)}x_{m(i)-1}, f_{n(i)}x_{n(i)-1}, a) \\
\leq& \limsup_{i\to\infty} \phi\big(\max\{d(x_{m(i)-1}, x_{n(i)-1}, a), d(x_{m(i)-1}, x_{m(i)}, a), d(x_{n(i)-1}, x_{n(i)}, a), \\
& \qquad d(x_{m(i)-1}, x_{n(i)}, a), d(x_{n(i)-1}, x_{m(i)}, a)\}\big) \\
\leq& \phi\big(\lim_{i\to\infty} \max\{d(x_{m(i)-1}, x_{n(i)-1}, a), d(x_{m(i)-1}, x_{m(i)}, a), d(x_{n(i)-1}, x_{n(i)}, a), \\
& \qquad d(x_{m(i)-1}, x_{n(i)}, a), d(x_{n(i)-1}, x_{m(i)}, a)\}\big) \\
=& \phi\big(\max\{\varepsilon, 0, 0, \varepsilon, \varepsilon\}\big) \\
<& \frac{\varepsilon}{2},
\end{aligned}
$$

which is a contradiction. Hence $\{x_n\}$ is Cauchy, and there is $u \in X$ such that $x_n \to u$ as $n \to \infty$ by the completeness of X. For each fixed $n \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $i > n$. By (2.1),

$$
\begin{aligned}
& d(f_n u, u, a) \\
\leq& d(f_n u, x_{i+1}, a) + d(f_n u, u, x_{i+1}) + d(x_{i+1}, u, a) \\
=& d(f_n u, f_{i+1}x_i, a) + d(f_n u, u, x_{i+1}) + d(x_{i+1}, u, a) \\
\leq& \phi\big(\max\{d(u, x_i, a), d(u, f_n u, a), d(x_i, x_{i+1}, a), d(u, x_{i+1}, a), d(f_n u, x_i, a)\}\big) \\
& + d(f_n u, u, x_{i+1}) + d(x_{i+1}, u, a).
\end{aligned}
$$

Let $i \to \infty$, then by Lemma 1.6, the above deduces to

$$
\begin{aligned}
& d(f_n u, u, a) \\
\leq& \limsup_{i\to\infty} \phi\big(\max\{d(u, x_i, a), d(u, f_n u, a), d(x_i, x_{i+1}, a), d(u, x_{i+1}, a), d(f_n u, x_i, a)\}\big) \\
\leq& \phi\big(\lim_{i\to\infty} \max\{d(u, x_i, a), d(u, f_n u, a), d(x_i, x_{i+1}, a), d(u, x_{i+1}, a), d(f_n u, x_i, a)\}\big) \\
=& \phi(d(f_n u, u, a)), \forall\, a \in X,
\end{aligned}
$$

which implies that

$$d(f_n u, u, a) = 0, \forall\, a \in X,$$

hence

$$f_n u = u, \forall n \in \mathbb{N}.$$

Therefore u is a common fixed point of $\{f_i\}_{i\in\mathbb{N}}$. Suppose that $v \in X$ is another common fixed point of $\{f_i\}_{i\in\mathbb{N}}$, then thee exists $b \in X$ such that $d(u,v,b) > 0$, hence by (2.1),

$$\begin{aligned} d(u,v,b) =& d(f_1u, f_2v, b) \\ \leq& \phi\big(\max\{d(u,v,b), d(u, f_1u, b), d(v, f_2v, b), d(u, f_2v, b), d(f_1u, v, b)\}\big) \\ =& \phi(d(u,v,b)), \end{aligned}$$

hence by the property of ϕ,

$$0 < d(u,v,b) < \frac{d(u,v,b)}{2}.$$

This is a contradiction. Hence u is the unique common fixed point of $\{f_i\}_{i\in\mathbb{N}}$.

From Theorem 2.1, we obtain the following result.

Theorem 2.2. *Let (X,d) be a complete 2-metric space, $\{f_i\}_{i\in\mathbb{N}}$ a family of self mappings on X and $m_i \in \mathbb{N}$ for all $i \in \mathbb{N}$. Suppose that for each $i, j \in \mathbb{N}$ with $i \neq j$ and $x,y,a \in X$,*

$$d(f_i^{m_i}x, f_j^{m_j}y, a) \leq \phi\big(\max\{d(x,y,a), d(x, f_i^{m_i}x, a), d(y, f_j^{m_j}y, a), d(x, f_j^{m_j}y, a), d(y, f_i^{m_i}x, a)\}\big),$$

where ϕ is the function in Theorem 2.1. Then $\{f_i\}_{i\in\mathbb{N}}$ have a unique common fixed point.

Proof. Let $F_i = f_i^{m_i}$ for all $i \in \mathbb{N}$, then $\{F_i\}_{i\in\mathbb{N}}$ satisfy the all conditions of Theorem 2.1. Hence $\{F_i\}_{i\in\mathbb{N}}$ have a unique common fixed point $u \in X$. Fix any $i \in \mathbb{N}$. Since $F_i(f_i(u)) = f_i(F_i(u)) = f_i(u)$, so $f_i(u)$ is a fixed point of F_i. Fix any $j \in \mathbb{N}$ with $j \neq i$, then for any $a \in X$,

$$\begin{aligned} & d(f_i(u), F_j(f_i(u)), a) \\ =& d(F_i(f_i(u)), F_j(f_i(u)), a) \\ \leq& \phi\big(\max\{d(f_i(u), f_i(u), a), d(f_i(u), F_i(f_i(u)), a), d(f_i(u), F_j(f_i(u)), a), \\ & \qquad d(f_i(u), F_j(f_i(u)), a), d(f_i(u), F_i(f_i(u)), a)\}\big) \\ =& \phi\big(d(f_i(u), F_j(f_i(u)), a)\big). \end{aligned}$$

If $f_i(u) \neq F_j(f_i(u))$, then $d(f_i(u), F_j(f_i(u)), a) > 0$ for some $a \in X$, hence from the above formula,

$$d(f_i(u), F_j(f_i(u)), a) < \frac{d(f_i(u), F_j(f_i(u)), a)}{2},$$

which is a contradiction. Hence

$$F_j(f_i(u)) = f_i(u), \forall j \neq i.$$

That is, $f_i(u)$ is a common fixed point of $\{F_j\}_{j\in\mathbb{N}}$ for all $i \in \mathbb{N}$. So $f_i(u) = u$ for all $i \in \mathbb{N}$ by uniqueness of common fixed points of $\{F_j\}_{j\in\mathbb{N}}$, hence u is a common fixed point of $\{f_i\}_{i\in\mathbb{N}}$. If v is also common fixed point of $\{f_i\}_{i\in\mathbb{N}}$, then v is also a common fixed point of $\{F_i\}_{i\in\mathbb{N}}$, hence $u = v$ by the uniqueness. Therefore u is the unique common fixed point of $\{f_i\}_{i\in\mathbb{N}}$.

Now, we give more general result than Theorem 2.2.

Theorem 2.3. *Let (X,d) be a complete 2-metric space, $\{f_{i,k}\}_{i\in\mathbb{N}}$ a family of self mappings on X and $m_{i,k} \in \mathbb{N}$ for all $i,k \in \mathbb{N}$. Suppose that for each $i, j, k \in \mathbb{N}$ with $i \neq j$ and $x,y,a \in X$,*

$$d(f_{i,k}^{m_{i,k}}x, f_{j,k}^{m_{j,k}}y, a) \leq \phi_k\big(\max\{d(x,y,a), d(x, f_{i,k}^{m_{i,k}}x, a), d(y, f_{j,k}^{m_{j,k}}y, a), d(x, f_{j,k}^{m_{j,k}}y, a), d(y, f_{i,k}^{m_{i,k}}x, a)\}\big),$$

where $\phi_k : [0,\infty) \to [0,\infty)$ is a mapping satisfying the property of ϕ in Theorem 2.1. If $f_{i_1,j_1}f_{i_2,j_2} = f_{i_2,j_2}f_{i_1,j_1}$ for all $i_1, i_2, j_1, j_1 \in \mathbb{N}$ with $j_1 \neq j_2$, then $\{f_i\}_{i\in\mathbb{N}}$ have a unique common fixed point.

Proof. For any fixed $k \in \mathbb{N}$, $\{f_{i,k}\}_{i\in\mathbb{N}}$ have a unique common fixed point u_k by Theorem 2.2. Now, we will prove that $u_\mu = u_\nu$ for all $\mu, \nu \in \mathbb{N}$. In fact, for each $i, j, \mu, \nu \in \mathbb{N}$ with $\mu \neq \nu$, since $f_{i,\mu}(u_\mu) = u_\mu$ and $f_{j,\nu}(u_\nu) = u_\nu$. Hence $f_{i,\mu}(f_{j,\nu}(u_\nu)) = f_{i,\mu}(u_\nu)$, therefore $f_{j,\nu}(f_{i,\mu}(u_\nu)) = f_{i,\mu}(u_\nu)$, i.e., $f_{i,\mu}(u_\nu)$ is a common fixed point of $\{f_{j,\nu}\}_{j\in\mathbb{N}}$. So $f_{i,\mu}(u_\nu) = u_\nu$ for all $i \in \mathbb{N}$ by the uniqueness of common fixed point of $\{f_{j,\nu}\}_{j\in\mathbb{N}}$. This means that u_ν is a common fixed point of $\{f_{i,\mu}\}_{i\in\mathbb{N}}$, hence $u_\nu = u_\mu$ by the uniqueness of common fixed point of $\{f_{i,\mu}\}_{i\in\mathbb{N}}$. Let $u^* = u_\mu$, then obviously, u^* is the unique common fixed point of $\{f_{i,k}\}_{i,k\in\mathbb{N}}$.

A mapping $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if ψ is continuous and non-decreasing and $\psi(t) = 0 \Leftrightarrow t = 0$.

Next, we will give another common fixed point theorem under another contractive condition.

Theorem 2.4. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a family of self mappings on a complete 2-metric space (X,d) satisfying $f_i(X)\subset f_{i+1}(X)$ for all $n\in\mathbb{N}$. Suppose that for each $i,j,k\in\mathbb{N}$ with $i\neq j, i\neq k, j\neq k$ and $x,y,z,a\in X$,*

$$\psi(d(f_ix,f_jy,a))\leq\psi(d(f_jy,f_kz,a))-\varphi(d(f_jy,f_kz,a)), \tag{2.11}$$

where ψ is an altering distance function and $\varphi:[0,\infty)\to[0,\infty)$ is a lower semi-continuous function such that $\varphi(t)=0\Leftrightarrow t=0$. Then $\{f_i\}_{i\in\mathbb{N}}$ have a unique common fixed point.

Proof. Take any $x_0\in X$. By the condition $f_i(X)\subset f_{i+1}(X)$ for all $n=1,2,\cdots$, we construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows $f_nx_{n-1}=f_{n+1}x_n=y_n,\ \forall n=1,2,3,\cdots$.

Take $i=n+2, j=n+1, k=n, x=x_{n+1}, y=x_n, z=x_{n-1}$, then by (2.11), for any $a\in X$,

$$\psi(d(f_{n+2}x_{n+1},f_{n+1}x_n,a))\leq\psi(d(f_{n+1}x_n,f_nx_{n-1},a))-\varphi(d(f_{n+1}x_n,f_nx_{n-1},a)),$$

that is,

$$\psi(d(y_{n+1},y_n,a))\leq\psi(d(y_n,y_{n-1},a))-\varphi(d(y_n,y_{n-1},a))\leq\psi(d(y_n,y_{n-1},a)), \tag{2.12}$$

hence using the non-decreasing property of ψ, we obtain

$$d(y_{n+1},y_n,a)\leq d(y_n,y_{n-1},a),\forall a\in X, n=2,3,\cdots. \tag{2.13}$$

So for any fixed $a\in X$, $\{d(y_n,y_{n-1},a)\}$ is non-increasing, hence there is $r(a)\geq 0$ such that

$$\lim_{n\to\infty}d(y_n,y_{n-1},a)=r(a)$$

Let $n\to\infty$ in the both sides of the first inequality in (2.12), then

$$\psi(r(a))\leq\psi(r(a))-\liminf_{n\to\infty}\varphi(d(y_n,y_{n+1},a))\leq\psi(r(a))-\varphi(\lim_{n\to\infty}d(y_n,y_{n+1},a))=\psi(r(a))-\varphi(r(a)),$$

hence $\varphi(r(a))=0$, which implies that $r(a)=0$. Therefore

$$\lim_{n\to\infty}d(y_n,y_{n-1},a)=0,\forall a\in X. \tag{2.14}$$

Take $a=y_{n-1}$ in (2.12), then we obtain

$$\psi(d(y_{n+1},y_n,y_{n-1}))\leq\psi(d(y_n,y_{n-1},y_{n-1}))=\psi(0)=0,\forall n=1,2,\cdots,$$

hence

$$d(y_{n+2},y_{n+1},y_n)=0,\forall n=1,2,\cdots. \tag{2.15}$$

Fix any $\alpha\in\mathbb{N}$, then $d(y_\alpha,y_{\alpha+1},y_{\alpha+2})=0$ by (2.15). Suppose that $d(y_\alpha,y_n,y_{n+1})=0$, where $n>\alpha+1$. Take $i=n+3,j=n+2,k=n+1,x=x_{n+2},y=x_{n+1},z=x_n,a=y_\alpha$, then by (2.11),

$$\begin{aligned}\psi(d(y_{n+2},y_{n+1},y_\alpha))=&\psi(d(f_{n+3}x_{n+2},f_{n+2}x_{n+1},y_\alpha))\\ \le&\psi(d(f_{n+2}x_{n+1},f_{n+1}x_n,y_\alpha))-\varphi(d(f_{n+2}x_{n+1},f_{n+1}x_n,y_\alpha))\\ =&\psi(d(y_{n+1},y_n,y_\alpha))-\varphi(d(y_{n+1},y_n,y_\alpha))\\ =&\psi(0)-\varphi(0)=0.\end{aligned}$$

Hence using the property of ψ and (2.15), we have

$$d(y_\alpha,y_n,y_{n+1})=0,\ \forall n\ge\alpha\ge 1. \tag{2.16}$$

For all $k>n>m$, using (2.16), we obtain

$$\begin{aligned}&d(y_m,y_n,y_k)\\ \le&d(y_m,y_n,y_{k-1})+d(y_m,y_{k-1},y_k)+d(y_n,y_{k-1},y_k)=d(y_m,y_n,y_{k-1})\\ \le&\cdots\le d(y_m,y_n,y_{n+1})=0.\end{aligned}$$

Hence we have the following fact

$$d(y_m,y_n,y_k)=0,\forall m,n,k\in\mathbb{N}. \tag{2.17}$$

Suppose that $\{y_n\}$ is not a Cauchy sequence, then there exist $a\in X$ and $\varepsilon>0$ such that for any $i\in\mathbb{N}$ there exist $m(i),n(i)\in\mathbb{N}$ with $m(i),n(i)>i$ satisfying

(i) $m(i)>n(i)+1$ and $n(i)\to\infty$ as $i\to\infty$;

(ii) $d(y_{m(i)},y_{n(i)},a)>\varepsilon$, but $d(y_{m(i)-1},y_{n(i)},a)\le\varepsilon,i=1,2,\cdots$.

Using (2.14) and (2.17) and the following fact

$$d(y_{m(i)},y_{n(i)},a)\le d(y_{m(i)},y_{m(i)-1},a)+d(y_{m(i)-1},y_{n(i)},a)+d(y_{m(i)},y_{n(i)},y_{m(i)-1}),$$

we obtain

$$\lim_{i\to\infty}d(y_{m(i)},y_{n(i)},a)=\lim_{i\to\infty}d(y_{m(i)-1},y_{n(i)},a)=\varepsilon. \tag{2.18}$$

Since the following two inequalities hold

$$|d(y_{m(i)},y_{n(i)},a)-d(y_{m(i)},y_{n(i)-1},a)|\leq d(y_{n(i)-1},y_{n(i)},a)+d(y_{m(i)},y_{n(i)},y_{n(i)-1})$$

and

$$|d(y_{m(i)-1},y_{n(i)-1},a)-d(y_{m(i)},y_{n(i)-1},a)|\leq d(y_{m(i)-1},y_{m(i)},a)+d(y_{m(i)},y_{m(i)-1},y_{n(i)-1}),$$

so by (2.14), (2.17) and (2.18), for each $a\in X$,

$$\lim_{n\to\infty}d(y_{m(i)},y_{n(i)},a)=\lim_{n\to\infty}d(y_{m(i)-1},y_{n(i)},a)=\lim_{i\to\infty}d(y_{m(i)},y_{n(i)-1},a)=\lim_{i\to\infty}d(y_{m(i)-1},y_{n(i)-1},a)=\varepsilon. \tag{2.19}$$

Take $i=m(i)+1, j=n(i)+1, k=m(i), x=x_{m(i)}, y=x_{n(i)}, z=x_{m(i)-1}$, then by (2.11), for each $a\in X$,

$$\psi(d(f_{m(i)+1}x_{m(i)},f_{n(i)+1}x_{n(i)},a)\leq\psi(d(f_{n(i)+1}x_{n(i)},f_{m(i)}x_{m(i)-1},a))-\varphi(d(f_{n(i)+1}x_{n(i)},f_{m(i)}x_{m(i)-1},a)),$$

that is,

$$\psi(d(y_{m(i)},y_{n(i)},a)\leq\psi(d(y_{n(i)},y_{m(i)-1},a))-\varphi(d(y_{n(i)},y_{m(i)-1},a)).$$

Let $i\to\infty$, then by (2.19) and the above formula,

$$\psi(\varepsilon)\leq\psi(\varepsilon)-\liminf_{i\to\infty}\varphi(d(y_{n(i)},y_{m(i)-1},a))\leq\psi(\varepsilon)-\varphi(\lim_{i\to\infty}d(y_{n(i)},y_{m(i)-1},a))=\psi(\varepsilon)-\varphi(\varepsilon),$$

hence $\varphi(\varepsilon)=0$, which implies that $\varepsilon=0$. This is a contradiction, hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists $u\in X$ such that $y_n\to u$ as $n\to\infty$. Fix any $n\in\mathbb{N}$ and take $l\in\mathbb{N}$ satisfying $l>n+1$. Let $i=n, j=l+1, k=l, x=u, y=x_l, z=x_{l-1}$, then by (2.11),

$$\psi(d(f_nu,f_{l+1}x_l,a))\leq\psi(d(f_{l+1}x_l,f_lx_{l-1},a))-\varphi(d(f_{l+1}x_l,f_lx_{l-1},a)),\ \forall a\in X,$$

that is,

$$\psi(d(f_nu,y_l,a))\leq\psi(d(y_l,y_{l-1},a))-\varphi(d(y_l,y_{l-1},a)),\forall a\in X$$

Let $l\to\infty$, then the above formula deduces to

$$\psi(d(f_nu,u,a))\leq\psi(0))-\liminf_{l\to\infty}\varphi(d(y_l,y_{l-1},a))\leq\psi(0))-\varphi(\lim_{l\to\infty}d(y_l,y_{l-1},a))=\psi(0))-\varphi(0)=0.$$

Hence $f_n u = u$ for all $n = 1, 2, \cdots$, so u is a common fixed point of $\{f_i\}_{i\in\mathbb{N}}$. Suppose that v is also a common fixed point of $\{f_i\}_{i\in\mathbb{N}}$. Take $i = 1, j = 2, k = 3, x = u, y = z = v$, then by (2.11), for each $a \in X$,

$$\psi(d(f_1 u, f_2 v, a)) \leq \psi(d(f_2 v, f_3 v, a)) - \varphi(d(f_2 v, f_3 v, a)),$$

that is,

$$\psi(d(u, v, a)) \leq \psi(d(v, v, a)) - \varphi(d(v, v, a)) = \psi(0) - \varphi(0) = 0,$$

so $u = v$. Hence u is the unique common fixed point of $\{f_i\}_{i\in\mathbb{N}}$.

From Theorem 2.4, we obtain the following particular forms.

Theorem 2.5. *Let (X, d) be a complete 2-metric space, $\{f_i\}_{i\in\mathbb{N}}$ a family of self mappings on X satisfying $f_i(X) \subset f_{i+1}(X)$ for all $n = 1, 2, \cdots$. Suppose that for each $i, j, k \in \mathbb{N}$ with $i \neq j, i \neq k, j \neq k$ and $x, y, z, a \in X$,*

$$d(f_i x, f_j y, a) \leq d(f_j y, f_k z, a) - \varphi(d(f_j y, f_k z, a)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$. Then $\{f_i\}_{i\in\mathbb{N}}$ have a unique common fixed point.

Proof. Let $\psi = 1_X$, then the conclusion follows from Theorem 2.4.

Theorem 2.6. *Let (X, d) be a complete 2-metric space, $\{f_i\}_{i\in\mathbb{N}}$ a family of self mappings on X satisfying $f_i(X) \subset f_{i+1}(X)$ for all $n = 1, 2, \cdots$. Suppose that for each $i, j, k \in \mathbb{N}$ with $i \neq j, i \neq k, j \neq k$ and $x, y, z, a \in X$,*

$$d(f_i x, f_j y, a) \leq h\, d(f_j y, f_k z, a),$$

where $h \in [0, 1)$. Then $\{f_i\}_{i\in\mathbb{N}}$ have a unique common fixed point.

Proof. Let $\varphi(t) = (1 - h)t$ for all $t \in [0, \infty)$, then the conclusion follows from Theorem 2.5.

Conflict of Interests

The author declares that there is no conflict of interests.

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