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# A NOTE ON THE PAPER "COMMON FIXED POINT THEOREMS FOR THREE MAPS IN DISCONTINUOUS $G_b$-METRIC SPACES"

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**Abstract.** In this paper, fixed points three nonlinear operators are investigated. Common fixed point theorems are established in a complete $G_b$-metric space. The result presented in this paper improves the corresponding results in [1].

**Keywords:** $G$-metric space; $b$-metric space; Common fixed point.

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## 1. Main results

In [1], Roshan *et al.* obtained a common fixed point theorem in a complete $G_b$-metric space. After carefully reading the paper, the authors find that the proof of $d_{n+1} \leq d_n$ in Theorem 2.1 [1] turned out to be not comprehensive. They only proved $d_{3n+1} \leq d_{3n}$ and $d_{3n+2} \leq d_{3n+1}$ and declared that $d_{n+1} \leq d_n$, which has a skip.

Next, we give a new proof.

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**Theorem 1.1** Let $(X,G)$ be a complete $G_b$-metric space. Let $A$, $B$, $C$: $X \to X$ be mappings which satisfy the following condition:

$$\psi(2s^4G(Ax,By,Cz)) \leq \psi(M(x,y,z)) - \varphi(M(x,y,z)) \tag{1.1}$$

for all $x,y,z \in X$, where $\psi$, $\varphi$: $[0,\infty) \to [0,\infty)$ are two mappings such that $\psi$ is continuous nondecreasing, $\varphi$ is a lower semi-continuous function with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ and

$$M(x,y,z) = \max\{G(x,y,z), G(x,Ax,By), G(y,By,Cz), G(z,Cz,Ax)\}.$$

Then, either one of $A$, $B$, and $C$ has a fixed point, or, the maps $A$, $B$ and $C$ have a unique common fixed point.

**Proof.** Choose $x_0 \in X$. Define the sequence $\{x_n\}$ as $x_{3n+1} = Ax_{3n}$, $x_{3n+2} = Bx_{3n+1}$ and $x_{3n+3} = Cx_{3n+2}$ for all $n = 0,1,2,\ldots$. If $x_{3n} = x_{3n+1}$, then $x_{3n}$ is a fixed point of $A$. If $x_{3n+1} = x_{3n+2}$, then $x_{3n+1}$ is a fixed point of $B$. If $x_{3n+2} = x_{3n+3}$, then $x_{3n+2}$ is a fixed point of $C$. Now, assume that $x_n \neq x_{n+1}$ for all $n$. Let $d_n = G(x_n, x_{n+1}, x_{n+2})$. we obtain from (1.1) that

$$\begin{aligned}
\psi(d_{3n+1}) &\leq \psi(2s^4 d_{3n+1}) = \psi(2s^4 G(x_{3n+1}, x_{3n+2}, x_{3n+3})) \\
&= \psi(2s^4 G(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2})) \\
&\leq \psi(M(x_{3n}, x_{3n+1}, x_{3n+2})) - \varphi(M(x_{3n}, x_{3n+1}, x_{3n+2})),
\end{aligned}$$

where

$$\begin{aligned}
M(x_{3n}, x_{3n+1}, x_{3n+2}) &= \max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, Ax_{3n}, Bx_{3n+1}), \\
&\quad G(x_{3n+1}, Bx_{3n+1}, Cx_{3n+2}), G(x_{3n+2}, Cx_{3n+2}, Ax_{3n})\} \\
&= \max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n+1}, x_{3n+2}), \\
&\quad G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n+2}, x_{3n+3}, x_{3n+1})\} \\
&= \max\{d_{3n}, d_{3n}, d_{3n+1}, d_{3n+1}\} \\
&= \max\{d_{3n}, d_{3n+1}\}.
\end{aligned}$$

We prove that $d_{3n+1} \leq d_{3n}$ for each $n \in \mathbb{N}$. If $d_{3n+1} > d_{3n}$ for some $n \in \mathbb{N}$, then we have $\psi(d_{3n+1}) \leq \psi(d_{3n+1}) - \varphi(d_{3n+1})$, which implies that $d_{3n+1} = 0$, a contradiction to $d_{3n+1} > 0$. Also, we have

$$\begin{aligned}
\psi(d_{3n+2}) &\leq \psi(2s^4 d_{3n+2}) = \psi(2s^4 G(x_{3n+2}, x_{3n+3}, x_{3n+4})) \\
&= \psi(2s^4 G(Bx_{3n+1}, Cx_{3n+2}, Ax_{3n+3})) \\
&= \psi(2s^4 G(Ax_{3n+3}, Bx_{3n+1}, Cx_{3n+2})) \\
&\leq \psi(M(x_{3n+3}, x_{3n+1}, x_{3n+2})) - \varphi(M(x_{3n+3}, x_{3n+1}, x_{3n+2})),
\end{aligned}$$

where

$$\begin{aligned}
M(x_{3n+3}, x_{3n+1}, x_{3n+2}) &= \max\{G(x_{3n+3}, x_{3n+1}, x_{3n+2}), G(x_{3n+3}, Ax_{3n+3}, Bx_{3n+1}), \\
&\quad G(x_{3n+1}, Bx_{3n+1}, Cx_{3n+2}), G(x_{3n+2}, Cx_{3n+2}, Ax_{3n+3})\} \\
&= \max\{G(x_{3n+3}, x_{3n+1}, x_{3n+2}), G(x_{3n+3}, x_{3n+4}, x_{3n+2}), \\
&\quad G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n+2}, x_{3n+3}, x_{3n+4})\} \\
&= \max\{d_{3n+1}, d_{3n+2}, d_{3n+1}, d_{3n+2}\} \\
&= \max\{d_{3n+1}, d_{3n+2}\}.
\end{aligned}$$

Similarly, if $d_{3n+2} > d_{3n+1}$ for some $n \in \mathbb{N}$, then we have $\psi(d_{3n+2}) \leq \psi(d_{3n+2}) - \varphi(d_{3n+2})$, which implies that $d_{3n+2} = 0$, a contradiction to $d_{3n+2} > 0$. Also, we have

$$\begin{aligned}
\psi(d_{3n+3}) &\leq \psi(2s^4 d_{3n+3}) = \psi(2s^4 G(x_{3n+3}, x_{3n+4}, x_{3n+5})) \\
&= \psi(2s^4 G(Cx_{3n+2}, Ax_{3n+3}, Bx_{3n+4})) \\
&= \psi(2s^4 G(Ax_{3n+3}, Bx_{3n+4}, Cx_{3n+2})) \\
&\leq \psi(M(x_{3n+3}, x_{3n+4}, x_{3n+2})) - \varphi(M(x_{3n+3}, x_{3n+4}, x_{3n+2})),
\end{aligned}$$

where

$$\begin{aligned} M(x_{3n+3}, x_{3n+4}, x_{3n+2}) &= \max\{G(x_{3n+3}, x_{3n+4}, x_{3n+2}), G(x_{3n+3}, Ax_{3n+3}, Bx_{3n+4}), \\ &\quad G(x_{3n+4}, Bx_{3n+4}, Cx_{3n+2}), G(x_{3n+2}, Cx_{3n+2}, Ax_{3n+3})\} \\ &= \max\{G(x_{3n+3}, x_{3n+4}, x_{3n+2}), G(x_{3n+3}, x_{3n+4}, x_{3n+5}), \\ &\quad G(x_{3n+4}, x_{3n+5}, x_{3n+3}), G(x_{3n+2}, x_{3n+3}, x_{3n+4})\} \\ &= \max\{d_{3n+2}, d_{3n+3}, d_{3n+3}, d_{3n+2}\} \\ &= \max\{d_{3n+2}, d_{3n+3}\}. \end{aligned}$$

Similarly, if $d_{3n+3} > d_{3n+2}$ for some $n \in \mathbb{N}$, then we have $\psi(d_{3n+3}) \leq \psi(d_{3n+3}) - \varphi(d_{3n+3})$, which implies that $d_{3n+3} = 0$, a contradiction to $d_{3n+3} > 0$. Hence, we have $0 < d_{n+1} \leq d_n$ for each $n \in \mathbb{N}$. The rest proof process is the same with which was given in [1]. We, therefore, omit the proof.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

## References

[1] J.R. Roshan *et al.*, Common fixed point theorems for three maps in discontinuous $G_b$-metric spaces, Acta Mathe. Sci. 34 (2014), 1643-1654.