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COMMON FIXED POINT THEOREMS OF MULTIVALUED MAPPINGS

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Abstract: In the present paper, we prove the existence of the common fixed point for multivalued maps in cone metric spaces. Our results generalize some recent results in the literature.

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1. Introduction

Recently, Huang and Zhang [1] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently many authors have studied the strong convergence to a fixed point with contractive constant in Cone metric space, see [2],[8],[9],[10] and [11]. Seong Hoon Cho and Mi Sun Kim [6] have proved certain fixed point theorems by using multivalued mapping in the setting of contractive constant in metric spaces. In this paper, we obtain common fixed point for a pair of multivalued maps satisfying a generalized contractive type conditions in Cone metric spaces.

Let E be a Banach space and a subset P of E is said to be a cone if it satisfies that following conditions,

i) P is closed, non-empty and $P \neq \{0\}$;

ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b;

iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

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The partial ordering $\leq$ with respect to the cone P by $x \leq y$ if and only if $y - x \in P$. If $y - x \in$ interior of P, then it is denoted by $x \ll y$. The cone P is said to be Normal if a number $K > 0$ such that for all $x, y \in E, 0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ .The cone P is called regular of every increasing sequence which is bounded above is convergent and every decreasing sequence which is bounded below is convergent.

Definition 1.1. Let X be a non-empty set of E. Suppose that the map $d: X \times X \to E$ satisfies;

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called cone metric on X and (X, d) is called Cone metric space.

Example 1.2. Let $E = R^2, P = \{(x, y) \in E: x, y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \to E$ defined by

$$d(x, y) = (|x - y|, \propto |x - y|)$$

where $\propto \geq 0$ is a constant. Then (X, d) is a Cone metric space [1].

Definition 1.3. Let (X, d) be a Cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

Definition 1.4. Let (X, d) is said to be a complete cone metric space, if every Cauchy sequence is convergent in X.

Let (X, d) be a metric space. We denote by $CB(X)$ the family of nonempty closed bounded subset of X and let $C(X)$ denote the set of all nonempty compact subsets of X. Let $H(.,.)$ be the Hausdorff distance on $CB(X)$. That is, for $A, B \in CB(X)$,

$$H(A, B) = \quad max\left\{ Sup_{a \in A} d(a, B), Sup_{b \in B} d(A, b) \right\}$$

where $d(a, B) = \inf\{d(a, b): b \in B\}$ is the distance from the point a to the subset B.

Theorem 1.5[7]. A multivalued mapping $T: X \to CB(X)$ is called a contraction mapping if there exists $k \in (0,1)$ such that

$H(Tx, Ty) \leq kd(x, y) \forall x, y \in X$ and $x \in X$ is said to be a fixed point of T if $x \in T(X)$.

2. Fixed Point

In this section, we shall give some results which generalizes the result of [3], [4], [5], [7], [12] and [13].

Theorem 2.1. Let (X,d) be a complete cone metric space and let mapping $T_1,T_2{:}\,X \to C(X)$ satisfying the following conditions

(i) For each $x \in X,\ T_1(x),T_2(x) \in CB(X)$.

(ii) $H(T_1(x),T_2(y)) \leq \propto d(x,y) + \beta d(x,T_1(x)) + \gamma d(y,T_2(y))$

where $\propto,\beta,\gamma$ are non negative real numbers and $\propto + \beta + \gamma < 1$. $Then\ there\ exists\ p \in X\ \ such\ that\ p \in T_1(x) \cap T_2(x\,)$.

Proof. Let $x_o \in X,\ \ T_1(x_o)$ is a nonempty closed bounded subset of X. Choose $x_1 \in T_1(x_o)$, for this x_1 by the same reason mentioned above $T_2(x_1)$ is nonempty closed bounded subset of X. Since $x_1 \in T_1(x_o)$ and $T_1(x_o)$ and $T_2(x_1)$ are closed bounded subset of $X, \exists\, x_2 \in T_2(x_1)$ such that

$$d(x_{1,}x_2) \leq H\big(T_1(x_o),T_2(x_1)\big) + q$$

where $q = max\left\{\frac{\propto+\beta}{1-\gamma},\frac{\propto+\gamma}{1-\beta}\right\}$.

Hence $q \in (0,1)$. Then we have

$$\begin{aligned} d(x_{1,}x_2) &\leq H\big(T_1(x_o),T_2(x_1)\big) + q \\ &\leq \propto d(x_{0,}x_1) + \beta d\left(x_0,T_1(x_o))+\gamma d\big(x_{1,}T_2(x_1)\big)\right) + q \\ &\leq \propto d(x_{0,}x_1) + \beta d(x_0,x_1) + \gamma d(x_1,x_2) + q \end{aligned}$$

$$d(x_1,x_2) \leq \frac{\propto + \beta}{1-\gamma} d(x_o,x_1) + q$$

$$d(x_1,x_2) \leq q d(x_o,x_1) + q.$$

For this $x_{2,}\ T_1(x_2)$ is a nonempty closed bounded subset of X.Since $x_2 \in T_2(x_1)$ and $T_2(x_1)$ and $T_1(x_2)$ are closed bounded subset of $X, \exists\, x_3 \in T_1(x_2)$ such that

$$\begin{aligned} d(x_{2,}x_3) &\leq H\big(T_1(x_2),T_2(x_1)\big) + q^2 \\ &\leq \propto d(x_{2,}x_1) + \beta d\left(x_2,T_1(x_2))+\gamma d\big(x_{1,}T_2(x_1)\big)\right) + q^2 \\ &\leq \propto d(x_{1,}x_2) + \beta d(x_2,x_3) + \gamma d(x_1,x_2) + q^2 \end{aligned}$$

$$\begin{aligned} d(x_2,x_3) &\leq \frac{\propto + \gamma}{1-\beta} d(x_1,x_2) + q^2 \\ &\leq q\{q d(x_o,x_1) + q\} + q^2 \\ &\leq q^2 d(x_o,x_1) + 2q^2. \end{aligned}$$

Continuing this process, we get a sequence $\{x_n\}$ such that $x_{n+1} \in T_2(x_n)$ or $x_{n+1} \in T_1(x_n)$ and $d(x_{n+1,} x_n) \leq q^n d(x_{0,} x_1) + nq^n$.

Let $0 \ll c$ be given, choose a natural number N_1 such that $q^n d(x_{0,} x_1) + nq^n \ll c$ for all $n \geq N_1$ this implies $d(x_{n+1,} x_n) \ll c$.

$\{x_n\}$ is a Cauchy sequence in $\{X, d)$ is a complete cone metric space, there exists $p \in X$ such that $x_n \to p$.

Choose a natural number N_2 $such\ that$ $d(x_n, p) \ll \frac{c(1-\gamma)}{2m}$

and $d(x_{n-1}, p) \ll \frac{c(1-\gamma)}{2m\propto}$ for all $n \geq N_2$.

$$d(T_1(p), p) \leq d(p, x_n) + d(x_n, T_1(p))$$
$$\leq d(p, x_n) + H(T_2(x_{n-1}), T_1(p))$$
$$\leq d(p, x_n) + \propto d(x_{n-1}, p) + \beta d(x_{n-1}, T_2(x_{n-1}))$$
$$+\gamma d(p, T_1(p))$$
$$\leq d(p, x_n) + \propto d(x_{n-1}, p) + \beta d(x_{n-1}, x_n)$$
$$+\gamma d(p, T_1(p))$$
$$(1-\gamma) d(p, T_1(p)) \leq d(p, x_n) + \propto d(x_{n-1}, p)$$
$$+\beta d(x_{n-1}, x_n)$$
$$d(p, T_1(p)) \leq \frac{1}{1-\gamma} d(p, x_n) + \frac{\propto}{1-\gamma} d(p, x_{n-1})$$
$$+\frac{\beta}{1-\gamma} d(x_{n-1}, x_n) \qquad \text{for all } n \geq N_2.$$

$d(T_1(p), p) \ll \frac{c}{m}$ for all $m \geq 1$, we get

$\frac{c}{m} - d(T_1(p), p) \in P$ and as $m \to \infty$, we get $\frac{c}{m} \to 0$ and P is closed $-d(T_1(p), p) \in P$, but $d(T_1(p), p) \in P$. Therefore $d(T_1(p), p) = 0$ and so $p \in T_1(p)$.

Similarly, it can be established that $p \in T_2(p)$. Hence $p \in T_1(p) \cap T_2(p)$.

Theorem 2.2. Let (X, d) be a complete cone metric space and let mapping $T_1, T_2 \colon X \to C(X)$ satisfying the following conditions

(i) For each $x \in X, T_1(x), T_2(x) \in CB(X)$.

(ii) $H(T_1(x), T_2(y)) \leq \propto d(x, y) + \beta d(x, T_2(y)) + \gamma d(y, T_1(x))$

where $\propto, \beta, \gamma$ are non-negative real numbers and $\propto + \beta + \gamma < 1$. Then there exists $p \in X$ such that $p \in T_1(x) \cap T_2(x)$.

Proof. Let $x_0 \in X$, $T_1(x_0)$ is a nonempty closed bounded subset of X. Choose $x_1 \in T_1(x_o)$, for this x_1 by the same reason mentioned above $T_2(x_1)$ is nonempty closed bounded subset of X. Since $x_1 \in T_1(x_0)$ and $T_1(x_0)$ and $T_2(x_1)$ are closed bounded subset of $X, \exists\, x_2 \in T_2(x_1)$ such that

$$d(x_{1,}x_2) \le H\big(T_1(x_o), T_2(x_1)\big) + q$$

where $q = max\left\{\frac{\propto+\beta}{1-\beta}, \frac{\propto+\gamma}{1-\gamma}\right\}$.

Hence $q \in (0,1)$. Then we have

$$d(x_{1,}x_2) \le H\big(T_1(x_o), T_2(x_1)\big) + q$$

$$\le \propto d(x_{0,}x_1) + \beta d\left(x_0, T_2(x_1)) + \gamma d\big(x_{1,}T_1(x_0)\big)\right) + q$$

$$\le \propto d(x_{0,}x_1) + \beta d(x_0, x_2) + \gamma d(x_1, x_1) + q$$

$$\le \propto d(x_{0,}x_1) + \beta[d\,(x_0, x_1) + d(x_1, x_2)] + q$$

$$d(x_1, x_2) \le \frac{\propto +\beta}{1-\beta} d(x_o, x_1) + q$$

$$d(x_1, x_2) \le q d(x_o, x_1) + q.$$

For this $x_{2,}T_1(x_2)$ is a nonempty closed bounded subset of X.Since $x_2 \in T_2(x_1)$ and $T_2(x_1)$ and $T_1(x_2)$ are closed bounded subset of $X, \exists\, x_3 \in T_1(x_2)$ such that

$$d(x_{2,}x_3) \le H\big(T_1(x_2), T_2(x_1)\big) + q^2$$

$$\le \propto d(x_{2,}x_1) + \beta d\left(x_2, T_2(x_1)) + \gamma d\big(x_{1,}T_1(x_2)\big)\right) + q^2$$

$$\le \propto d(x_{2,}x_1) + \beta d(x_2, x_2) + \gamma d(x_1, x_3) + q^2$$

$$\le \propto d(x_{1,}x_2) + \gamma[d(x_1, x_2) + d(x_2, x_3)] + q^2$$

$$d(x_2, x_3) \le \frac{\propto +\gamma}{1-\gamma} d(x_1, x_2) + q^2$$

$$\le q\{q d(x_o, x_1) + q\} + q^2$$

$$\le q^2 d(x_o, x_1) + 2q^2.$$

Continuing this process, we get a sequence $\{x_n\}$ such that $x_{n+1} \in T_2(x_n)$ or $x_{n+1} \in T_1(x_n)$ and

$$d\big(x_{n+1,}x_n\big) \le q^n d\big(x_{0,}x_1\big) + nq^n.$$

Let $0 \ll c$ be given, choose a natural number N_1 such that $q^n d\big(x_{0,}x_1\big) + nq^n \ll c$ for all $n \ge N_1$ this implies $d\big(x_{n+1,}x_n\big) \ll c$.

$\{x_n\}$ is a Cauchy sequence in $\{X, d)$ is a complete cone metric space, there exists $p \in X$ such that $x_n \to p$.

Choose a natural number N_2 such that $d(x_n, p) \ll \frac{c(1-\beta)}{2m(1+\gamma)}$

and $d(x_{n-1}, p) \ll \frac{c(1-\beta)}{2m(\propto+\beta)}$ for all $n \geq N_2$.

$$d(T_1(p), p) \leq d(p, x_n) + d(x_n, T_1(p))$$
$$\leq d(p, x_n) + H(T_2(x_{n-1}), T_1(p))$$
$$\leq d(p, x_n) + \propto d(x_{n-1}, p) + \beta d(x_{n-1}, T_1(p)) + \gamma d(p, T_2(x_{n-1}))$$
$$\leq d(p, x_n) + \propto d(x_{n-1}, p) + \beta d(x_{n-1}, T_1(p)) + \gamma d(p, x_n)$$
$$\leq (1+\gamma) d(p, x_n) + (\propto + \beta) d(x_{n-1}, p) + \beta d(T_1(p), p)$$

$d(T_1(p), p) \leq \frac{1+\gamma}{1-\beta} d(x_n, p) + \frac{\propto+\beta}{1-\beta} d(x_{n-1}, p)$ for all n$\geq N_2$.

$d(T_1(p), p) \ll \frac{c}{m}$ for all $m \geq 1$, we get $\frac{c}{m} - d(T_1(p), p) \in P$ and as $m \to \infty$, we get $\frac{c}{m} \to 0$ and P is closed $-d(T_1(p), p) \in P$, but $d(T_1(p), p) \in P$. Therefore $d(T_1(p), p) = 0$ and so $p \in T_1(p)$.

Similarly, it can be established that $p \in T_2(p)$. Hence $p \in T_1(p) \cap T_2(p)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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