Available online at http://scik.org
Adv. Fixed Point Theory, 6 (2016), No. 1, 24-42
ISSN: 1927-6303

ON THE CONTROLLABILITY OF SOME PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN BANACH SPACES

KHALIL EZZINBI[1], PATRICE NDAMBOMVE[2,*]

[1]Université Cadi Ayyad, Faculté des Sciences Semlalia,
Département de Mathématiques, B.P . 2390 Marrakech, Morocco
[2]African University of Science and Technology (AUST),
Mathematics Institute, P.M.B 681, Garki, Abuja F.C.T, Nigeria

Abstract. This work concerns the study of the controllability for some partial functional integrodifferential equation with infinite delay in Banach spaces. We give sufficient conditions that ensure the controllability of the system by supposing that its undelayed part admits a resolvent operator in the sense of Grimmer, and by making use of the measure of noncompactness and the Mönch fixed-point theorem. As a result, we obtain a generalization of a host of important results in the literature, without assuming the compactness of the resolvent operator. An example is given for illustration.

Keywords: Controllability; Functional integrodifferential equation; Infinite delay; Resolvent operator; Mönch's fixed-point Theorem.

2010 AMS Subject Classification: 93B05, 45K05, 47H10.

1. Introduction

*Corresponding author
E-mail address: ezzinbi@uca.ma (K. Ezzinbi)
Received September 16, 2015

In this work, we study the controllability for some systems that arise in the analysis of heat conduction in materials with memory and viscosity [15, 7]. The interesting thing about materials with memory is that they act adaptively to their environment. They can easily be shaped into different forms at a low temperature, but return to their original shape on heating. Steering such systems from an initial state (initial condition) to a desired terminal one (boundary condition) by choosing appropriately a control, is of interest to many engineers and scientists. Such systems take the form of the following abstract model of partial functional integrodifferential equation with infinite delay in a Banach space $(X, \|\cdot\|)$:

$$(1) \qquad \begin{cases} x'(t) = Ax(t) \; + \; \displaystyle\int_0^t \gamma(t-s)x(s)ds \; + \; f(t,x_t) + Cu(t) \;\; \text{for } t \in I = [0,b], \\ x_0 = \varphi \in \mathscr{B}, \end{cases}$$

where $A : \mathscr{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space X; for $t \geq 0$, $\gamma(t)$ is a closed linear operator with domain $\mathscr{D}(\gamma(t)) \supset \mathscr{D}(A)$. The control u belongs to $L^2(I,U)$ which is a Banach space of admissible controls, where U is a Banach space. The operator $C \in \mathscr{L}(U,X)$, where $\mathscr{L}(U,X)$ denotes the Banach space of bounded linear operators from U into X, and the phase space $\mathscr{B}$ is a linear space of functions mapping $]-\infty,0]$ into X satisfying axioms which will be described later, for every $t \geq 0$, x_t denotes the history function of $\mathscr{B}$ defined by

$$x_t(\theta) = x(t+\theta) \;\; \text{for } -\infty \leq \theta \leq 0,$$

$f : I \times \mathscr{B} \to X$ is a continuous function satisfying some conditions. In the literature devoted to equations with finite delay, the phase space is the space of continuous functions on $[-r,0]$, for some $r > 0$, endowed with the uniform norm topology. But when the delay is unbounded, the selection of the phase space $\mathscr{B}$ plays an important role in both qualitative and quantitative theories. A usual choice is a normed space satisfying some suitable axioms, which was introduced by Hale and Kato [14].

The controllability problem for nonlinear integrodifferential systems in infinite dimensional Banach spaces has been studied by several authors: see for instance [4],[5], [6], [11, 12, 16, 17, 1, 19, 2, 20] and the references therein. Many authors have also studied the controllability problem of nonlinear differential systems with delay in infinite dimensional Banach spaces: see for instance [4], [16], [19], [2], [20], etc and the references therein. In [2], the authors obtained a controllability result for nonlinear functional evolution equations with infinite delay using the nonlinear alternative of Leray-Schauder type. In [17], the authors obtained the controllability results for an impulsive functional differential system with finite delay using Schaefer's fixed-point theorem. In [20], S. Selvi and M. M. Arjunan proved the controllability for impulsive differential systems with finite delay using Mönch's fixed-point Theorem, and in [4], K. Balachandran and R. Sakthivel studied the controllability of functional semilinear integrodifferential systems in Banach spaces using Schaefer's fixed point Theorem with the compactness assumption on the semigroup. The particular cases in which $\gamma(t) = 0$ and $A = A(t)$ were considered by K. Balachandran and R. Sakthivel [4] (for the finite delay case); S. Baghli, M. Benchohra and K. Ezzinbi [2]; and many others. In this work, we extend and complement the works of K. Balachandran and R. Sakthivel [4] and Baghli *et al* [2] by considering some integrodifferential equation when $\gamma(t) \neq 0$, and without any compactness assumption.

Integrodifferential equations appear in many areas of applications such as Electronics, Engineering, Physical Sciences, Fluid Dynamics, etc. During the last decades, these integrodifferential systems have received considerable attention. In recent years, many authors have worked on the existence and regularity of solutions of nonlinear functional integrodifferential equations with infinite delay, using the resolvent operator theory, see e.g., [13] and the references contained in it.

R. Grimmer in [7], proved the existence and uniqueness of resolvent operators that give the variation of parameters formula for the solutions, for these integrodifferential equations. In [8], W. Desch, R. Grimmer and W. Schappacher proved that the compactness of the resolvent operator is equivalent to that of the semigroup. In this work, we use the fact that the operator-norm continuity of the resolvent operator is equivalent to that of the semigroup. In fact, we assume that the resolvent operator admitted by the linear undelayed part of equation (1) is operator-norm continuous. This property allows us to drop the compactness assumption on the operator semigroup, considered by the authors in [4, 2], and prove that the operator solution satisfies the Mönch condition. We prove the controllability result using the Mönch's fixed-point Theorem and the Hausdorff measure of noncompactness. This method enables us overcome the resolvent operator case considered in this work. In contrary to the evolution semigroup case considered by the authors in [16, 20], here the semigroup property can not be used because resolvent operators in general are not semigroups.

To the best of our knowledge, up to now no work has reported on controllability of partial functional integrodifferential equation (1) with infinite delay in Banach spaces. It has been an untreated topic in the literature, and this fact is what motivates the present work.

The rest of the work is organized as follows: Section 2 is devoted to preliminary results. In this section, we give the definition of resolvent operator. This allows us to define the mild solution of equation (1). In Section 3, we study the controllability of equation (1). In Section 4, we give an example to illustrate the obtained results.

2. Integrodifferential equations, measure of noncompactness and Mönch's theorem

In this section, we introduce some definitions and Lemmas that will be used throughout the paper. Let $I=[0,b],\ \ b>0$ and let X be a Banach space. A measurable function $x:I\to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable. We denote by $L^1(I,X)$ the Banach space of Bochner integrable functions $x:I\to X$ normed by

$$\|x\|_{L^1}=\int_0^b\|x(t)\|dt.$$

Consider the following linear homogeneous equation:

$$\begin{cases} x'(t)=Ax(t)+\displaystyle\int_0^t\gamma(t-s)x(s)ds \ \text{ for } \quad \mathrm{t}\geq 0,\\ x(0)=x_0\in X, \end{cases} \tag{2}$$

where A and $\gamma(t)$ are closed linear operators on a Banach space X.

In the sequel, we assume A and $\big(\gamma(t)\big)_{t\geq 0}$ satisfy the following conditions:

$(\mathbf{H_1})$ A is a densely defined closed linear operator in X. Hence $\mathscr{D}(A)$ is a Banach space equipped with the graph norm defined by, $|y| = \|Ay\| + \|y\|$ which will be denoted by $(X_1, |\cdot|)$.

$(\mathbf{H_2})$ $\big(\gamma(t)\big)_{t\geq 0}$ is a family of linear operators on X such that $\gamma(t)$ is continuous when regarded as a linear map from $(X_1, |\cdot|)$ into $(X, \|\cdot\|)$ for almost all $t \geq 0$ and the map $t \mapsto \gamma(t)y$ is measurable for all $y \in X_1$ and $t \geq 0$, and belongs to $W^{1,1}(\mathbb{R}^+, X)$. Moreover there is a locally integrable function $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\gamma(t)y\| \;\leq\; b(t)|y| \quad \text{and} \quad \left\|\frac{d}{dt}\gamma(t)y\right\| \;\leq\; b(t)|y|\;.$$

Remark 2.1. Note that $(\mathbf{H_2})$ is satisfied in the modelling of Heat Conduction in materials with memory and viscosity. More details can be found in [10].

Let $\mathscr{L}(X)$ be the Banach space of bounded linear operators on X.

Definition 2.2. [13] A resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation (2) is a bounded operator valued function

$$R : \; [0, +\infty) \;\longrightarrow\; \mathscr{L}(X)$$

such that

(i) $R(0) = Id_X$ and $\|R(t)\| \leq Ne^{\beta t}$ for some constants N and β.

(ii) For all $x \in X$, the map $t \mapsto R(t)x$ is continuous for $t \geq 0$.

(iii) Moreover for $x \in X_1$, $R(\cdot)x \in \mathscr{C}^1(\mathbb{R}^+; X) \cap \mathscr{C}(\mathbb{R}^+; X_1)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t \gamma(t-s)R(s)x\,ds \\ &= R(t)Ax + \int_0^t R(t-s)\gamma(s)x\,ds. \end{aligned}$$

Observe that the map defined on $\mathbb{R}^+$ by $t \mapsto R(t)x_0$ solves equation (2) for $x_0 \in \mathscr{D}(A)$.

Theorem 2.3. [7] Assume that $(\mathbf{H_1})$ and $(\mathbf{H_2})$ hold. Then, the linear equation (2) has a unique resolvent operator $\big(R(t)\big)_{t\geq 0}$.

Remark 2.4. In general, the resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation (2) does not satisfy the semigroup law, namely,

$$R(t+s) \neq R(t)R(s) \qquad \text{for some } \ t,s>0\,.$$

We have the following theorem that establishes the equivalence between the operator-norm continuity of the C_0-semigroup and the resolvent operator for integral equations.

Theorem 2.5. [12] *Let A be the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ and let $\big(\gamma(t)\big)_{t\geq 0}$ satisfy $(\mathbf{H_2})$. Then the resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation (2) is operator-norm continuous (or continuous in the uniform operator topology) for $t>0$ if and only if $\big(T(t)\big)_{t\geq 0}$ is operator-norm continuous for $t>0$.*

In this work, we will employ an axiomatic definition of the phase space $\mathscr{B}$ introduced by Hale and Kato in [14]. Thus, $(\mathscr{B},\|\cdot\|_{\mathscr{B}})$ will be a normed linear space of functions mapping $]-\infty,0]$ into X and satisfying the following axioms:

- $(\mathbf{A_1})$ There exist positive constant H and functions $K:\mathbb{R}^+\to\mathbb{R}^+$ continuous and $M:\mathbb{R}^+\to\mathbb{R}^+$ locally bounded, such that for $a>0$, if $x:]-\infty,a]\to X$ is continuous on $[0,a]$ and $x_0\in\mathscr{B}$, then for every $t\in[0,a]$, the following conditions hold:
  - **(i)** $x_t\in\mathscr{B}$,
  - **(ii)** $\|x(t)\|\leq H\|x_t\|_{\mathscr{B}}$, which is equivalent to $\|\varphi(0)\|\leq H\|\varphi\|_{\mathscr{B}}$ for every $\varphi\in\mathscr{B}$,
  - **(iii)** $\|x_t\|_{\mathscr{B}}\leq K(t)\sup\limits_{0\leq s\leq t}\|x(s)\|+M(t)\|x_0\|_{\mathscr{B}}$.
- $(\mathbf{A_2})$ For the function x in $(\mathbf{A_1})$, $t\to x_t$ is a $\mathscr{B}$-valued continuous function for $t\in[0,a]$.
- $(\mathbf{A_3})$ The space $\mathscr{B}$ is complete.

Example 2.6. [13] *Let the spaces*

BC the space of bounded continuous functions defined from $(-\infty,0]$ to X;

BUC the space of bounded uniformly continuous functions defined from $(-\infty,0]$ to X;

$C^\infty := \Big\{\phi\in BC:\ \lim_{\theta\to-\infty}\phi(\theta)\ \textit{exists}\Big\}$;

$C^0 := \Big\{\phi\in BC:\ \lim_{\theta\to-\infty}\phi(\theta)=0\Big\}$, *be endowed with the uniform norm*

$$\|\phi\| = \sup_{\theta\leq 0}\|\phi(\theta)\|.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions $(\mathbf{A_1})-(\mathbf{A_3})$.

Definition 2.7. Let $u \in L^2(I,U)$ and $\varphi \in \mathscr{B}$. A function $x :]-\infty, b] \to X$ is called a mild solution of equation (1) if $x \in \mathscr{C}([0,b];X)$ and satisfies the following integral equation

$$x(t) = \begin{cases} R(t)\varphi(0) + \int_0^t R(t-s)\,[f(s,x_s)+Cu(s)]\,ds \ \text{ for } t \in I, \\ \varphi(t) \ \text{ for } \ -\infty \le t \le 0. \end{cases} \tag{3}$$

Definition 2.8. Equation (1) is said to be controllable on the interval I if for every $\varphi \in \mathscr{B}$ and $x_1 \in X$, there exists a control $u \in L^2(I,U)$ such that a mild solution x of equation (1) satisfies the condition $x(b) = x_1$.

For proving the main result of the paper we recall some properties of the measure of non-compactness and the Mönch fixed-point Theorem.

Definition 2.9. [3] Let D be a bounded subset of a normed space Y. The Hausdorff measure of noncompactness (shortly MNC) is defined by

$$\beta(D) = \inf\Big\{\varepsilon > 0 : D \; has \; a \; finite \; cover \; by \; balls \; of \; radius \; less \; than \; \varepsilon\Big\}.$$

Theorem 2.10. [3] *Let D, D_1, D_2 be bounded subsets of a Banach space Y. The Hausdorff MNC has the following properties:*

- **(i)** *If $D_1 \subset D_2$, then $\beta(D_1) \le \beta(D_2)$, (monotonicity).*
- **(ii)** $\beta(D) = \beta(\overline{D})$.
- **(iii)** $\beta(D) = 0$ *if and only if D is relatively compact.*
- **(iv)** $\beta(\lambda D) = |\lambda|\beta(D)$ *for any $\lambda \in \mathbb{R}$, (Homogeneity)*
- **(v)** $\beta(D_1 + D_2) \le \beta(D_1) + \beta(D_2)$, *where $D_1 + D_2 = \{d_1 + d_2 : \ d_1 \in D_1, \ d_2 \in D_2\}$, (sub-additivity)*
- **(vi)** $\beta(\{a\} \cup D) = \beta(D)$ *for every $a \in Y$.*
- **(vii)** $\beta(D) = \beta(\overline{co}(D))$, *where $\overline{co}(D)$ is the closed convex hull of D.*
- **(viii)** *For any map $G : \mathscr{D}(G) \subseteq X \to Y$ which is Lipschitz continuous with a Lipschitz constant k, we have*

$$\beta(G(D)) \ \le \ k\beta(D),$$

for any subset $D \subseteq \mathscr{D}(G)$.

Let

$$R_b = \sup_{t\in[0,b]} \|R(t)\|, \;\; K_b = \sup_{t\in[0,b]} \|K(t)\|, \;\; M_b = \sup_{t\in[0,b]} \|M(t)\|.$$

We now state the following useful result for equicontinuous subsets of $\mathscr{C}([a,b];X)$, where X is a Banach space.

Lemma 2.11. [3] *Let* $M \subset \mathscr{C}([a,b];X)$ *be bounded and equicontinuous. Then* $\beta(M(t))$ *is continuous and*

$$\beta(M) = \sup\{\beta(M(t)); t \in [a,b]\}, \quad \textit{where } M(t) = \{x(t); x \in M\}.$$

Lemma 2.12. [3] *Let* $M \subset \mathscr{C}([a,b];X)$ *be bounded and equicontinuous. Then the set* $\overline{co}(M)$ *is also bounded and equicontinuous.*

To prove the controllability for equation (1), we need the following results.

Lemma 2.13. [11] *If* $(u_n)_{n\geq 1}$ *is a sequence of Bochner integrable functions from I into a Banach space Y with the estimation* $\|u_n(t)\| \leq \mu(t)$ *for almost all* $t \in I$ *and every* $n \geq 1$*, where* $\mu \in L^1(I,\mathbb{R})$*, then the function*

$$\psi(t) \;=\; \beta(\{u_n(t) : n \geq 1\})$$

belongs to $L^1(I,\mathbb{R}^+)$ *and satisfies the following estimation*

$$\beta\left(\left\{\int_0^t u_n(s)ds \,:\; n \geq 1\right\}\right) \;\leq\; 2\int_0^t \psi(s)ds.$$

We now state the following nonlinear alternative of Mönch's type for selfmaps, which we shall use in the proof of the controllability of equation (1).

Theorem 2.14. [9] (Mönch, 1980) *Let* $\mathscr{K}$ *be a closed and convex subset of a Banach space Z and* $0 \in \mathscr{K}$*. Assume that* $F : \mathscr{K} \to \mathscr{K}$ *is a continuous map satisfying Mönch's condition, namely,*

$$D \subseteq \mathscr{K} \textit{ be countable and } D \subseteq \overline{co}(\{0\} \cup F(D)) \implies \overline{D} \textit{ is compact}.$$

Then F has a fixed point.

3. Controllability results

In this section, we give sufficient conditions ensuring the controllability of equation (1). For that goal, we need to assume that:

($\mathbf{H_3}$)

(i) The following linear operator $W : L^2(I,U) \to X$ defined by

$$Wu = \int_0^b R(b-s)Cu(s)\,ds,$$

is surjective so that it induces an isomorphism between $L^2(I,U)/_{\mathrm{Ker}W}$ and X again denoted by W with inverse W^{-1} taking values in $L^2(I,U)/_{\mathrm{Ker}W}$, (see e.g.,[18]).

(ii) There exists a function $L_W \in L^1(I,\mathbb{R}^+)$ such that for any bounded set $Q \subset X$ we have

$$\beta((W^{-1}Q)(t)) \leq L_W(t)\beta(Q),$$

where β is the Hausdorff MNC.

($\mathbf{H_4}$) The function $f : I \times \mathscr{B} \longrightarrow X$ satisfies the following two conditions:

(i) $f(\cdot,\varphi)$ is measurable for $\varphi \in \mathscr{B}$ and $f(t,\cdot)$ is continuous for a.e $t \in I$,

(ii) for every positive integer q, there exists a function $l_q \in L^1(I,\mathbb{R}^+)$ such that

$$\sup_{\|\varphi\|_{\mathscr{B}} \leq q} \|f(t,\varphi)\| \leq l_q(t) \text{ for a.e. } t \in I \text{ and } \liminf_{q\to+\infty} \int_0^b \frac{l_q(t)}{q} dt = l < +\infty,$$

(iii) there exists a function $h \in L^1(I,\mathbb{R}^+)$ such that for any bounded set $D \subset \mathscr{B}$,

$$\beta(f(t,D)) \leq h(t) \sup_{-\infty<\theta\leq 0} \beta(D(\theta)) \text{ for a.e } t \in I,$$

where

$$D(\theta) = \{\phi(\theta) : \phi \in D\}.$$

$(\mathbf{H_5})$

$$\tau = \Big(1 + 2R_b M_2 \|L_W\|_{L^1}\Big)\Big(2R_b \|h\|_{L^1}\Big) < 1,$$

where $R_b = \sup\limits_{0 \le t \le b} \|R(t)\|$ and M_2 is such that $M_2 = \|C\|$.

Theorem 3.1. *Suppose that hypotheses $(\mathbf{H_3}) - (\mathbf{H_5})$ hold and equation (2) has a resolvent operator $\big(R(t)\big)_{t \ge 0}$ that is continuous in the operator-norm topology for $t > 0$. Then equation (1) is controllable on I provided that*

$$R_b(1 + R_b M_2 M_3 b) K_b l < 1, \tag{4}$$

where M_3 is such that $M_3 = \|W^{-1}\|$ and $K_b = \sup_{t \in [0,b]} \|K(t)\|$.

Proof. Using $(\mathbf{H_3})$ and given an arbitrary function x, we define the control as usual by the following formula:

$$u_x(t) \;=\; W^{-1}\left\{x_1 - R(b)\varphi(0) - \int_0^b R(b-s) f(s, x_s)\, ds\right\}(t) \qquad \text{for } t \in I.$$

For each $x \in \mathscr{C}([0,b], X)$ such that $x(0) = \varphi(0)$, we define its extension $\widetilde{x}$ from $]-\infty, b]$ to X as follows

$$\widetilde{x}(t) = \begin{cases} x(t) & \text{if } t \in [0,b], \\ \varphi(t) & \text{if } t \in]-\infty, 0]. \end{cases}$$

We define the following space

$$E_b = \Big\{x :]-\infty, b] \to X \text{ such that } x|_I \in \mathscr{C}([0,b], X) \text{ and } x_0 \in \mathscr{B}\Big\},$$

where $x|_I$ is the restriction of x to I. We show using this control that the operator $P : E_b \to E_b$ defined by

$$(Px)(t) = R(t)\varphi(0) + \int_0^t R(t-s)\big[f(s, \widetilde{x}_s) + C u_x(s)\big]\, ds \;\; \text{for } \; t \in I = [0,b]$$

has a fixed-point. This fixed point is then a mild solution of equation (1). Observe that $(Px)(b) = x_1$. This means that the control u_x steers the integrodifferential equation from φ

to x_1 in time b which implies the equation (1) is controllable on I. For each $\varphi \in \mathscr{B}$, we define the function $y \in \mathscr{C}([0,b],X)$ by $y(t) = R(t)\varphi(0)$ and its extension $\widetilde{y}$ on $]-\infty,0]$ by

$$\widetilde{y}(t) = \begin{cases} y(t) & \text{if } t \in [0,b], \\ \varphi(t) & \text{if } t \in]-\infty,0]. \end{cases}$$

For each $z \in \mathscr{C}([0,b],X)$, set $\widetilde{x}(t) = \widetilde{z}(t) + \widetilde{y}(t)$, where $\widetilde{z}$ is the extension by zero of the function z on $]-\infty,0]$. Observe that x satifies (3) if and only if $z(0) = 0$ and

$$z(t) = \int_0^t R(t-s)\big[f(s,\widetilde{z}_s + \widetilde{y}_s) + Cu_z(s)\big]\,ds \quad \text{for t} \in [0,\text{b}],$$

where

$$u_z(t) \;=\; W^{-1}\left\{x_1 - R(b)\varphi(0) - \int_0^b R(b-s)f(s,\widetilde{z}_s + \widetilde{y}_s)\,ds\right\}(t).$$

Now let

$$E_b^0 = \Big\{z \in E_b \text{ such that } z_0 = 0\Big\}.$$

Thus E_b^0 is a Banach space provided with the supremum norm. Define the operator $\Gamma : E_b^0 \to E_b^0$ by

$$(\Gamma z)(t) = \int_0^t R(t-s)\big[f(s,\widetilde{z}_s + \widetilde{y}_s) + Cu_z(s)\big]\,ds \quad \text{for } t \in [0,b].$$

Note that the operator P has a fixed point if and only if Γ has one. So to prove that P has a fixed point, we only need to prove that Γ has one. For each positive number q, let $B_q = \{z \in E_b^0 : \|z\| \le q\}$. Then, for any $z \in B_q$, we have by axiom $(\mathbf{A_1})$ that

$$\begin{aligned}
\|z_s + y_s\| &\le \|z_s\|_{\mathscr{B}} + \|y_s\|_{\mathscr{B}} \\
&\le K(s)\|z(s)\| + M(s)\|z_0\|_{\mathscr{B}} + K(s)\|y(s)\| + M(s)\|y_0\|_{\mathscr{B}} \\
&\le K_b\|z(s)\| + K_b\|R(t)\|\|\varphi(0)\| + M_b\|\varphi\|_{\mathscr{B}} \\
&\le K_b\|z(s)\| + K_bR_bH\|\varphi\|_{\mathscr{B}} + M_b\|\varphi\|_{\mathscr{B}} \\
&\le K_b\|z(s)\| + \Big(K_bR_bH + M_b\Big)\|\varphi\|_{\mathscr{B}} \\
&\le K_b q + \Big(K_bR_bH + M_b\Big)\|\varphi\|_{\mathscr{B}}.
\end{aligned}$$

Thus,

$$\|z_s + y_s\| \le K_b q + \Big(K_bR_bH + M_b\Big)\|\varphi\|_{\mathscr{B}} =: q'.$$

We shall prove the theorem in the following steps.

Step 1. We claim that there exists $q>0$ such that $\Gamma(B_q)\subset B_q$. We proceed by contradiction. Assume that it is not true. Then for each positive number q, there exists a function $z^q\in B_q$, such that $\Gamma(z^q)\notin B_q,\ i.e.,\ \|(\Gamma z^q)(t)\|>q$ for some $t\in[0,b]$. Now we have that

$$
\begin{aligned}
q &< \left\|(\Gamma z^q)(t)\right\| \\
&\le R_b\int_0^b \left\|f(s,\tilde{z}_s^q+\widetilde{y}_s)\right\|ds+R_b\int_0^b \|Cu_{z^q}(s)\|\,ds \\
&\le R_b\int_0^b \left\|f(s,\tilde{z}_s^q+\widetilde{y}_s)\right\|ds+R_b\int_0^b \left\|BW^{-1}\left[x_1-R(b)\varphi(0)-\int_0^b R(b-s)f(s,\tilde{z}_s^q)\,ds\right]\right\|ds \\
&\le bR_bM_2M_3\left(\|x_1\|+R_b\|\varphi(0)\|+R_b\int_0^b \|f(s,\tilde{z}_s^q)\|\,ds\right)+R_b\int_0^b \left\|f(s,\tilde{z}_s^q+\widetilde{y}_s)\right\|ds \\
&\le bR_bM_2M_3\left(\|x_1\|+R_bH\|\varphi\|_{\mathscr{B}}+R_b\int_0^b l_{q'}(s)\,ds\right)+R_b\int_0^b l_{q'}(s)\,ds,
\end{aligned}
$$

where $q':=K_b\,q+q_0$, with $q_0:=\Big(K_bR_bH+M_b\Big)\|\varphi\|_{\mathscr{B}}$. Hence

$$
q\le\Big(1+R_bM_2M_3b\Big)R_b\int_0^b l_{q'}(s)\,ds+R_bM_2M_3b\Big(\|x_1\|+R_bH\|\varphi\|_{\mathscr{B}}\Big).
$$

Dividing both sides by q and noting that $q'=K_bq+q_0\to+\infty$ as $q\to+\infty$, we obtain that

$$
1\le\Big(1+R_bM_2M_3b\Big)R_b\left(\frac{\int_0^b l_{q'}(s)\,ds}{q}\right)+\frac{R_bM_2M_3b\Big(\|x_1\|+R_bH\|\varphi\|_{\mathscr{B}}\Big)}{q}
$$

and

$$
\liminf_{q\to+\infty}\left(\frac{\int_0^b l_{q'}(s)\,ds}{q}\right)=\liminf_{q\to+\infty}\left(\frac{\int_0^b l_{q'}(s)\,ds}{q'}\,\frac{q'}{q}\right)=lK_b.
$$

Thus we have, $1\le\Big(1+R_bM_2M_3b\Big)R_bK_bl$, and this contradicts (4). Hence for some positive number q, we must have $\Gamma(B_q)\subset B_q$.

Step 2. $\Gamma: E_b^0\to E_b^0$ is continuous. In fact let $\Gamma:=\Gamma_1+\Gamma_2$, where

$$
(\Gamma_1 z)(t)=\int_0^t R(t-s)f(s,\widetilde{z}_s+\widetilde{y}_s)\,ds \quad\text{and}\quad (\Gamma_2 z)(t)=\int_0^t R(t-s)Cu_z(s)\,ds.
$$

Let $\{z^n\}_{n\geq 1} \subset E_b^0$ with $z^n \to z$ in E_b^0. Then there exists a number $q > 1$ such that $\|z^n(t)\| \leq q$ for all n and $a.e.\ t \in I$. So $z^n, z \in B_q$. By $(\mathbf{H_4}) - (\mathbf{i}),\ f(t, \tilde{z}_t^n + \widetilde{y}_t) \to f(t, \widetilde{z}_t + \widetilde{y}_t)$ for each $t \in [0, b]$. And by $(\mathbf{H_4}) - (\mathbf{ii})$,

$$\|f(t, \tilde{z}_t^n + \widetilde{y}_t) - f(t, \widetilde{z}_t + \widetilde{y}_t)\| \leq 2l_{q'}(t).$$

Then we have

$$\|\Gamma_1 z^n - \Gamma_1 z\|_{\mathscr{C}} \leq R_b \int_0^b \|f(s, \tilde{z}_s^n + \widetilde{y}_s) - f(s, \widetilde{z}_s + \widetilde{y}_s)\| \, ds \longrightarrow 0,\ as\ n \to +\infty$$

by dominated convergence Theorem. Also we have that

$$\|\Gamma_2 z^n - \Gamma_2 z\|_{\mathscr{C}} \leq R_b^2 M_2 M_3 b \int_0^b \|f(s, \tilde{z}_s^n) - f(s, \widetilde{z}_s)\| \, ds \longrightarrow 0,\ as\ n \to +\infty$$

by dominated convergence Theorem. Thus

$$\|\Gamma z^n - \Gamma z\| \leq \|\Gamma_1 z^n - \Gamma_1 z\| + \|\Gamma_2 z^n - \Gamma_2 z\| \longrightarrow 0,\ as\ n \to +\infty.$$

Hence Γ is continuous on E_b^0.

<u>**Step 3.**</u> $\Gamma(B_q)$ is equicontinuous on $[0, b]$. In fact let $t_1, t_2 \in I,\ \ t_1 < t_2$ and $z \in B_q$, we have

$$\begin{aligned}
&\|(\Gamma z)(t_2) - (\Gamma z)(t_1)\| \\
&\leq \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\| \|f(s, \widetilde{z}_s + \widetilde{y}_s) + Cu_z(s)\| \, ds \\
&\quad + \int_{t_1}^{t_2} \|R(t_2 - s)\| \|f(s, \widetilde{z}_s + \widetilde{y}_s) + Cu_z(s)\| \, ds \\
&\leq \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\| l_{q'}(s) \, ds \\
&\quad + \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\| M_2 M_3 \left(\|x_1\| + R_b H \|\varphi\|_{\mathscr{B}} + R_b \int_0^b l_{q'}(\tau) \, d\tau \right) ds \\
&\quad + \int_{t_1}^{t_2} \|R(t_2 - s)\| l_{q'}(s) \, ds \\
&\quad + \int_{t_1}^{t_2} \|R(t_2 - s)\| M_2 M_3 \left(\|x_1\| + R_b H \|\varphi\|_{\mathscr{B}} + R_b \int_0^b l_{q'}(\tau) \, d\tau \right) ds.
\end{aligned}$$

By the continuity of $\big(R(t)\big)_{t\geq 0}$ in the operator-norm toplogy, the dominated convergence Theorem, we conclude that the right hand side of the above inequality tends to zero and independent of z as $t_2 \to t_1$. Hence $\Gamma(B_q)$ is equicontinuous on I.

<u>**Step 4.**</u> We show that the Mönch's condition holds.

Suppose that $D \subseteq B_q$ is countable and $D \subseteq \overline{co}(\{0\} \cup \Gamma(D))$. We shall show that $\beta(D) = 0$, where β is the Hausdorff MNC. Without loss of generality, we may assume that $D = \{z^n\}_{n\geq 1}$. Since Γ maps B_q into an equicontinuous family, $\Gamma(D)$ is also equicontinuous on I.

By $(\mathbf{H_3}) - (\mathbf{ii})$, $(\mathbf{H_4}) - (\mathbf{iii})$ and Lemma 2.13, we have that

$$
\begin{aligned}
&\beta\Big(\{u_{z^n}(t)\}_{n\geq 1}\Big) \\
&\leq L_W(t)\beta\left(\{x_1 - R(b)\varphi(0)\}\right) + L_W(t)\beta\left(\left\{\int_0^b R(t-b)f\Big(s, \{\tilde{z}_s^n + \widetilde{y}_s\}_{n\geq 1}\Big)\,ds\right\}_{n\geq 1}\right) \\
&\leq 2R_b L_W(t)\left(\int_0^b h(s)\beta\left(\{\tilde{z}_s^n\}_{n\geq 1} + \{\widetilde{y}_s\}\right)ds\right) \\
&\leq 2R_b L_W(t)\left(\int_0^b h(s)\beta\left(\{\tilde{z}_s^n\}_{n\geq 1}\right)\,ds\right), \text{ since } \Big\{\widetilde{y}_s :\ s\in[0,b]\Big\} \text{ is compact} \\
&\leq 2R_b L_W(t)\left(\int_0^b h(s)\sup_{-\infty<\theta\leq 0}\beta\left(\{\tilde{z}_s^n(\theta)\}_{n\geq 1}\right)\,ds\right) \\
&\qquad\qquad \Big(\text{by Lemma 2.11, since } D = \{z^n\}_{n\geq 1} \text{ is equicontinuous}\Big) \\
&\leq 2R_b L_W(t)\left(\int_0^b h(s)\,ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\beta\Big(\{(\Gamma z^n)(t)\}_{n\geq 1}\Big) &\leq \beta\left(\left\{\int_0^t R(t-s)f(s,\{\tilde{z}_s^n + \widetilde{y}_s\}_{n\geq 1})\,ds\right\}_{n\geq 1}\right) \\
&\quad + \beta\left(\left\{\int_0^t R(t-s)u_{z^n}(s)\,ds\right\}_{n\geq 1}\right) \\
&\leq 2R_b\left(\int_0^b h(s)\,ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right) \\
&\quad + 2R_b M_2\left(\int_0^b L_W(s)\,ds\right)2R_b\left(\int_0^b h(s)\,ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right) \\
&\leq 2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right) \\
&\quad + 2R_b M_2\|L_W\|_{L^1}2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right).
\end{aligned}
$$

It follows that

$$\begin{aligned}
\beta\Big(\Gamma(D)(t)\Big) &\leq 2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)+2R_bM_2\|L_W\|_{L^1}2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)\\
&\leq \Big(1+2R_bM_2\|L_W\|_{L^1}\Big)2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)\\
&= \tau\sup_{0\leq t\leq b}\beta\Big(D(t)\Big).
\end{aligned}$$

Since D and $\Gamma(D)$ are equicontinuous on $[0,b]$ and D is bounded, it follows by Lemma 2.11 that $\beta\Big(\Gamma(D)\Big)\leq\tau\beta\Big(D\Big)$, where τ is as defined in ($\mathbf{H_5}$). Thus from the Mönch condition, we get that

$$\beta\Big(D\Big)\leq\beta\Big(\overline{co}(\{0\}\cup\Gamma(D)\Big)=\beta\Big(\Gamma(D)\Big)\leq\tau\beta\Big(D\Big),$$

and since $\tau<1$, this implies $\beta\Big(D\Big)=0$, which implies that D is relatively compact as desired in B_q and the Mönch condition is satisfied. We conclude by Theorem 2.14, that Γ has a fixed point z in B_q. Then $x=z+y$ is a fixed point of P in E_b and thus equation (1) is controllable on $[0,b]$.

we now illustrate our main result by the following example.

4. Example

(5)
$$\begin{cases}
\dfrac{\partial v}{\partial t}(t,\xi)=\dfrac{\partial v}{\partial \xi}(t,\xi)+\displaystyle\int_0^t\zeta(t-s)\frac{\partial v}{\partial \xi}(s,\xi)\,ds+\int_{-\infty}^0\alpha(\theta)g(t,v(t+\theta,\xi))\,d\theta+\eta\omega(t,\xi)\\
\qquad\qquad\qquad\text{for } t\in I=[0,1] \text{ and } \xi\in[0,\pi]\\
v(t,0)=v(t,\pi)=0\ \text{ for } t\in[0,1]\\
v(\theta,\xi)=\phi(\theta,\xi)\ \text{ for } \theta\in]-\infty,0] \text{ and } \xi\in[0,\pi],
\end{cases}$$

where $\eta>0$, $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and Lipschitzian with respect to the second variable, the initial data function $\phi:\mathbb{R}^-\times[0,\pi]\to\mathbb{R}$ is a given function, $\omega:[0,1]\times[0,\pi]\to\mathbb{R}$ continuous in t and $\omega(t,0)=\omega(t,\pi)=0$, $\alpha:\mathbb{R}^-\to\mathbb{R}$ is continuous, $\alpha\in L^1(\mathbb{R}^-,\mathbb{R})$ and $\zeta\in W^{1,1}(\mathbb{R}^+,\mathbb{R}^+)$.

Let $X = U = \mathscr{C}_0([0,\pi],\mathbb{R})$, the space of all continuous functions from $[0,\pi]$ to $\mathbb{R}$ vanishing at 0 and π equipped with the uniform norm topology, and the phase space $\mathscr{B} = BUC(\mathbb{R}^-, X)$, the the space of uniformly bounded continuous functions endowed with the following norm

$$\|\varphi\|_{\mathscr{B}} = \sup_{\theta \leq 0} \|\varphi(\theta)\|.$$

Then, the space $BUC(\mathbb{R}^-, X)$ satisfies axioms $(\mathbf{A_1})$, $(\mathbf{A_2})$ and $(\mathbf{A_3})$.

We define $A : \mathscr{D}(A) \subset X \to X$ by:

$$\begin{cases} \mathscr{D}(A) = \Big\{ y \in X : y' \text{ exists and } y' \in X \Big\} \\ Ay = y'. \end{cases}$$

Theorem 4.1. (Theorem 4.1.4, p. 82 of [21]) A is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on $\mathscr{C}_0([0,\pi],\mathbb{R})$.

Moreover, the C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ generated by A above and defined by

$$T(t)y(s) = y(t+s) \ \text{ for } y \in X,$$

is not compact for $t > 0$ and is operator-norm continuous for $t > 0$. Thus by Theorem 2.5, the corresponding resolvent operator is operator-norm continuous for $t > 0$.

Now define

$$x(t)(\xi) \ = \ v(t,\xi), \quad x'(t)(\xi) \ = \ \frac{\partial v(t,\xi)}{\partial t}, \quad \omega(t,\xi) = u(t)(\xi).$$

$$\varphi(\theta)(\xi) = \phi(\theta,\xi) \ \text{ for } \theta \in]-\infty,0] \text{ and } \xi \in [0,\pi].$$

$$f(t,\psi)(\xi) \ = \ \int_{-\infty}^{0} \alpha(\theta) g(t,\psi(\theta)(\xi))\, d\theta \ \text{ for } \theta \in]-\infty,0] \text{ and } \xi \in [0,\pi].$$

$C : X \to X$ be defined by $\Big(Cu(t)\Big)(\xi) \ = Cu(t)(\xi) = \eta\,\omega(t,\xi)$.

$$(\gamma(t)x)(\xi) \ = \ \zeta(t)\frac{\partial}{\partial \xi} v(t,\xi) \ \text{ for } \ t \in [0,1], \ \ x \in \mathscr{D}(A) \text{ and } \ \xi \in [0,\pi].$$

We suppose that $\varphi \in BUC(\mathbb{R}^-, X)$. Then, equation (5) is then transformed into the following abstract form

$$(6)\qquad \begin{cases} x'(t) = Ax(t) \ + \ \displaystyle\int_0^t \gamma(t-s)x(s)ds \ + \ f(t,x_t) + Cu(t) \ \text{ for } t \in I = [0,1], \\ \\ x_0 = \varphi \in \mathscr{B}. \end{cases}$$

Suppose there exists a continuous function $p \in L^1(I; \mathbb{R}^+)$ such that

$$|g(t,y_1) - g(t,y_2)| \leq p(t)|y_1 - y_2| \ \text{ for } t \in I \text{ and } y_1, y_2 \in \mathbb{R}.$$

and

$$g(t,0) = 0 \ \text{ for } t \in I.$$

One can see that f is Lipschitz continuous with respect to the second variable and moreover for $\varphi \in \mathscr{B}$, we have we have

$$\sup_{\|\varphi\|_{\mathscr{B}} \leq q} \left\| f(t,\varphi) \right\| \leq q \, \|\alpha\| \, p(t).$$

So f satisfies $(\mathbf{H_4}) - (\mathbf{i})$ and $(\mathbf{H_4}) - (\mathbf{ii})$ with $l_q(t) = q\,\|\alpha\|\, p(t)$. Also f satisfies $(\mathbf{H_4}) - (\mathbf{iii})$ by condition $(\mathbf{viii})$ of Theorem 2.10, since f is Lipschitz. Now for $\xi \in [0,\pi]$, the operator W is given by

$$(Wu)(\xi) = \eta \int_0^1 R(1-s)\omega(s,\xi)\, ds.$$

Assuming that W satisfies $(\mathbf{H_3})$, then all the conditions of Theorem 3.1 hold and equation (6) is controllable.

5. Conclusion

This paper contains the controllability of some partial functional integrodifferential differential equation with infinite delay in Banach spaces, obtained by using the Hausdorff Measure of Noncompactness and the Mönch fixed point theorem. The result shows that without assuming the compactness of the resolvent operator for the associated undelayed part, the Mönch fixed point theorem can effectively be used to obtain controllability results under some sufficient conditions.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] R. Atmania and S. Mazouzi, Controllability of semilinear integrodifferential equations with nonlocal conditions, Electron. J. Differential Equations. 2005 (2005), 1-9.

[2] S. Baghli, M. Benchohra, K. Ezzinbi, Controllability Results for Semilinear Functional and Neutral Functional Evolution Equations with Infinite Delay, Surveys in Mathematics and its Applications, (2009).

[3] J. Banas, K. Goebel, Measures of Noncompactness in Banach Spaces, vol. 60 of Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, (1980).

[4] K. Balachandran and R. Sakthivel, Controllability of functional semilinear integrodifferential systems in Banch spaces, J. Math. Anal. Appl. 255 (2001), 447-457.

[5] K. Balachandran and J. Y. Park, Existence of solutions and controllability of nonlinear integrodifferential systems in Banach spaces, Math. Probl. Eng. 2003 (2003), 65-79.

[6] Y. K. Chang, J. J. Nieto and W. S. Li, Controllability of semilinear differential systems with nonlocal initial conditions in Banach spaces, J. Optim. Theory Appl. 142 (2008), 267-273.

[7] R. Grimmer, Resolvent operators for integral equations in a Banach space, Tran. Amer. Math. Soc. 273 (1983), 333-349.

[8] W. Desch, R. Grimmer and W. Schappacher, Some considerations for linear integrodifferential equations, J. Math. Anal. Appl. 104 (1984), 219-234.

[9] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, Nonlinear Anal. 4 (1980), 985-999.

[10] J. Liang, J. H. Liu and Ti-Jun Xiao, Nonlocal problems for integrodifferential equations, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 15 (2008), 815-824.

[11] J. R. Wang, Z. Fan and Y. Zhou, Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces, J. Optim. Theory Appl. 154 (2012), 292-302.

[12] K. Ezzinbi, G. Degla and P. Ndambomve, Controllability for some partial functional integrodifferential equations with nonlocal condition in Banach spaces, Discuss. Math. Differ. Incl. Control Optim. 35 (2015) 25-46.

[13] K. Ezzinbi, H. Toure and I. Zabsonre, Existence and regularity of solutions for some partial functional integrodifferential equations in Banach spaces, Nonlinear Anal. 70 (2009), 2761-2771.

[14] Hale J and Kato J. Phase space for retarded equations with infinite delay, Funkcial. Ekvac. 21 (1978), 11-41.

[15] A. Lorenzi and E. Sinestrari, An inverse problem in the theory of materials with memory, Nonlinear Anal. 12 (1998), 1317-1335.

[16] J. A. Machado, C. Ravichandran, M. Rivero and J. J. Trujillo, Controllability results for impulsive mixed-type functional integro-differential evolution equations with nonlocal conditions, Fixed Point Theory Appl. 2013 (2013), 66.

[17] M. Li, M. Wang and F. Zhang, Controllability of impulsive functional differential systems in Banach spaces, Chaos, Solitons and Fractals 29 (2006), 175-181.

[18] M.D. Quinn and N. Carmichael, An approach to nonlinear control problem using fixed point methods,degree theory and pseudo-inverses, Numer. Funct. Anal. Optim. 7 (1984), 197-219.

[19] R. Sakthivel, N. I. Mahmudov and J. J. Nieto, Controllability for a class of fractional-order neutral evolution control systems, Appl. Math. Comput. 218 (2012), 10334-10340.

[20] S. Selvi and M. M. Arjunan, Controllability results for impulsive differential systems with finite delay, J. Nonlinear Sci. Appl. 5 (2012), 206-219.

[21] I. I. Vrabie, C_0-semigroups and applications, Math. Stud. (2003).