Available online at http://scik.org
Adv. Fixed Point Theory, 5 (2015), No. 4, 433-447
ISSN: 1927-6303

CONVERGENCE THEOREMS FOR COMMON FIXED POINTS OF A FINITE FAMILY OF TOTAL ASYMPTOTICALLY NONEXPANSIVE NONSELF MAPPINGS IN HYPERBOLIC SPACES

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Abstract. Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $P : X \to C$ be the nonexpansive retraction. $S_1, S_2, \dots, S_r : C \to X$ be uniformly L-Lipschitzian and $(\{v_n\}, \{u_n\}, \xi)$-total asymptotically nonexpansive nonself mappings. In this paper, we introduce and study an iterative process for finding common fixed points of the family $\{S_j\}_{j=1}^r$. Assume that $F = \cap_{j=1}^r F(S_j) \neq \emptyset$, under certain conditions, strong and $\triangle$-convergence of the sequence are proved.

Keywords: Total asymptotically nonexpansive nonself mapping; $\triangle$-convergence; Hyperbolic space; Nonexpansive retraction.

2010 AMS Subject Classification: 47H09, 47H05.

1. Introduction

Most of the problems in various disciplines of science are nonlinear in nature, whereas fixed point theory proposed in the setting of normed linear spaces or Banach spaces majorly depends on the linear structure of the underlying spaces. A nonlinear framework for fixed point theory

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Received August 29, 2015

is a metric space embedded with a convex structure. The class of hyperbolic spaces, nonlinear in nature, is a general abstract theoretic setting with rich geometrical structure for metric fixed point theory. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory.

In 1976, the concept of $\triangle$-convergence in general metric spaces was coined by Lim [1]. In recent years, Yang and Zhao [2] studied the strong and $\triangle$- convergence theorems for total asymptotically nonexpansive nonself-mappings in CAT(0) spaces. Li and Bo [3] modified a classical Kuhfittig iteration algorithm in the general setup of hyperbolic space, and proved a $\triangle$-convergence theorem for an implicit iterative scheme. Recently, Wan [4] extended Chang's [5] result from CAT(0) spaces to the general setup of uniformly convex hyperbolic spaces, and obtained convergence theorems of the mixed Agarwal-ÓRegan-Sahu type iterative scheme for approximating a common fixed point of total asymptotically nonexpansive mappings.

Inspired and motivated by these facts, a new type of multistep iterative sequence is introduced and studied in this paper. This iterative sequence can be viewed as an extension of İsa Yildirim, Murat Özdemir [6] from CAT(0) space to the general setup of uniformly convex hyperbolic space. Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $S_1, S_2, \ldots, S_r : C \to X$ be uniformly L-Lipschitzian and $(\{v_n\}, \{u_n\}, \xi)$-total asymptotically nonexpansive nonself mappings. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = PW(y_{n+r-2}, S_1(PS_1)^{n-1}y_{n+r-2}, \alpha_{1n}), \\ y_{n+r-2} = PW(y_{n+r-3}, S_2(PS_2)^{n-1}y_{n+r-3}, \alpha_{2n}), \\ y_{n+r-3} = PW(y_{n+r-4}, S_3(PS_3)^{n-1}y_{n+r-4}, \alpha_{3n}), \\ \cdots \\ y_{n+1} = PW(y_n, S_{r-1}(PS_{r-1})^{n-1}y_n, \alpha_{(r-1)n}), \\ y_n = PW(x_n, S_r(PS_r)^{n-1}x_n, \alpha_{rn}), \end{cases} \tag{1.1}$$

where $P : X \to C$ be the nonexpansive retraction. The iterative sequence (1.1) is called the projection type multistep iteration for a finite family of total nonexpansive nonself-mapping in hyperbolic space.

The purpose of paper is to construct a multistep iterative scheme for approximation common fixed point of a finite family of total asymptotically nonexpansive nonself-mapping in the general setup of hyperbolic spaces. Under a limit condition, we establish some strong and $\triangle$-convergence results.

2. Preliminaries

In this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [7], which is more restrictive than the hyperbolic space introduced in Goebel and Kirk [8] and more general than the hyperbolic space in Reich and Shafrir [9]. Concretely, (X,d,W) is called a hyperbolic space if (X,d) is a metric space and $W: X\times X\times[0,1]\to X$ is a function satisfying

(1) $d(z,W(x,y,\alpha))\leq(1-\alpha)d(z,x)+\alpha d(z,y)$;

(2) $d(W(x,y,\alpha),W(x,y,\beta))=|\alpha-\beta|d(x,y)$;

(3) $W(x,y,\alpha)=W(y,x,1-\alpha)$;

(4) $d(W(x,z,\alpha),W(y,w,\alpha))\leq(1-\alpha)d(x,y)+\alpha d(z,w)$

for all $x,y,z,w\in X$ and $\alpha,\beta\in[0,1]$. A nonempty subset C of a hyperbolic space X is convex if $W(x,y,\alpha)\in X(\forall x,y\in X)$ and $\alpha\in[0,1]$. The class of hyperbolic spaces contains normed spaces and convex subsets thereof, the Hilbert ball equipped with the hyperbolic metric [10], Hadamardmanifolds as well as CAT(0) spaces in the sense of Gromov [11].

A hyperbolic space X is uniformly convex if for $u,x,y\in X$, $r>0$ and $\varepsilon\in(0,2]$, there exists $\delta\in(0,1]$ such that

$$d(W(x,y,\frac{1}{2}),u)\leq(1-\delta)r,$$

provided that $d(x,u)\leq r$, $d(y,u)\leq r$ and $d(x,y)\geq\varepsilon r$.

A map $\eta:(0,+\infty)\times(0,2]\to(0,1]$ is called modulus of uniform convexity if $\delta=\eta(r,\varepsilon)$ for given $r>0$. Besides, η is monotone if it decreases with r, that is,

$$\eta(r_2,\varepsilon)\leq\eta(r_1,\varepsilon),\forall r_2\geq r_1.$$

Let C be a nonempty subset of a metric space (X,d). Recall that a mapping $T: C\to X$ is said to be nonexpansive if

$$d(Tx,Ty)\leq d(x,y),\forall x,y\in C.$$

Recall that C is said to be a retraction of X if there exists a continuous map $P : X \to C$ such that $Px = x, \forall x \in C$. A map $P : X \to C$ is said to be a retraction if $P^2 = P$. Consequently, if P is a retraction, then $Py = y$ for all y in the range of P.

Let C be a nonempty and closed subset of a metric space (X,d), a map $P : X \to C$ is a retraction, a mapping $T : C \to X$ is said to be

(1)$(\{v_n\},\{u_n\},\xi)$-total asymptotically nonexpansive nonself mappings [12] if there exist nonnegative sequences $\{v_n\}$, $\{u_n\}$ with $v_n \to 0$, $u_n \to 0$, and a strictly increasing continuous function $\xi : [0,\infty) \to [0,\infty)$ with $\xi(0) = 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq d(x,y) + v_n\xi(d(x,y)) + u_n, \ \ \forall x,y \in C, \quad n \geq 1. \tag{2.1}$$

(2) asymptotically nonexpansive nonself-mapping if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim\limits_{n\to\infty} k_n = 1$, such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq k_n d(x,y), \ \ \forall x,y \in C, \quad n \geq 1.$$

(3) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq Ld(x,y), \ \ \forall x,y \in C, \quad n \geq 1.$$

Remark 2.1 From the definitions above, we know that each nonexpansive mapping is an asymptotically nonexpansive nonself-mapping, an asymptotically nonexpansive nonself-mapping is total asymptotically nonexpansive nonself mappings, and each asymptotically nonexpansive nonself-mapping is uniformly $L = \sup_{n\geq 1}\{k_n\}$-Lipschitzian.

To study our results in the general setup of hyperbolic spaces, we first collect some basic concepts. Let $\{x_n\}$ be a bounded sequence in hyperbolic space X. For $x \in X$, define a continuous functional $r(\cdot,\{x_n\}) : X \to [0,+\infty)$ by

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}.$$

The asymptotic radius $r_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is given by

$$r_C(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in C\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

The asymptotic center $A_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is set

$$A_C(\{x_n\}) = \{x \in C : r(x,\{x_n\}) = r_C(\{x_n\})\}.$$

A sequence $\{x_n\}$ in hyperbolic space X is said to $\triangle$-convergence to $x \in X$, if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we call x the $\triangle$-limit of $\{x_n\}$.

Lemma 2.1. [13] *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η, and let C be a nonempty, closed and convex subset of X. Then every bounded sequence$\{x_n\}$ in X has a unique asymptotic center with respect to C.*

Lemma 2.2. [13,14] *Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\beta_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$, $\{y_n\}$ are sequences in X such that* $\limsup_{n\to\infty} d(x_n,x) \le c$*,* $\limsup_{n\to\infty} d(y_n,x) \le c$ *and* $\limsup_{n\to\infty} d(W(x_n,y_n,\beta_n),x) = c$ *for some $c \ge 0$, then* $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Lemma 2.3. [4] *Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let C be a nonempty closed convex subset of X. $S : C \to X$ be uniformly L-Lipschitzian and $(\{v_n\},\{u_n\},\xi)$-total asymptotically nonexpansive nonself mappings. $P : X \to C$ be the nonexpansive retraction. Let $\{x_n\}$ be a bounded approximation fixed point sequence in C, i.e,* $\lim_{n\to\infty} d(x_n,Sx_n) = 0$ *and $\{x_n\} \rightharpoonup p$. Then we have $Sp = p$.*

Lemma 2.4. [15] *Let $\{a_n\}$, $\{b_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \le (1+b_n)a_n + t_n$ for all $n \ge 1$. If $\sum_{n=1}^{\infty} b_n < +\infty$, $\sum_{n=1}^{\infty} t_n < +\infty$, then* $\lim_{n\to\infty} a_n$ *exists.*

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $S_1, S_2, \ldots, S_r : C \to X$ be uniformly L-Lipschitzian and $(\{v_n\}, \{u_n\}, \xi)$-total asymptotically nonexpansive nonself mappings. Let $P : X \to C$ be the nonexpansive retraction Let the sequence $\{x_n\}$ be defined iteratively by (1.1). Assume that $F = \cap_{j=1}^{r} F(S_j) \neq \emptyset$ and the following conditions are satisfied:*

(1) $\sum\limits_{n=1}^{\infty} v_n < \infty$, $\sum\limits_{n=1}^{\infty} u_n < \infty$.

(2)Suppose that $\{\alpha_{jn}\}, j = 1, 2, \ldots, r$ are real sequences in $[a, b]$ for some $a, b \in (0, 1)$.

(3)There exists a constant $M > 0$ such that $\xi(t) \leq Mt, t \geq 0$.

Then the sequence $\{x_n\}$ $\triangle$-converges to a point $q \in F$.

Proof. We divide the proof into four steps.

Step 1. We prove that

$$\lim_{n\to\infty} d(x_n, p), \forall p \in F, \lim_{n\to\infty} d(x_n, F) \text{ exist}. \tag{3.1}$$

Let $p \in F$. Using (1.1) and (2.1), we have that

$$\begin{aligned} d(S_r(PS_r)^{n-1}x_n, p) &\leq d(x_n, p) + v_n \xi(d(x_n, p)) + u_n \\ &\leq d(x_n, p) + v_n M d(x_n, p) + u_n \\ &= (1 + v_n M) d(x_n, p) + u_n. \end{aligned} \tag{3.2}$$

By (1.1) and (3.2), we obtain that

$$\begin{aligned} d(y_n, p) &= d(PW(x_n, S_r(PS_r)^{n-1}x_n, \alpha_{rn}), p) \\ &\leq d(W(x_n, S_r(PS_r)^{n-1}x_n, \alpha_{rn}), p) \\ &\leq (1 - \alpha_{rn}) d(x_n, p) + \alpha_{rn} d(S_r(PS_r)^{n-1}x_n, p) \\ &\leq (1 - \alpha_{rn}) d(x_n, p) + \alpha_{rn}((1 + v_n M) d(x_n, p) + u_n) \\ &\leq d(x_n, p) + \alpha_{rn} v_n M d(x_n, p) + \alpha_{rn} u_n \\ &\leq (1 + v_n M) d(x_n, p) + u_n. \end{aligned} \tag{3.3}$$

For $1 \leq j \leq r-1$, we have

$$\begin{aligned} d(S_j(PS_j)^{n-1}y_{n+r-j-1},p) &\leq d(y_{n+r-j-1},p)+v_n\xi(d(y_{n+r-j-1},p))+u_n \\ &\leq d(y_{n+r-j-1},p)+v_nMd(y_{n+r-j-1},p)+u_n \\ &= (1+v_nM)d(y_{n+r-j-1},p)+u_n. \end{aligned} \tag{3.4}$$

From (3.4) with $j=r-1$, we have

$$\begin{aligned} d(y_{n+1},p) &= d(PW(y_n,S_{r-1}(PS_{r-1})^{n-1}y_n,\alpha_{(r-1)n}),p) \\ &\leq d(W(y_n,S_{r-1}(PS_{r-1})^{n-1}y_n,\alpha_{(r-1)n}),p) \\ &\leq (1-\alpha_{(r-1)n})d(y_n,p)+\alpha_{(r-1)n}d(S_{r-1}(PS_{r-1})^{n-1}y_n,p) \\ &\leq (1-\alpha_{(r-1)n})d(y_n,p)+\alpha_{(r-1)n}((1+v_nM)d(y_n,p)+u_n) \\ &\leq d(y_n,p)+\alpha_{(r-1)n}v_nMd(y_n,p)+\alpha_{(r-1)n}u_n \\ &\leq (1+v_nM)d(y_n,p)+u_n. \end{aligned} \tag{3.5}$$

By induction, for $2 \leq j \leq r-1$, we have

$$\begin{aligned} d(y_{n+r-j},p) &= d(PW(y_{n+r-j-1},S_j(PS_j)^{n-1}y_{n+r-j-1},\alpha_{jn}),p) \\ &\leq d(W(y_{n+r-j-1},S_j(PS_j)^{n-1}y_{n+r-j-1},\alpha_{jn}),p) \\ &\leq (1-\alpha_{jn})d(y_{n+r-j-1},p)+\alpha_{jn}d(S_j(PS_j)^{n-1}y_{n+r-j-1},p) \\ &\leq (1-\alpha_{jn})d(y_{n+r-j-1},p)+\alpha_{jn}((1+v_nM)d(y_{n+r-j-1},p)+u_n) \\ &\leq (1+v_nM)d(y_{n+r-j-1},p)+u_n \\ &\leq (1+v_nM)^{r-j+1}d(x_n,p)+u_n[1+(1+v_nM)+\ldots+(1+v_nM)^{r-j}]. \end{aligned} \tag{3.6}$$

Using (3.3) and (3.6), for $2 \leq j \leq r$, we have

$$d(y_{n+r-j},p) \leq (1+v_nM)^{r-j+1}d(x_n,p)+u_n[1+(1+v_nM)+\ldots+(1+v_nM)^{r-j}]. \tag{3.7}$$

From (1.1), (3.4) and (3.6), we have

$$\begin{aligned}
d(x_{n+1},p) &= d(PW(y_{n+r-2},S_1(PS_1)^{n-1}y_{n+r-2},\alpha_{1n}),p)\\
&\le d(W(y_{n+r-2},S_1(PS_1)^{n-1}y_{n+r-2},\alpha_{1n}),p)\\
&\le (1-\alpha_{1n})d(y_{n+r-2},p)+\alpha_{1n}d(S_1(PS_1)^{n-1}y_{n+r-2},p)\\
&\le (1-\alpha_{1n})d(y_{n+r-2},p)+\alpha_{1n}((1+v_nM)d(y_{n+r-2},p)+u_n)\\
&\le (1+v_nM)d(y_{n+r-2},p)+u_n \qquad (3.8)\\
&\le (1+v_nM)^r d(x_n,p)+u_n[1+(1+v_nM)+\ldots+(1+v_nM)^{r-1}]\\
&\le [1+\binom{r}{1}v_nM+\binom{r}{2}v_n^2M^2+\ldots+\binom{r}{r}v_n^rM^r]d(x_n,p)+u_nr(1+v_nM)^{r-1}\\
&\le (1+\beta_rv_nM)d(x_n,p)+u_nr(1+v_nM)^{r-1}\\
&\le (1+\sigma_n)d(x_n,p)+\delta_n,
\end{aligned}$$

where $\beta_r=\binom{r}{1}+\binom{r}{2}+\ldots+\binom{r}{r}$, $\sigma_n=\beta_rv_nM$, $\delta_n=u_nr(1+v_nM)^{r-1}$, Since $\sum_{n=1}^{\infty}\sigma_n<\infty$, $\sum_{n=1}^{\infty}\delta_n<\infty$, it follows from Lemma 2.4 that (3.1) is proved, and $\{x_n\}$ is also bounded.

Step 2. We prove that

$$\lim_{n\to\infty}d(x_n,S_r(PS_r)^{n-1}x_n)=0,\lim_{n\to\infty}d(y_{n+r-j-1},S_j(PS_j)^{n-1}y_{n+r-j-1})=0,1\le j\le r-1. \quad (3.9)$$

Assume that

$$\lim_{n\to\infty}d(x_n,p)=c\ge 0. \quad (3.10)$$

Using (3.2), (3.4) and (3.7), we have

$$\limsup_{n\to\infty}d(S_r(PS_r)^{n-1}x_n,p)\le c,\limsup_{n\to\infty}d(y_{n+r-j},p)\le c,2\le j\le r. \quad (3.11)$$

and

$$\limsup_{n\to\infty}d(S_j(PS_j)^{n-1}y_{n+r-j-1},p)\le c,1\le j\le r-1. \quad (3.12)$$

Using (3.8), we have

$$\begin{aligned}
d(x_{n+1},p) &\le d(W(y_{n+r-2},S_1(PS_1)^{n-1}y_{n+r-2},\alpha_{1n}),p)\\
&\le (1+\sigma_n)d(x_n,p)+\delta_n.
\end{aligned}$$

By (3.10), we have

$$\lim_{n\to\infty} d(W(y_{n+r-2}, S_1(PS_1)^{n-1}y_{n+r-2}, \alpha_{2n}), p) = c. \tag{3.13}$$

It follows from (3.11)-(3.13) and Lemma 2.2 that

$$\lim_{n\to\infty} d(y_{n+r-2}, S_1(PS_1)^{n-1}y_{n+r-2}) = 0. \tag{3.14}$$

From (3.8), we have

$$d(x_{n+1}, p) \le (1+v_nM)d(y_{n+r-2}, p) + u_n.$$

It follows that

$$\liminf_{n\to\infty} d(y_{n+r-2}, p) \ge c. \tag{3.15}$$

Using (3.11) and (3.15), we have

$$\lim_{n\to\infty} d(y_{n+r-2}, p) = c. \tag{3.16}$$

Using (3.6) with $j = 2$, we have

$$\begin{aligned} d(y_{n+r-2}, p) &\le d(W(y_{n+r-3}, S_2(PS_2)^{n-1}y_{n+r-3}, \alpha_{2n}), p) \\ &\le (1+v_nM)^{r-1}d(x_n, p) + u_n[1 + (1+v_nM) + \ldots + (1+v_nM)^{r-2}]. \end{aligned} \tag{3.17}$$

Using (3.16) and (3.10), we have

$$\lim_{n\to\infty} d(W(y_{n+r-3}, S_2(PS_2)^{n-1}y_{n+r-3}, \alpha_{2n}), p) = c. \tag{3.18}$$

It follows from (3.11), (3.12), (3.18) and Lemma 2.2 that

$$\lim_{n\to\infty} d(y_{n+r-3}, S_2(PS_2)^{n-1}y_{n+r-3}) = 0. \tag{3.19}$$

By induction, we get for $2 \le j \le r-1$

$$\lim_{n\to\infty} d(y_{n+r-j-1}, S_j(PS_j)^{n-1}y_{n+r-j-1}) = 0, \lim_{n\to\infty} d(y_{n+r-j}, p) = c. \tag{3.20}$$

From (3.5), we see

$$\liminf_{n\to\infty} d(y_n, p) \ge c. \tag{3.21}$$

By (3.11), we have

$$\lim_{n\to\infty} d(y_n, p) = c. \tag{3.22}$$

From (3.3), we obtain

$$\lim_{n\to\infty} d(W(x_n, S_r(PS_r)^{n-1}x_n, \alpha_{rn}), p) = c. \tag{3.23}$$

From (3.10), (3.11), (3.24) and Lemma 2.2, we have that

$$\lim_{n\to\infty} d(x_n, S_r(PS_r)^{n-1}x_n) = 0. \tag{3.24}$$

So (3.9) is proved.

Step 3. We show that

$$\lim_{n\to\infty} d(x_n, S_j x_n) = 0, j = 1,2,\ldots,r. \tag{3.25}$$

From $y_n = PW(x_n, S_r(PS_r)^{n-1}x_n, \alpha_{rn})$ and (3.24), we have

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, W(x_n, S_r(PS_r)^{n-1}x_n, \alpha_{rn})) \\ &\leq (1-\alpha_{rn})d(x_n, x_n) + \alpha_{rn} d(x_n, S_r(PS_r)^{n-1}x_n) \\ &\leq d(x_n, S_r(PS_r)^{n-1}x_n) \to 0 (n\to\infty). \end{aligned} \tag{3.26}$$

Since S_{r-1} is uniformly L-Lipschitzian, it follows from (3.20) and (3.26) that,

$$\begin{aligned} & d(x_n, S_{r-1}(PS_{r-1})^{n-1}x_n) \\ & \leq d(x_n, y_n) + d(y_n, S_{r-1}(PS_{r-1})^{n-1}y_n) + d(S_{r-1}(PS_{r-1})^{n-1}y_n, S_{r-1}(PS_{r-1})^{n-1}x_n) \\ & \leq d(x_n, y_n) + d(y_n, S_{r-1}(PS_{r-1})^{n-1}y_n) + Ld(y_n, x_n) \to 0(n\to\infty). \end{aligned} \tag{3.27}$$

From (3.26) and (3.10), we find

$$\begin{aligned} d(x_n, y_{n+1}) &\leq d(x_n, W(y_n, S_{r-1}(PS_{r-1})^{n-1}y_n, \alpha_{(r-1)n})) \\ &\leq (1-\alpha_{(r-1)n})d(x_n, y_n) + \alpha_{(r-1)n} d(y_n, S_{r-1}(PS_{r-1})^{n-1}y_n) \\ &\to 0(n\to\infty). \end{aligned} \tag{3.28}$$

Since S_{r-2} is uniformly L-Lipschitzian. it follows from (3.20) and (3.28) that,

$$\begin{aligned} & d(x_n, S_{r-2}(PS_{r-2})^{n-1}x_n) \\ & \leq d(x_n, y_{n+1}) + d(y_{n+1}, S_{r-2}(PS_{r-2})^{n-1}y_{n+1}) \\ & + d(S_{r-2}(PS_{r-2})^{n-1}y_{n+1}, S_{r-2}(PS_{r-2})^{n-1}x_n) \\ & \leq (1+L)d(x_n, y_{n+1}) + d(y_{n+1}, S_{r-2}(PS_{r-2})^{n-1}y_{n+1}) \to 0(n\to\infty). \end{aligned} \tag{3.29}$$

Continuing in this fashion we have

$$\lim_{n\to\infty} d(x_n, S_j(PS_j)^{n-1}x_n) = 0, j = 1,2,\dots,r. \tag{3.30}$$

and

$$\lim_{n\to\infty} d(x_n, y_{n+j-1}) = 0, j = 1,2,\dots,r-1, \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.31}$$

Since S_1 is uniformly L-Lipschitzian. Denote as $(PS_1)^{1-1}$ the identity maps from C onto itself. Thus by the inequality (3.30) and (3.31), we have

$$\begin{aligned} d(x_n, S_1x_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, S_1(PS_1)^n x_{n+1}) \\ &+ d(S_1(PS_1)^n x_{n+1}, S_1(PS_1)^n x_n) + d(S_1(PS_1)^n x_n, S_1x_n) \\ &\le (1+L)d(x_n, x_{n+1}) + d(x_{n+1}, S_1(PS_1)^n x_{n+1}) \\ &+ d(S_1(PS_1)^{1-1}(PS_1)^n x_n, S_1(PS_1)^{1-1}x_n) \\ &\le (1+L)d(x_n, x_{n+1}) + d(x_{n+1}, S_1(PS_1)^n x_{n+1}) + Ld((PS_1)^n x_n, x_n) \\ &\le (1+L)d(x_n, x_{n+1}) + d(x_{n+1}, S_1(PS_1)^n x_{n+1}) + Ld(S_1(PS_1)^{n-1}x_n, x_n) \\ &\to 0 (n\to\infty). \end{aligned} \tag{3.32}$$

Similarily, we may show that (3.25).

Step 4. We prove that $\{x_n\}$ $\triangle$-converges to a point $q \in F$. Since $\{x_n\}$ is bounded, by Lemma 2.1, it has a unique asymptotic center $A_C(\{x_n\}) = \{q\}$, that is $x_n \rightharpoonup q$ [4]. If $\{w_n\}$ is any sequence of$\{x_n\}$ with $A_C(\{w_n\}) = \{w\}$. Since $\lim_{n\to\infty} d(x_n, S_jx_n) = 0, j = 1,2,\dots,r$, it follows from we have Lemma 2.3 that $w \in F$. By the uniqueness of asymptotic center,we have $w = q$. It implies that q is the unique asymptotic center of $\{w_n\}$ for each subsequence $\{w_n\}$ of $\{x_n\}$, that is $\{x_n\}$ $\triangle$-converges to a point $q \in F$.

Let $r = 2$, we have the following theorem.

Theorem 3.2. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $S_1, S_2 : C \to X$ be uniformly L-Lipschitzian and $(\{v_n\}, \{u_n\}, \xi)$-total asymptotically nonexpansive nonself mappings. Let $P : X \to C$ be the nonexpansive retraction. Let the sequence $\{x_n\}$ be defined iteratively by* *the following manner:*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = PW(y_n, S_1(PS_1)^{n-1}y_n, \alpha_n), \\ y_n = PW(x_n, S_2(PS_2)^{n-1}x_n, \beta_n). \end{cases}$$

Assume that $F = F(S_1) \cap F(S_2) \neq \emptyset$ *and the following conditions are satisfied:*

(1) $\sum\limits_{n=1}^{\infty} v_n < \infty$, $\sum\limits_{n=1}^{\infty} u_n < \infty$.

(2)Suppose that $\{\alpha_n\}, \beta_n\}$ *are real sequences in* $[a,b]$ *for some* $a, b \in (0,1)$.

(3)There exists a constant $M > 0$ *such that* $\xi(t) \leq Mt, t \geq 0$.

Then the sequence $\{x_n\}$ $\triangle$*-converges to a point* $q \in F$.

Theorem 3.2 developed the result in [4].

Example 1. Let $\mathbf{R}$ be the real line with the usual norm $|\cdot|$ and $C = [-1,1]$. Define two mapping $T_1, T_2 : C \to C$ by

$$T_1 x = \begin{cases} -2sin\frac{x}{2}, & x \in [0,1], \\ 2sin\frac{x}{2}, & x \in [-1,0), \end{cases}$$

and

$$T_2 x = \begin{cases} x, & x \in [0,1], \\ -x, & x \in [-1,0). \end{cases}$$

It is proved in [16] that both T_1 and T_2 are asymptotically nonexpansive mappings with $k_n = 1, \forall n \geq 1$. Therefore, they are total asymptotically nonexpansive mappings with $v_n = v_n = 0, \forall n \geq 1, \xi(r) = r, \forall r \geq 0$. Additionally, they are uniformly L-Lipschitzian mapping with $L = 1$. $F(T_1) = \{0\}$ and $F(T_2) = [0,1]$. Let

$$\alpha_n = \frac{n}{2n+1}, \ \ \beta_n = \frac{n}{3n+1}, \forall n \geq 1. \tag{3.33}$$

Therefore, the conditions of Theorem 3.2 are fulfilled.

Example 2. Let $\mathbf{R}$ be the real line with the usual norm $|\cdot|$ and $C = [0, \infty)$. Define two mapping $T_1, T_2 : C \to C$ by

$$T_1 x = sinx, \ \ T_2 x = x.$$

It is proved in [17] that both T_1 and T_2 are total asymptotically nonexpansive mappings with $v_n = \frac{1}{n^2}, v_n = \frac{1}{n^3}0, \forall n \geq 1$. Additionally, they are uniformly L-Lipschitzian mapping with $L =$

1. $F(T_1) = \{0\}$ and $F(T_2) = [0,\infty)$. Let $\{\alpha_n\},\{\beta_n\}$ be the same as in (3.33) Therefore, the conditions of Theorem 3.2 are fulfilled.

Theorem 3.3. *Under the assumption of Theorem 3.1, if one of $\{S_j\}_{j=1}^r$ is either complete continuous or semi-compact, then the sequence of $\{x_n\}$ defined by (1.1) converges strongly(i.e., in the metric topology) to a common fixed point $q \in F$.*

Proof. By Theorem 3.1, we have $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, S_j x_n) = 0, j = 1,2,\ldots,r$.

If one of $\{S_j\}_{j=1}^r$ is semi-compact, say $\{S_k\}$, $k \in \{1,2,\ldots,r\}$, $\lim_{n\to\infty} d(x_n, S_k x_n) = 0$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that x_{n_i} converges strongly to $q \in C$. Then by the continuity of $\{S_j\}$, we get

$$d(q, S_j q) = \lim_{i\to\infty} d(x_{n_i}, S_j x_{n_i}) = 0, j = 1,2,\ldots,r,$$

which implies that $q \in F$. It follows from Theorem 3.1 that $\lim_{n\to\infty} d(x_n, q)$ exists and thus $\lim_{n\to\infty} d(x_n, q) = 0$.

If one of $\{S_j\}_{j=1}^r$ is complete continuous, say $\{S_k\}$, then exists a subsequence $\{S_k x_{n_i}\}$ of $\{S_k x_n\}$ such that $\{S_k x_{n_i}\}$ converges strongly to $q \in C$. By Theorem 3.1, $\lim_{i\to\infty} d(x_{n_i}, S_k x_{n_i}) = 0$, we have that $\lim_{i\to\infty} d(x_{n_i}, q) = 0$.

Theorem 3.4. *Under the assumption of Theorem 3.1, if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$with $f(0) = 0, f(t) > 0, \forall t > 0$ such that*

$$f(d(x_n, F)) \le d(x_n, S_1 x_n) + d(x_n, S_2 x_n) + \ldots + d(x_n, S_r x_n). \tag{3.34}$$

Then the sequence of $\{x_n\}$ defined by (3.1) converges strongly(i.e., in the metric topology) to a common fixed point $q \in F$.

Proof. By (3.25) and (3.34), we obtain that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0,\infty) \to [0,\infty)$with $f(0) = 0, f(t) > 0, \forall t > 0$, we have

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{3.35}$$

$\{x_n\}$ is bounded, there exists a constant K such that $d(x_n, p) \le K$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, given $\varepsilon > 0$, there exists a nature number N such that

$\sum_{n=N}^{\infty}(\sigma_n K+\delta_n)<\frac{\varepsilon}{4}$, and find $p\in F$ such that $d(x_N,p)<\frac{\varepsilon}{4}$. For all $n\geq N$ and $m\geq 1$, we have

$$\begin{aligned} d(x_{n+m},x_n) &\leq d(x_{n+m},p)+d(x_n,p)\\ &\leq d(x_N,p)+d(x_N,p)+2\sum_{n=N}^{\infty}(\sigma_n K+\delta_n)\\ &\leq \frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{2}=\varepsilon. \end{aligned}$$

we show that $\{x_n\}$ is a Cauchy sequence in C. C is complete, we can assume that $\{x_n\}$ converges strongly to some $q\in C$. By (3.35) implies that $\lim_{j\to\infty} d(q,F)=0$.. F is closed, hence $q\in F$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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