Available online at http://scik.org
Adv. Fixed Point Theory, 6 (2016), No. 1, 1-9
ISSN: 1927-6303

COINCIDENCE AND COMMON FIXED POINT THEOREMS IN A MULTIPLICATIVE METRIC SPACE

Q.H. KHAN*, M. IMDAD

Department of Mathematics, Aligarh Muslim University Aligarh 202002, India

Abstract. In this paper, we prove a unique common fixed point theorem for two pairs of weakly compatible mappings on multiplicative metric spaces without any continuity requirement. Examples on the uniqueness of common fixed points are provided. The results obtained in this paper improve the corresponding results announced recently.

Keywords: Coincidentally commuting mappings; Coincidence point; Multiplicative metric space; Common fixed point.

2010 AMS Subject Classification: 54H25, 47H10.

1. Introduction

A bulk of literature exists with commuting pairs such as weakly commuting [9], compatible mappings [6], compatible mappings of type A [6], R-weak commutativity [8]. Jungck and Rhoades [5] also Dhage [2] termed a pair of self-mappings to be coincidentally commuting (or weakly compatable) if they merely commute at thier coincidence point. One may note that this

*Corresponding author

E-mail address: qhkhan.ssitm@gmail.com.

Received August 18, 2015

notion does not involve the metric of the underlying set. The following one way implications are obviously true but thier converse are not .

Commuting maps $\Rightarrow$ Weakly Commuting maps $\Rightarrow$ Compatible maps $\Rightarrow$ Coincidentally commuting maps

Ozavsar gave the concept of multiplicative contraction mappings and proved fixed point theorem of such mappings on a complete metric space in [7]. Gu proved the common fixed point theorems of weak commutative mappings on a complete metric space in [3].

2. Preliminaries

In this section some definitions are given which will be used in this paper.

Definition 2.1. [1] Let X be a nonempty set. A multiplicative metric is a mapping

$d : X \times X \to R^+$ satisfying the following conditions:

(i) $d(x,y) \geq 1$ for all $x,y \in X$ and $d(x,y) = 1$, if and only if $x = y$.

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$.

(iii) $d(x,y) \leq d(x,z).d(z,y)$ for all $x,y,z \in X$ (multiplicative triangle inequality).

Definition 2.2. [7] Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x) = \{y/d(x,y) < \varepsilon\}, \varepsilon > 1$ there exists a natural number N such that $n \geq N$, then $x_n \in B_\varepsilon(x)$. The sequence$\{x_n\}$ is said to be multiplicative converging to x, denoted by $x_n \to x(n \to \infty)$.

Definition 2.3. [7] Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. The sequence is called a multiplicative Cauchy sequence if it hold that for all $\varepsilon{>}1$, there exists $N \in \mathbb{N}$ such that $d(x_n,x_m){<}\varepsilon$ for all $m,n > N$.

Definition 2.4. [7] A multiplicative metric space is complete, if every multiplicative Cauchy sequence in it is multiplicative convergence to x $\in X$.

Definition 2.5. [3] Suppose that S,T are two self-mappings of a multiplicative metric space (X,d); S,T are called commutative mappings if it holds for that for all $x \in X$, $STx = TSx$.

Definition 2.6. [3] Suppose that S,T are two self-mappings of a multiplicative metric space (X,d); S,T are called Weak commutative mappings if it holds for that for all $x \in X$, $d(STx,TSx) \leq d(Tx,Sx)$.

Definition 2.7. [1] Let (X,d) be a multiplicative metric space. A mapping $f : X \to X$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0,1)$ such that $d(f(x_1), f(x_2)) \leq d(x_1,x_2)^{\lambda}$ for all $x,y \in X$.

Definition 2.8. [7] (Multiplicative continuity) Let (X,d_X) and (X,d_Y) be two multiplicative metric spaces and $f : X \to Y$ be a function. If f holds the requirement that, for every $\varepsilon > 1$ there exists $\delta > 1$ such that $f(B_\delta(x)) \subset f(B_\varepsilon(f(x))$ then we call f multiplicative continuous at $x \in X$.

Definition 2.9. [7] (Semi-Multiplicative continuity) Let (X,d) be a multiplicative metric space, (Y,d) be a metric space and $f : X \to Y$ be a function. If f holds the requirement that for all $\varepsilon > 0$ there exists $\delta > 1$ such that $f(B_\delta(x)) \subset f(B_\varepsilon(f(x))$ then we call f semi-multiplicative continuous at $x \in X$. Similarly a function $g : Y \to X$ is also said to be semi-multiplicative continuous at $y \in Y$ if it satisfies a similar requirement.

Definition 2.10. [7] (Multiplicative convergence) Let (X,d) be a multiplicative metric space, (x_n) be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x))$, there exists a natural number N such that $n \geq N \Rightarrow x_n \in B_\varepsilon(x)$, then the sequence (x_n) is said to be multiplicative convergent to x denoted by $x_n \to x(n \to \infty)$.

Lemma 2.11. [7] *Let (X,d) be a multiplicative metric space, (x_n) be a sequence in X and $x \in X$. Then $x_n \to x(n \to \infty)$ if and only if $d(x_n,x) \to 1(n \to \infty)$.*

Lemma 2.12. [7] *Let (X,d) be a multiplicative metric space, (x_n) be a sequence in X. If the sequence (x_n) is multiplicative convergent, then the multiplicative limit point is unique.*

Lemma 2.13. [7] *Let (X,d_X) and (X,d_Y) be two multiplicative metric spaces, and $f : X \to Y$ be a mapping and (x_n) be any sequence in X. Then f is multiplicative continuous at the point $x \in X$ if and only if $f(x_n) \to f(x)$ for every sequence (x_n) with $x_n \to x(n \to \infty)$.*

Theorem 2.14. [7] *Let (X,d) be a multiplicative metric space and (x_n) be a sequence in X. The sequence is multiplicative convergent, then it is a multiplicative Cauchy sequence.*

Lemma 2.15. [7] *Let (X,d) be a multiplicative metric space and (x_n) be a sequence in X. Then (x_n) is a multiplicative Cauchy sequence if and only if $d(x_m,x_n) \to 1$ $(m,n \to \infty)$.*

Theorem 2.16. [7] *Let (X,d) be a multiplicative metric space. A mapping $f : X \to X$ multiplicative contraction. If (X,d) is complete, then f has unique common fixed point.*

Recently, He, Song and Chen proved the fixed point theorem for mappings of two pairs of weak commutative mappings on a multiplicative metric.

Theorem A. [10] *Let S,T,A, and B be self mappings of a complete multiplicative metric space X, they satisfy the following conditions:*

(i) $S(X) \subset B(X)$, $T(X) \subset A(X)$,

(ii) A and S are weak commutative, B and T also are weak commutative,

(iii) One of S,T,A and B is continuous,

(iv) $d(Sx,Ty) \leq \{max\{d(Ax,By),d(Ax,Sx),d(Ty,By),d(Sx,By),d(Ax,Ty)\}\}^{\lambda}$,

$\lambda \in (0,\frac{1}{2})$ $\forall$ $x,y \in X$

Then S,T,A and B have a unique common fixed point.

3. Main results

In this section, we improve Theorem A by relaxing the continututy requirement of the maps completely and to reduce the commutativity requirement of the maps to coincidence point only.

Theorem 3.1. *Let A,B,I and J be self-mappings of a multiplicative metric space (X,d) satisfying $A(X) \subset J(X)$, $B(X) \subset I(X)$ and*

$$d(Ax,By) \leq \max\{d(Ix,Jy),d(Ix,Ax),d(By,Jy),d(Ax,Jy),d(Ix,By)\}^{\lambda},$$

$$\lambda \in (0,\frac{1}{2}) \ \forall \ x,y \in X. \tag{3.1.1}$$

If one of A(X), B(X), I(X), J(X)is complete subspace of X, then the following conclusion hold

(i) (A,I) has coincidence point,

(ii) (B,J) has coincidence point,

Further if the pairs (A,I) and (B,J) are coincidently commuting, then A, B, I and J have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X, since $A(X) \subset J(X)$. We can find a point x_1 in X such that $Ax_0 = Jx_1$. Also since $B(X) \subset I(X)$, we can choose a point x_2 with $Bx_1 = Ix_2$. Using this argument repeatedly, one can construct a sequence z_n such that $z_{2n} = Ax_{2n} = Jx_{2n+1}$ and $z_{2n+1} = Bx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2$

$$d(z_{2n+1}, z_{2n+2}) = d(Bx_{2n+1}, Ax_{2n+2}) = d(Ax_{2n+2}, Bx_{2n+1})$$

$$\leq \{max\{d(Ix_{2n+2}, Jx_{2n+1}), d(Ix_{2n+2}, Ax_{2n+2}), d(Bx_{2n+1}, Jx_{2n+1}),$$

$$d(Ax_{2n+2}, Jx_{2n+1}), d(Ix_{2n+2}, Bx_{2n+1})\}\}^{\lambda}$$

$$\leq \{max\{d(z_{2n+1}, z_{2n}), d(z_{2n+1}, z_{2n+2}), d(z_{2n+1}, z_{2n}), d(z_{2n+2}, z_{2n}), d(z_{2n+1}, z_{2n+2})\}\}^{\lambda}$$

$$= d^{\lambda}(z_{2n}, z_{2n+1}).d^{\lambda}(z_{2n+1}, z_{2n+2})$$

$$d(z_{2n+1}, z_{2n+2}) \leq d^{\frac{\lambda}{1-\lambda}}(z_{2n}, z_{2n+1}).$$

Letting $\frac{\lambda}{1-\lambda} = h$, we have

$$d(z_{2n+1}, z_{2n+2}) \leq d^{h}(z_{2n}, z_{2n+1}). \tag{3.1.2}$$

We can also write

$$d(z_{2n+2}, z_{2n+3}) \leq d^{h}(z_{2n+1}, z_{2n+2}). \tag{3.1.3}$$

From (3.1.2) and (3.1.3) we know that

$$d(z_n, z_{n+1}) \leq d^{h}(z_{n-1}, z_n) .... \leq d^{h^n}(z_1, z_0), \forall n \geq z.$$

Letting $m, n \in N$, such that $m \geq n$, t we get

$$d(z_m, z_n) \leq d(z_m, z_{m-1}).d(z_{m-1}, z_{m-2})....d(z_{n+1}, z_n)$$

$$\leq d^{h^{m-1}}(z_1, z_0).d^{h^{m-2}}(z_1, z_0).....d^{h^n}(z_1, z_0)$$

$$\leq d^{\frac{h^n}{1-h}}(z_1, z_0).$$

This implies that $d(z_m, z_n) \to 1 (m, n \to \infty)$. Hence z_n is a multiplecative Cauchy sequence in X. Now suppose that $I(X)$ is a complete subspace of X, then observing that the subsequence z_n which contained in $I(X)$ must get a limit z in $I(X)$. Let $u \in I^{-1}(z) = Iu = z$.

Now we prove that $Au = z$ by using $y = x_{2n+1}$ in (3.1.1) then,

$$d(Au, Bx_{2n+1}) \leq \{max\{(d(Iu, Jx_{2n+1}), d(Iu, Au), d(Bx_{2n+1}, Jx_{2n+1}),$$

$$d(Au, Jx_{2n+1}), d(Iu, Bx_{2n+1})\}\}^{\lambda},$$

which on letting $n \to \infty$ and using $Iu = z$, we get

$$d(Au, z) \leq \{max\{d(Au, z), d(z, Au), d(z, z), d(Au, z), d(z, z)\}\}^{\lambda}$$

$$\leq \{max\{d(Au, z), 1\}\}^{\lambda}$$

$$= d^{\lambda}(Au, z),$$

which implies $d(Au, z) = 1$, i.e $Au = z = Iu$. Since $A(X) \subset J(X)$, implies that $z \in J(X)$, Let $v \in J^{-1}(z)$. Then $Jv = z$. Again using this argument it can easily show that $Bv = z$ yielding there by $Jv = Bv = z$. The remaining two case certain essentially to the previous case. Indeed if $B(X)$ is complete then $z \in B(X) \subset I(X)$ and if $A(X)$ is complete then $z \in A(X) \subset J(X)$.

Moreover if the pair (A, I) and (B, J) are coincidentally commuting at u and v, respectively

(i) $z = Au = Iu = Bv = Jv$,

(ii) $Az = A(Iu) = I(Au) = Iz$,

(iii) $Bz = B(Jv) = J(Bv) = Jz$,

Now we show that $Az = z$

$$d(Az, Bx_{2n+1}) \leq \{max\{(d(Iz, Jx_{2n+1}), d(Iz, Az), d(Bx_{2n+1}, Jx_{2n+1}),$$

$$d(Az, Jx_{2n+1}), d(Iz, Bx_{2n+1})\}\}^{\lambda},$$

which on letting n$\to \infty$

$$d(Az, z) \leq \{max\{(d(Iz, z), d(Iz, Az), d(z, z), d(Az, z), d(Iz, z))\}^{\lambda}$$

$$\leq \{max\{d(Az, z), 1\}\}^{\lambda}$$

$$= d^{\lambda}(Az, z).$$

This implies that $Az = z = Iz$.

Similarly we can show by using that $Bz = z = Jz$.

Now we prove that A,B,I and J have a unique common fixed point. Suppose, w is also a common fixed point of A,B,I and J.

$$d(z,w) = d(Az,Bw) \leq \{max\{d(Iz,Bw),d(Iz,Az),d(Jw,Bw),d(Az,Jw),d(Iz,Bw))\}\}^{\lambda}$$

$$\leq \{max\{d(z,w),1\}\}^{\lambda}$$

$$= d^{\lambda}(z,w).$$

This implies that $d(z,w) = 1$, i.e, $z = w$.

This is a contradiction so A,B,I and J have a unique common fixed point.

Remark 3.2. Theorem 3.1 mainly generalizes and extends the corresponding result in He, Song and Chen [10,Theorem 3.2].

Threom 3.2. *Let A, B, I and J be four self-mappings of a multiplicative metric space (X,d) satisfying $A^m(X) \subset J^s(X)$, $B^n(X) \subset I^r(X)$ and*

$$d(A^m x,B^n y) \leq \{max\{d(I^r x,J^s y),d(I^r x,A^m x),d(B^n y,J^s y),d(A^m x,J^s y),$$

$$d(I^r x,B^n y)\}\}^{\lambda}\ \lambda \in (0,1/2)\ \forall\, x,y \in X \text{ and } p,q,r,s \in \mathbb{Z} \tag{3.2.1}$$

If one of A(X), B(X), I(X), J(X) is complete subspace of X, then the following conclusion hold and

(i) (A,I) has coincidence point,

(ii)(B,J) has coincidence point,

Further if the pairs (A, I) and (B, J) are coincidently commuting , then A, B, I and J have a unique common fixed point.

Proof. Since (A,I) and (B,J) are coincidentaly commuting so A^m and B^n commuting with I^n and J^s. Thus by Theorem 3.1 there esists a unique z in X, such that

$$z = A^m z = B^n z = I^r z = J^s z,$$

$$Az = A(A^m z) = A^m(Az), Az = A(I^r z) = I^r(Az),$$

$$Bz = B(B^n z) = B^n(Bz), Bz = B(J^s z) = J^s(Jz),$$

which show that Az is common fixed point of A^m and I^r and Bz is common fixed point of B^r and J^s. Now consider $x = Az$ and $y = Bz$ in (3.2.1)

$$d(Az,Bz) = d(A^m x, B^n y)$$

$$\leq \{max\{d(I^r(Az),J^s(Bz)),d(I^r(Az),A^m(Az)),d(B^n(Bz),J^s(Bz)),$$

$$d(A^m(Az),J^s(Bz)),d(I^r(Az),B^n(Bz))\}\}^\lambda.$$

$$\leq \{max\{d(Sz,Tz),d(Sz,Sz),d(Tz,Tz),d(Sz,Tz),d(Sz,Tz)\}\}^\lambda$$

$$\leq \{max\{d(Sz,Tz),1,1,d(Sz,Tz),d(Sz,Tz)\}\}^\lambda$$

$$\leq \{d^\lambda(Az,Bz)\}.$$

This yields $Az = Bz$. Hence it is the common fixed point of A^m and I^r. Also Bz is a common fixed point of B^n and J^s. Let $x = Bz$ and $y = Jz$ in (3.2.1) we obtained $Bz = Jz$ and hence, it is the common fixed point of A^m, B^n, $I^r and$ J^s. By uniqueness of z in X we can show that $z = Az = Bz = Iz = Jz$. This completes the proof.

4. Related examples

In this section, we furnish examples demonstrating the validity of the hypothese and degree of generality of the results proved herein.

Example 4.1. *Consider X = [0,6] and (X,d) be a multiplicative metric space. Defne the mapping d : X × X→ R^+ by d(x,y) = $e^{|x-y|}$ for all x, y ∈ X. Define the self maps A, B, I and J on X.*

$$A_0 = 0, Ax = 1, 0 < x \leq 6,$$

$$B_0 = 0, Bx = 3, 0 < x \leq 6,$$

$$I_0 = 0, Ix = 5, 0 < x < 6, I6 = 6,$$

$$J_0 = 0, Jx = 6, 0 < x < 6, J6 = 1.$$

One may note that all four maps A, B, I and J are discontinuous ,also the pairs (A,I) and (B,J) are coincidently commuting . Clearly A(X) = $\{0,1\} \subset \{0,1,6\}$ *= J(X), and B(X) =* $\{0,3\} \subset \{0,3,5\}$

= I(X) and A(X), B(X), I(X), J(X) are complete. Let $\lambda = \frac{1}{5}$ according to the inequality of Theorem 3.1.

$$d(Ax,By) \leq \{max\{d(Ix,Jy),d(Ix,Ax),d(By,Jy),d(Ax,Jy),d(Ix,By)\}\}^{\lambda}$$

$$\lambda \in (0,1)\ \forall\ x,y \in X,$$

$$e^{|1-3|} \leq \{max\{e^{|5-6|},e^{|5-1|},e^{|3-6|},e^{|5-3|},e^{|6-1|}\}^{\lambda}.$$

So the contractivity condition of Theorem 3.1 is true. Therefore, for all the conditions of Theorem 3.1 are satisfied and hence A,B,I and J have a unique common fixed point (namely 0).

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] A.E. Bashirov, E.M. Kurplnara, A. Ozyapici, Multiplicative calculus and itsapplications, J. Math. Anal. Appl. 337 (2008), 36-48.

[2] B.C. Dhage, On common fixed ponts of coincidentally commuting mappings in D-metric space, Indian J. Pure Appl. Math. 30 (1999), 395-406.

[3] F. Gu, L.M. Cui, Y.H. Wu, Some fixed point theorems for new contractive type mappings, J. Qiqihar Univ. 19 (2003), 85-89.

[4] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986), 771-779.

[5] G. Jungck, B. E. Rhoades, Fixed points for set-valued functions without continuity, Indian J. Pure. Appl. Math. 29 (1998), 227-238.

[6] G. Jungck, P. P. Murthy, Y. J. Cho, Comatible mappings of type(A) and common fixed points, Math. Japan. 38 (1993), 381-390.

[7] M. Ozavsar, A. C. Cevikel, Fixed point of multiplicative contraction mappings on multiplicative metric space, arXiv:12055131v1[math.GN](2012).

[8] R. P. Pant, R-Week commutativity and common fixed points, Soohow J. Math. 25 (1999), 37-42.

[9] S. Sessa, On a weak commutativity condition in fixed point considerations, Publ. Inst. Math. (Beogard), 32 (1982), 146-153.

[10] X. He, M. Song, D. Chen, Common fixed points for weak commutative mappings on a multiplicative metric space, Fixed point Theorey Appl. 2014 (2014), Article ID 48.