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ON ϕ-COINCIDENCE AND COMMON ϕ- FIXED POINTS IN ORDERED METRIC SPACES

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Abstract. The aim of this work is to obtain some existence and uniqueness fixed point theorems for mixed g-monotone mapping in any number of variables under general contractive conditions in partially ordered metric spaces by using the condition of weak compatibility. Our results are different, more natural and generalizations of many results on multidimensional fixed points. For illustration of the effectiveness of our generalizations, some examples are equipped in this paper.

Keywords: ϕ-coincidence point; Common ϕ- fixed point; Ordered metric spaces; Mixed g-monotone property; Weakly compatible mappings.

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1. Introduction and preliminaries

Bhaskar and Lakshmikantham [4] introduced the notion of mixed monotone property and coupled fixed point for contractive operator of the form $F : X \times X \to X$, in the framework of partially ordered metric spaces and then established some existence and uniqueness for fixed and coupled

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fixed points of F (x is a fixed point of F if $F(x,x) = x$). They also illustrated these important results by proving the existence and uniqueness of the solution for a periodic boundary value problem. Afterwards, In [12] Lakshmikantham and *Ćirić* extended these results by defining the mixed g-monotone property and proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings $F : X \times X \to X$ and $g : X \to X$ in partially ordered metric spaces as mentioned in the next dialogue. Then many authors focused on coupled fixed point theory and proved remarkable results (see e.g., [9, 13, 18])

Definition 1.1. [7] A tripled $(X,d,\preceq)$ is called an ordered metric space iff

(i): (X,d) is a metric space,

(ii): $(X,\preceq)$ is a partially ordered set.

The results of Lakshmikantham and *Ćirić* are as follows:

Definition 1.2. [12] Let $(X,\preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1,x_2 \in X, \ \ g(x_1) \preceq g(x_2) \ \text{ implies } \ F(x_1,y) \preceq F(x_2,y)$$

and

$$y_1,y_2 \in X, \ \ g(y_1) \preceq g(y_2) \ \text{ implies } \ F(x,y_1) \succeq F(x,y_2).$$

If g is the identity mapping, we obtain the Bhaskar and Lakshmikantham's mixed monotonicity notion for the mapping F.

Definition 1.3. [12] An element $(x,y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $g : X \to X$ if

$$F(x,y) = g(x), \ \ F(y,x) = g(y).$$

Also, if g is the identity mapping, then (x,y) is called a coupled fixed point of the mapping F.

Definition 1.4. [12] Let X be a non-empty set and $F : X \times X \to X$ and $g : X \to X$. We say F and g are commuting if,

$$F(gx,gy) = gF(x,y)$$

for all $x, y \in X$.

Theorem 1.1. [12] *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Assume that there is a function $\varphi : [0,+\infty) \to [0,+\infty)$ with $\varphi(t) < t$ and $\lim_{r\to t_+} \varphi(r) < t$ for each $t >$?, and also suppose that $F : X \times X \to X$ and $g : X \to X$ are mappings such that F has the mixed g-monotone property and*

$$d\big(F(x,y),F(u,v)\big) \leq \varphi\left(\frac{d(g(x),g(u))+d(g(y),g(v))}{2}\right),$$

for all $x,y,u,v \in X$, for which $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$.
Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F, and also suppose that either

- **(a):** *F is continuous, or*
- **(b):** *X has the following properties*
  - **(i):** *If a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
  - **(ii):** *If a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0,y_0)$ and $g(y_0) \succeq F(y_0,x_0)$, then there exist $x,y \in X$ such that $g(x) = F(x,y)$ and $g(y) = F(y,x)$, i.e., F and g have a coupled coincidence point.

The authors in [12] endowed the product space $X \times X$ with the following partial order:

$$\text{for } (x,y),\ (u,v) \in X\times X,\ \ (x,y) \preceq (u,v) \ \Leftrightarrow\ x \preceq u \text{ and } y \succeq v.$$

They also considered some additional conditions on the product space $X \times X$ to ensure the existence and uniqueness of a coupled common fixed point.

Theorem 1.2. [12] *In addition to the hypothesis of Theorem 1.1, suppose that for every $(x,y),(x^*,y^*) \in X \times X$, there exists $(u,v) \in X \times X$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. Then F and g have a unique coupled common fixed point, i.e., there exists a unique $(x,y) \in X \times X$ such that*

$$x = g(x) = F(x,y),\ \ y = g(y) = F(y,x).$$

Berinde and Borcut [2] introduced the concept of tripled fixed point for nonlinear contractive mappings of the form $F : X^3 \to X$, in partially ordered complete metric spaces and obtained existence and uniqueness theorems of tripled fixed points for some general classes of contractive type mappings (see also, [5]).

Definition 1.5. [2] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \times X \to X$. The mapping F is said to has the mixed monotone property if, for any $x, y, z \in X$,

$$x_1, x_2 \in X,\ \ x_1 \preceq x_2 \ \ \Rightarrow \ \ F(x_1, y, z) \preceq F(x_2, y, z),$$

$$y_1, y_2 \in X,\ \ y_1 \preceq y_2 \ \ \Rightarrow \ \ F(x, y_1, z) \succeq F(x, y_2, z),$$

$$z_1, z_2 \in X,\ \ z_1 \preceq z_2 \ \ \Rightarrow \ \ F(x, y, z_1) \preceq F(x, y, z_2).$$

Definition 1.6. [2] An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F : X^3 \to X$ if

$$F(x, y, z) = x,\ \ F(y, x, y) = y \ \text{ and } \ F(z, y, x) = z.$$

Also, Berinde and Borcut [2] used the following partial order on the product space X^3 :

$$(x, y, z) \leq (u, v, w) \ \ \Leftrightarrow \ \ x \preceq u,\ \ y \succeq v,\ \ z \preceq w,$$

to prove the following result.

Theorem 1.3. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ have the mixed monotone property and there exist $j, r, l \geq 0$ with $j + r + l < 1$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + rd(y, v) + ld(z, w),$$

for any $x, y, z \in X$ for which $x \preceq u$, $v \preceq y$ and $z \preceq w$. Suppose either F is continuous or X has the following properties

(1): *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*

(2): *if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that, $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, z_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that:

$$F(x, y, z) = x,\ \ F(y, x, y) = y \ \text{ and } \ F(z, y, x) = z,$$

that is, F has a triple fixed point.

After that, in the paper of Karapinar and Luong [10], the quadruple fixed point is considered and some new related fixed point theorems are obtained.

Definition 1.7. [10] Let $(X,\preceq)$ be a partially ordered set and $F : X^4 \to X$ be a mapping. We say that F has the mixed monotone property if $F(x,y,z,w)$ is monotone non-decreasing in x and z, and monotone non-increasing in y and w, that is, for any $x,y,z,w \in X$,

$$\begin{aligned}
x_1,x_2 \in X,\ x_1 \preceq x_2 &\Rightarrow F(x_1,y,z,w) \preceq F(x_2,y,z,w),\\
y_1,y_2 \in X,\ y_1 \preceq y_2 &\Rightarrow F(x,y_1,z,w) \succeq F(x,y_2,z,w),\\
z_1,z_2 \in X,\ z_1 \preceq z_2 &\Rightarrow F(x,y,z_1,w) \preceq F(x,y,z_2,w),\\
w_1,w_2 \in X,\ w_1 \preceq w_2 &\Rightarrow F(x,y,z,w_1) \succeq F(x,y,z,w_1).
\end{aligned}$$

Definition 1.8. [10] An element $(x,y,z,w) \in X^4$ is called a quadruple fixed point of $F : X^4 \to X$ if

$$F(x,y,z,w) = x, F(y,z,w,x) = y, F(z,w,x,y) = z \text{ and } F(w,x,y,z) = w.$$

Finally, about $N-$ fixed point or multidimensional case, we have to distinguish between two types of definitions, to see the difference between them one can read the note of Karapinar et al. [11].

- $*$**:** In some cases, the arguments are ordered, for instance, the following notion was given in [8] and also mentioned in Paknazar et al., (Definition (1.12)):

  $$x_i = F(x_i,x_{i-1},\ldots,x_2,x_1,x_2\ldots,x_{n-i+1}) \text{ for all } i \in \{1,2,\ldots,n\}.$$

  This definition can be interpreted as an extension of the second equation of Berinde and Borcut's notion, that is, $y = F(y,x,y)$.
- $*$**:** In other cases, the arguments are permuted, for instance, the notion of N-fixed point introduced in Paknazar et al.[15] (Definition ?(2.1)) is as follows:

  $$x_i = F(x_i,x_{i+1},\ldots,x_{n-1},x_n,x_1\ldots,x_{i-1}) \text{ for all } i \in \{1,2,\ldots,n\}.$$

  This notion generalizes Karapinar and Luong's quadrupled concept.

Recently, Rold$\acute{a}$n et al. [17] obtained some existence and uniqueness theorems that extend the previous mentioned results for nonlinear mappings of any number of arguments, not necessarily permuted or ordered, in the framework of partially ordered metric spaces by using weaker contraction conditions.

Next we state some definitions and results that we use to perform this work.

Definition 1.9. [17] Let $g : X \to X$ be mapping and $(X,d,\preceq)$ be an ordered metric space, then X is said to have the sequential g- monotone property if it verifies the following properties:

(i): If $\{x_m\}_{m\geq 0}$ is a non-decreasing sequence in X and $\lim_{m\to\infty} x_m = x$, then $gx_m \preceq gx$ for all $m \geq 0$,

(ii): If $\{y_m\}_{m\geq 0}$ is a non-increasing sequence in X and $\lim_{m\to\infty} y_m = y$, then $gy_m \succeq gy$ for all $m \geq 0$.

If g is the identity mapping, then X is said to have the sequential monotone property.

Definition 1.10. [17] We say that F and g are commuting if $gF(x_1,\ldots,x_n) = F(gx_1,\ldots,gx_n)$ for all $x_1,\ldots,x_n \in X$, and they are weakly compatible if they commute at their coincidence points.

Fix a partition $\{A,B\}$ of the set $\Lambda_n = \{1,2,\ldots,n\}$, that is, $A\cup B = \Lambda_n$ and $A\cap B = \emptyset$, we will denote

$$\Omega_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A \quad \text{and} \quad \sigma(B) \subseteq B\}$$

and

$$\acute{\Omega}_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B \quad \text{and} \quad \sigma(B) \subseteq A\}.$$

If $(X,\preceq)$ is a partially ordered space, $x,y \in X$ and $i \in \Lambda_n$, we will use the following notation

$$x \preceq_i y \Leftrightarrow \begin{cases} x \preceq y, & i \in A, \\ x \succeq y, & i \in B. \end{cases}$$

Definition 1.11. [17] Let $(X,\preceq)$ be a partially ordered set and $F : X^n \to X$ be a mapping. We say that F has the mixed g-monotone property if F is g-monotone non-decreasing in arguments of A and g-monotone non-increasing in arguments of B, that is, for all $x_1,x_2,\dots,x_n,y,z \in X$ and $i \in \Lambda_n$.

$$gy \preceq gz \Rightarrow F(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n) \preceq_i F(x_1,\dots,x_{i-1},z,x_{i+1},\dots,x_n).$$

Henceforth, let $\sigma_1,\dots,\sigma_n,\tau:\Lambda_n \to \Lambda_n$ be n+1 mappings and ϕ be the (n+1)-tuple $(\sigma_1,\dots,\sigma_n,\tau)$.

Definition 1.12. [17] A point $(x_1,x_2,\dots,x_n) \in X^n$ is called a $\phi-$coincidence point of the mappings F and g if

$$F(x_{\sigma_i(1)},\dots,x_{\sigma_i(n)}) = gx_{\tau(i)} \text{ for all } i.$$

If g is the identity mapping on X, then $(x_1,\dots,x_n) \in X^n$ is called a $\phi-$ fixed point of the mappings F.

Definition 1.13. A point $(x_1,x_2,\dots,x_n) \in X^n$ is called a common $\phi-$fixed point of the mappings F and g if

$$F(x_{\sigma_i(1)},\dots,x_{\sigma_i(n)}) = gx_{\tau(i)} = x_{\tau(i)} \text{ for all } i.$$

Theorem 1.4. [17] *Let $(X,d,\preceq)$ be a complete ordered metric space. Let $\phi = (\sigma_1,\sigma_2,\dots,\sigma_n,\tau)$ be a $(n+1)$-tuple of mappings from $\{1,2,\dots,n\}$ into itself such that $\tau \in \Omega_{A,B}$ is a permutation and verifying that $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F: X^n \to X$ and $g: X \to X$ be two mappings such that F has the mixed g-monotone property on X, $F(X^n) \subseteq g(X)$ and g commutes with F. Assume that there exists $k \in [0,1)$ verifying*

$$d(F(x_1,x_2,\dots,x_n),F(y_1,y_2,\dots,y_n)) \leq k \max_{1\leq i\leq n} d(gx_i,gy_i)$$

for which $gx_i \preceq_i gy_i$ for all i. Suppose either F is continuous or X has the sequential g-monotone property. If there exist $x_0^1,\dots.x_0^n \in X$ verifying

$$gx_0^{\tau(i)} \preceq_i F(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\dots,x_0^{\sigma_i(n)}), \textit{ for all } i.$$

Then F and g have, at least, one $\phi-$ coincidence point.

In this paper, Inspired by Theorem 1.4, we prove a ϕ- coincidence point and common ϕ- fixed point theorems of contractive type mappings that not necessarily commuting but only weakly compatible.

2. Main results

Theorem 2.1. *Let $(X, \preceq, d)$ be an ordered metric space and $\phi = (\sigma_1, \ldots, \sigma_n, \tau)$ be a $(n+1)$-tuple of mappings from Λ_n into itself such that $\tau \in \Omega_{A,B}$ is a permutation, $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \acute{\Omega}_{A,B}$ if $i \in B$. Let $F : X^n \to X$ and $g : X \to X$ be two given mappings such that F has the mixed g-monotone property, $F(X^n) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Assume that there exist $a_i \in R, i \in \Lambda_n$ verifying $\sum_{i=1}^n a_i < 1$ and*

$$d(F(x^1, \ldots, x^n), F(y^1, \ldots, y^n)) \leq \sum_{i=1}^n a_i d(gx^i, gy^i), \tag{2.1}$$

for which $gx^i \preceq_i gy^i$. If there exist $x_0^1, \ldots, x_0^n \in X$ such that

$$gx_0^{\tau(i)} \preceq_i F(x_0^{\sigma_i(1)}, \ldots, x_0^{\sigma_i(n)}), \text{ for all } i \in \Lambda_n \tag{2.2}$$

and X has the sequential g-monotone property. Then F and g have, at least, one $\phi-$ coincidence point.

Proof. As τ is a permutation of the set $\{1, 2, \ldots, n\}$, then $\{\tau(1), \ldots, \tau(n)\} = \{1, \ldots, n\}$. Also, for the point $(x_0^1, \ldots, x_0^n)$ there exists $(x_1^1, \ldots, x_1^n) \in X^n$ such that

$$gx_1^{\tau(i)} = F(x_0^{\sigma_i(1)}, \ldots, x_0^{\sigma_i(n)}), \text{ for all } i \in \Lambda_n,$$

this can be done because $F(X^n) \subseteq g(X)$. For this point $(x_1^1, \ldots, x_1^n)$ we can find another point $(x_2^1, \ldots, x_2^n) \in X^n$ such that

$$gx_2^{\tau(i)} = F(x_1^{\sigma_i(1)}, \ldots, x_1^{\sigma_i(n)}), \text{ for all } i \in \Lambda_n.$$

Continuing this process we can construct the sequences $\{x_m^1\}_{m\geq 0}$, …, $\{x_m^n\}_{m\geq 0}$ such that

$$gx_{m+1}^{\tau(i)} = F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(n)}), \text{ for all } m \geq 0 \text{ and } i \in \Lambda_n.$$

By induction methodology for $m \geq 0$ we shall prove that

$$gx_m^i \preceq_i gx_{m+1}^i, \text{ for all } i \in \Lambda_n. \tag{2.3}$$

Indeed, from equation (2.2) we have

$$gx_0^{\tau(i)} \preceq_i F(x_0^{\sigma_i(1)}, \ldots, x_0^{\sigma_i(n)}) = gx_1^{\tau(i)}.$$

Since $\tau(i) \in A \Leftrightarrow i \in A$, $\tau(i) \in B \Leftrightarrow i \in B$ and $\{\tau(i)\}_{i=1}^{n} = \{i\}_{i=1}^{n}$, we get

$$gx_0^{\tau(i)} \preceq_{\tau(i)} gx_1^{\tau(i)}, \text{ or } gx_0^{i} \preceq_i gx_1^{i}, \text{ for all } i \in \Lambda_n. \tag{2.4}$$

Suppose that (2.3) is true for some $m \geq 0$ and we are going to prove it for $m+1$. Now we have to distinguish between wether $i \in A$ or $i \in B$,

(Case 1): Suppose that $i \in A$ $(\sigma_i \in \Omega_{A,B})$.
$gx_{m+1}^{\tau(i)} = F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)})$, for this argument $x_m^{\sigma_i(j)}$ we have two subcases,

(I): If $j \in A$ (where F is g-monotone non-decreasing), $\sigma_i(j) \in A$ (i.e., $gx_m^{\sigma_i(j)} \preceq gx_{m+1}^{\sigma_i(j)}$). Thus, $F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \preceq F(x_m^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)})$.

(II): $j \in B$ (where F is g-monotone non-increasing), $\sigma_i(j) \in B$ (i.e., $gx_m^{\sigma_i(j)} \succeq gx_{m+1}^{\sigma_i(j)}$). Thus, $F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \preceq F(x_m^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)})$.

That is,

$$F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \preceq F(x_m^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \ \forall \ i \in A, j \in \Lambda_n$$

and

$$\begin{aligned}
gx_{m+1}^{\tau(i)} &= F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \\
&\preceq F(x_{m+1}^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \\
&\vdots \\
&\preceq F(x_{m+1}^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \\
&\vdots \\
&\preceq F(x_{m+1}^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_{m+1}^{\sigma_i(n)}) = gx_{m+2}^{\tau(i)}.
\end{aligned}$$

Therefore

$$gx_{m+1}^{i} \preceq gx_{m+2}^{i}, \ \ i \in A. \tag{2.5}$$

(Case 2): If $i \in B$ $(\sigma_i \in \Omega'_{A,B})$.
$gx_{m+1}^{\tau(i)} = F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)})$, for this argument $x_m^{\sigma_i(j)}$ we have two subcases,

(I): If $j \in A$ (where F is g-monotone non-decreasing), $\sigma_i(j) \in B$ (i.e., $gx_m^{\sigma_i(j)} \succeq gx_{m+1}^{\sigma_i(j)}$). Thus, $F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \succeq F(x_m^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)})$.

(II): $j \in B$ (where F is g-monotone non-increasing), $\sigma_i(j) \in A$ (i.e., $gx_m^{\sigma_i(j)} \preceq gx_{m+1}^{\sigma_i(j)}$).

Thus, $F(x_m^{\sigma_i(1)},\dots,x_m^{\sigma_i(j)},\dots,x_m^{\sigma_i(n)}) \succeq F(x_m^{\sigma_i(1)},\dots,x_{m+1}^{\sigma_i(j)},\dots,x_m^{\sigma_i(n)})$.

We conclude that

$$F(x_m^{\sigma_i(1)},\dots,x_m^{\sigma_i(j)},\dots,x_m^{\sigma_i(n)}) \succeq F(x_m^{\sigma_i(1)},\dots,x_{m+1}^{\sigma_i(j)},\dots,x_m^{\sigma_i(n)}) \ \forall \ i \in B, j \in \Lambda_n$$

and

$$\begin{aligned} gx_{m+1}^{\tau(i)} &= F(x_m^{\sigma_i(1)},\dots,x_m^{\sigma_i(j)},\dots,x_m^{\sigma_i(n)}) \\ &\succeq F(x_{m+1}^{\sigma_i(1)},\dots,x_{m+1}^{\sigma_i(j)},\dots,x_{m+1}^{\sigma_i(n)}) = gx_{m+2}^{\tau(i)}. \end{aligned}$$

Therefore

$$gx_{m+1}^{i} \succeq gx_{m+2}^{i}, \ \ i \in B. \tag{2.6}$$

From inequalities (2.5) and (2.6), we get

$$gx_{m+1}^{i} \preceq_i gx_{m+2}^{i}.$$

Thus, inequality (2.3) is true for any $i \in \Lambda_n$ and $m \geq 0$. Note that the inequality (2.3) implies ($gx_{m-1}^{\sigma_i(j)} \preceq_j gx_m^{\sigma_i(j)}$, if $i \in A$, or $gx_{m-1}^{\sigma_i(j)} \succeq_j gx_m^{\sigma_i(j)}$, if $i \in B$).

Then we use the previous fact and the contraction condition (2.1) to assert that the sequences $\{gx_m^i\}_{m\geq 0}$ are Cauchy for all $i \in \Lambda_n$ as follows:

$$\begin{aligned} d(gx_m^{\tau(i)},gx_{m+1}^{\tau(i)}) =& d(F(x_{m-1}^{\sigma_i(1)},\dots,x_{m-1}^{\sigma_i(n)}),F(x_m^{\sigma_i(1)},\dots,x_m^{\sigma_i(n)})) \leq \sum_{j=1}^{n} a_j d(gx_{m-1}^{\sigma_i(j)},gx_m^{\sigma_i(j)}) \\ &\leq \sum_{j=1}^{n} a_j \max_{1\leq i\leq n} d(gx_{m-1}^{\tau(i)},gx_m^{\tau(i)}). \end{aligned}$$

Define $\delta_m = \max_{1\leq i\leq n} d(gx_m^{\tau(i)},gx_{m+1}^{\tau(i)})$ and taking maximum above implies:

$$\begin{aligned} &\delta_m \leq \sum_{j=1}^{n} a_j \delta_{m-1}, \\ &\delta_m \leq \lambda \delta_{m-1}, \ \ \lambda = \sum_{j=1}^{n} a_j < 1, \\ &\delta_m \leq \cdots \leq \lambda^m \delta_0 \\ &\Rightarrow d(gx_m^{\tau(i)},gx_{m+1}^{\tau(i)}) \leq \delta_m \leq \lambda^m \delta_0. \end{aligned} \tag{2.7}$$

For a fixed i we use the triangle inequality and (2.7) to obtain:

$$\begin{aligned} d(gx_m^{\tau(i)}, gx_{m+p}^{\tau(i)}) &\leq d(gx_m^{\tau(i)}, gx_{m+1}^{\tau(i)}) + d(gx_{m+1}^{\tau(i)}, gx_{m+2}^{\tau(i)}) + \cdots + d(gx_{m+p-1}^{\tau(i)}, gx_{m+p}^{\tau(i)}) \\ &\leq (\lambda^m + \lambda^{m+1} + \cdots + \lambda^{m+p-1})\delta_0 \\ &\leq \lambda^m(1 + \lambda + \cdots + \lambda^{p-1})\delta_0 \\ &\leq \lambda^m \frac{1-\lambda^p}{1-\lambda}\delta_0 \ \ \to 0 \ \text{as} \ m \to \infty. \end{aligned}$$

Therefore, $\{gx_m^i\}_{m\geq 0}$ are Cauchy sequences (for all $i \in \Lambda_n$) in $g(X)$. By the completeness of $g(X)$, there exist $\{gx^1, \ldots, gx^n\} \in g(X)$, such that

$$gx_m^i \to gx^i, \ \text{as} \ n \to \infty \ \text{for all} \ i \in \Lambda_n. \tag{2.8}$$

Finally, we claim that the point $(x^1, \ldots, x^n)$ is $\phi-$ coincidence point of F and g. Suppose that X has the sequential g- monotone property, by (2.3) and (2.8) we have $gx_m^i \preceq_i gx_{m+1}^i$ and $gx_m^i \to gx^i$, as $m \to \infty$ for all $i \in \Lambda_n$, implying $gx_m^i \preceq_i gx^i$ and $(gx_m^{\sigma_i(j)} \preceq_j gx^{\sigma_i(j)}$ or $gx_m^{\sigma_i(j)} \succeq_j gx^{\sigma_i(j)})$. Now consider

(2.9)

$$\begin{aligned} d(F(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), gx^{\tau(i)}) &\leq d(F(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(n)})) + d(gx_{m+1}^{\tau(i)}, gx^{\tau(i)}) \\ &\leq \sum_{j=1}^{n} a_j d(gx_m^{\sigma_i(j)}, gx^{\sigma_i(j)}) + d(gx_{m+1}^{\tau(i)}, gx^{\tau(i)}). \end{aligned}$$

By (2.8), there exist $m_0, m_1, \ldots, m_n \in N$ such that

$$d(gx_{m+1}^{\tau(i)}, gx^{\tau(i)}) < \frac{\varepsilon}{2} \ \forall \ m \geq m_0, \ \ for\, \varepsilon > 0$$

and

$$d(gx_m^{\sigma_i(j)}, gx^{\sigma_i(j)}) < \frac{\varepsilon}{2na_j} \ \forall \ m \geq m_j \ \ , j \in \Lambda_n.$$

Taking $m \geq \mu = \max\{m_0, m_1, \ldots, m_n\}$ and using (2.9), we get

$$\begin{aligned} d(F(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), gx^{\tau(i)}) &\leq (a_1 d(gx_m^{\sigma_i(1)}, gx^{\sigma_i(1)}) + \cdots + a_n d(gx_m^{\sigma_i(n)}, gx^{\sigma_i(n)})) \\ &\leq (a_1 \frac{\varepsilon}{2na_1} + \cdots + a_n \frac{\varepsilon}{2na_n}) + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since ε is arbitrary, then $F(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}) = gx^{\tau(i)}$. That is, $(x^1, \ldots, x^n)$ is a $\phi-$ coincidence point of F and g.

Remark 2.1. The sequential monotonicity of the set X can be replaced by the continuity of the commutative mappings F and g.
Indeed, $g(gx_{m+1}^{\tau(i)}) = g(F(x_m^{\sigma_i(1)},\ldots,x_m^{\sigma_i(n)})) = F(gx_m^{\sigma_i(1)},\ldots,gx_m^{\sigma_i(n)})$, and then taking limit at $m\to\infty$ and using (2.8) implies, $g(gx^{\tau(i)}) = F(gx^{\sigma_i(1)},\ldots,gx^{\sigma_i(n)})$. That is $(gx^1,\ldots,gx^n)$ is a ϕ-coincidence point for F and g.

Remark 2.2. Theorem 2.1 is different from corollary 10 in [17] for many reasons. First, we do not assume the completeness of the whole space X, only we need the completeness of the subspace $g(X)$. Second, we omit the continuity condition of the mapping g. Third, we weaken the commutativity condition of the mappings F and g, we use the weak-compatibility. Finally, our proof is essentially different.

The conditions of Theorem 2.1 are not enough to prove the existence and uniqueness of the common ϕ-fixed point for the mappings F and g. We apply the condition of weak- compatibility of F and g to obtain the following fixed point theorem. For a product space X^n, we define a partial ordering in the following way: for all $(x^1,\ldots,x^n),(y^1,\ldots,y^n)\in X^n$,

$$(x^1,\ldots,x^n)\preceq(y^1,\ldots,y^n)\Leftrightarrow x^i\preceq_i y^i \text{ for all } i\in\Lambda_n$$

and $(x^1,\ldots,x^n)=(y^1,\ldots,y^n)\Leftrightarrow x^i=y^i,\ \forall\ i$. Let Φ be the set of all $\phi-$ coincidence points of F and g.

Theorem 2.2. *In addition to the the hypothesis of Theorem 2.1 suppose that for any two non comparable elements $(x^1,\ldots,x^n)$, $(y^1,\ldots,y^n)\in\Phi$ there exists $(u^1,\ldots,u^n)$ such that:* $(F(u^{\sigma_1(1)},\ldots,u^{\sigma_1(n)}),\ldots,F(u^{\sigma_n(1)},\ldots,u^{\sigma_n(n)}))$ *is comparable, at the same time, to* $(F(x^{\sigma_1(1)},\ldots,x^{\sigma_1(n)}),\ldots,F(x^{\sigma_n(1)},\ldots,x^{\sigma_n(n)}))$ *and* $(F(y^{\sigma_1(1)},\ldots,y^{\sigma_1(n)}),\ldots,F(y^{\sigma_n(1)},\ldots,y^{\sigma_n(n)}))$. *Provided that F and g are weakly compatible. Then F and g have a unique common ϕ- fixed point.*

Proof. Due to Theorem 2.1 the set Φ is non-empty. Assume that $(x^1,\ldots,x^n)$ and $(y^1,\ldots,y^n)$ are two ϕ- coincidence points of F and g, that is ,

$$gx^{\tau(i)} = F(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)})$$

and

$$gy^{\tau(i)} = F(y^{\sigma_i(1)}, \dots, y^{\sigma_i(n)}), \ \forall \ i \in \Lambda_n.$$

If $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$ are comparable, say, $x^i \preceq_i y^i$, then $d(gx^{\tau(i)}, gy^{\tau(i)}) = 0 \ \forall \, i$, i.e., $gx^i = gy^i \ \forall \, i$.

If not, by the assumption there is $(u^1, \dots, u^n) \in X^n$ such that $\big(F(u^{\sigma_1(1)}, \dots, u^{\sigma_1(n)}), \dots, F(u^{\sigma_n(1)}, \dots, u^{\sigma_n(n)})\big)$ is comparable, at the same time, to $\big(F(x^{\sigma_1(1)}, \dots, x^{\sigma_1(n)}), \dots, F(x^{\sigma_n(1)}, \dots, x^{\sigma_n(n)})\big)$ and $\big(F(y^{\sigma_1(1)}, \dots, y^{\sigma_1(n)}), \dots, F(y^{\sigma_n(1)}, \dots, y^{\sigma_n(n)})\big)$. Put $u_0^i = u^i \ \forall \, i$ and apply the same argument in Theorem 2.1, one can determine the sequences $\{u_m^1\}_{m\geq 0}$, $\{u_m^2\}_{m\geq 0}$, ..., $\{u_m^n\}_{m\geq 0}$ such that

$$gu_{m+1}^{\tau(i)} = F(u_m^{\sigma_i(1)}, \dots, u_m^{\sigma_i(n)}), \quad \text{for all } \ m, i.$$

Further, set $x_0^i = x^i$ and $y_0^i = y^i \ \forall \, i$, we can define the sequences $\{x_m^i\}_{m\geq 0}$ and $\{y_m^i\}_{m\geq 0}, i \in \Lambda_n$ with

$$gx_{m+1}^{\tau(i)} = F(x_m^{\sigma_i(1)}, \dots, x_m^{\sigma_i(n)}) \ \text{ and } \ gy_{m+1}^{\tau(i)} = F(y_m^{\sigma_i(1)}, \dots, y_m^{\sigma_i(n)}) \ \forall \, m, i.$$

Since $gx_0^{\tau(i)} = F(x_0^{\sigma_i(1)}, \dots, x_0^{\sigma_i(n)}) = gx_1^{\tau(i)}$, we have

$$gx_0^{\tau(i)} \preceq_i gx_1^{\tau(i)}, \ \text{ and } \ gx_0^{\tau(i)} \succeq_i gx_1^{\tau(i)}.$$

Obviously, one can use the mathematical induction to claim that

$$gx_m^{\tau(i)} \preceq_i gx_{m+1}^{\tau(i)} \ \text{ and } \ gx_m^{\tau(i)} \succeq_i gx_{m+1}^{\tau(i)} \ \text{ for all } \ m \geq 0 \ \text{ and } \ i \in \Lambda_n.$$

Therefore,

$$gx_m^{\tau(i)} = gx^{\tau(i)} \ \text{ for all } \ m \geq 0 \ \text{ and } \ i \in \Lambda_n. \tag{2.10}$$

Similarly, We have

$$gy_m^{\tau(i)} = gy^{\tau(i)}. \tag{2.11}$$

Using the hypothesis of the theorem we have the inequality

$$F(u^{\sigma_i(1)}, \dots, u^{\sigma_i(n)}) \preceq_i F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)})$$

or

$$F(u^{\sigma_i(1)}, \dots, u^{\sigma_i(n)}) \succeq_i F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)}).$$

Consider the first inequality holds, that is, $gu_1^{\tau(i)} \preceq_i gx^{\tau(i)}$, then $gu_m^{\tau(i)} \preceq_i gx^{\tau(i)}$. The second inequality is same, then, $gu_m^{\tau(i)}$ is comparable to $gx^{\tau(i)}$. By a similar way one can assert that $gu_m^{\tau(i)}$ is comparable to $gy^{\tau(i)}$.

$$\begin{aligned} d(gu_{m+1}^{\tau(i)}, gx^{\tau(i)}) =& d(F(u_m^{\sigma_i(1)}, \ldots, u_m^{\sigma_i(n)}), F(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)})) \leq \sum_{j=1}^{n} a_j d(gu_m^{\sigma_i(j)}, gx^{\sigma_i(j)}) \\ &\leq \sum_{j=1}^{n} a_j \max_{1\leq i\leq n} d(gu_m^{\tau(i)}, gx^{\tau(i)}). \end{aligned}$$

Define $\delta_m = \max_{1\leq i\leq n} d(gu_m^{\tau(i)}, gx^{\tau(i)})$ and taking maximum above implies:

$$\begin{aligned} &\delta_{m+1} \leq \sum_{j=1}^{n} a_j \delta_m, \\ &\delta_{m+1} \leq \lambda \delta_m, \ \ \lambda = \sum_{j=1}^{n} a_j < 1, \\ &\delta_{m+1} \leq \cdots \leq \lambda^m \delta_0 \to 0 \ \text{ as } \ m \to \infty \\ &\Rightarrow gu_m^{\tau(i)} \to gx^{\tau(i)} \ \text{ for all } \ i \in \{1, \ldots, n\}. \end{aligned} \tag{2.12}$$

By a similar way, we get

$$gu_m^{\tau(i)} \to gy^{\tau(i)} \ \text{ for all } \ i \in \{1, \ldots, n\}. \tag{2.13}$$

The uniqueness of the limit gives

$$gx^i = gy^i \ \text{ or } \ F(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}) = F(y^{\sigma_i(1)}, \ldots, y^{\sigma_i(n)}).$$

Now apply the condition of weak compatibility

$$ggx^{\tau(i)} = gF(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}) = F(gx^{\sigma_i(1)}, \ldots, gx^{\sigma_i(n)}),$$

that means $(gx^i)_{i=1}^n$ is another ϕ- coincidence point of F and g, by the above fact that $ggx^i = gx^i$ for all i . Set $gx = \xi$, then $\xi^{\tau(i)} = g\xi^{\tau(i)} = F(\xi^{\sigma_i(1)}, \ldots, \xi^{\sigma_i(n)})$ Therefore, $(\xi^1, \ldots, \xi^n)$ is a common fixed point of F and g.

Example 2.1. *Let $X=[0,1]$ with the usual order and metric, i.e., $(X,\leq,d)$ contain an ordered metric space. Define the mappings $F:X^n\to X$ and $g:X\to X$ by*

$$\begin{aligned} F(x_1,x_2,\ldots,x_n) &= \frac{x_1^2}{2}-\frac{x_2^2}{4}+\frac{x_3^2}{8}-\frac{x_4^2}{16}\cdots+\frac{x_n^2}{2^n}, \\ g(x) &= x,\ \forall\ x\in X. \end{aligned}$$

Consider $\phi=\{\tau,\sigma_1,\ldots,\sigma_n\}$ be an $n+1$-tuple of permutations from $\Lambda_n=\{1,2,\ldots,n\}$ into itself in the form:

$$\begin{aligned} \tau=\sigma_1 &= \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 2 & 3 & \ldots & n \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 2 & 1 & 2 & \ldots & n \end{pmatrix} \\ &\vdots \\ \sigma_i &= \begin{pmatrix} 1 & 2 & 3 & \ldots & i & \ldots & n \\ i & i-1 & i-2 & \ldots & 1 & \ldots & n-i+1 \end{pmatrix} \\ &\vdots \\ \sigma_n &= \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ n & n-1 & n-2 & \ldots & 1 \end{pmatrix}. \end{aligned}$$

Let A be the set of odd numbers and B be the set of even numbers in Λ_n. One can easily see that all conditions of theorems 2.1 and 2.2 hold as in the following:

First, $\sigma_i\in\Omega_{A,B}$ if $i\in A$ and $\sigma_i\in\acute{\Omega}_{A,B}$ if $i\in B$.

Second, F has the mixed g-monotone property, that is,

$$gx\leq gy \text{ implies } F(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)\leq_i F(a_1,\ldots,a_{i-1},y,a_{i+1},\ldots,a_n).$$

To claim this we have to consider two cases

- **•:** *if $i\in A$, then* $F(a_1,a_2\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)=\frac{a_1^2}{2}-\frac{a_2^2}{4}+\cdots-\frac{a_{i-1}^2}{2^{i-1}}+\frac{x^2}{2^i}-\frac{a_{i+1}^2}{2^{i+1}}+\cdots+\frac{a_n^2}{2^n}$ $\leq\frac{a_1^2}{2}-\frac{a_2^2}{4}+\cdots-\frac{a_{i-1}^2}{2^{i-1}}+\frac{y^2}{2^i}-\frac{a_{i+1}^2}{2^{i+1}}+\cdots+\frac{a_n^2}{2^n}=F(a_1,a_2,\ldots,a_{i-1},y,a_{i+1},\ldots,a_n)$,
- **•:** *if $i\in B$, then* $F(a_1,a_2\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)=\frac{a_1^2}{2}-\frac{a_2^2}{4}+\cdots+\frac{a_{i-1}^2}{2^{i-1}}-\frac{x^2}{2^i}+\frac{a_{i+1}^2}{2^{i+1}}-\cdots+\frac{a_n^2}{2^n}$ $\geq\frac{a_1^2}{2}-\frac{a_2^2}{4}+\cdots+\frac{a_{i-1}^2}{2^{i-1}}-\frac{y^2}{2^i}+\frac{a_{i+1}^2}{2^{i+1}}+\cdots+\frac{a_n^2}{2^n}=F(a_1,a_2,\ldots,a_{i-1},y,a_{i+1},\ldots,a_n)$.

Third, for the contractive condition (2.1), we have for all $(x_1,\dots,x_n),(y_1,\dots,y_n)\in X^n$

$$\begin{aligned} d(F(x_1,x_2,\dots,x_n)F(y_1,y_2,\dots,y_n)) &= \left| \frac{x_1^2}{2}-\frac{x_2^2}{4}+\cdots+\frac{x_n^2}{2^n}-(\frac{y_1^2}{2}-\frac{y_2^2}{4}+\cdots+\frac{y_n^2}{2^n}) \right| \\ &\le \frac{1}{2}\mid x_1^2-y_1^2\mid+\frac{1}{4}\mid y_2^2-x_2^2\mid+\cdots+\frac{1}{2^n}\mid x_n^2-y_n^2\mid \\ &\le \frac{1}{2}d(gx_1,gy_1)+\frac{1}{4}d(gx_2,gy_2)+\cdots+\frac{1}{2^n}d(gx_n,gy_n) \\ &\le \Sigma_{i=1}^n a_i d(gx_i,gy_i). \end{aligned}$$

Where, $gx_i\le_i gy_i$ *and* $a_i=\frac{1}{2^i}$ *for all* $i\in\Lambda_n$ *(note,* $\Sigma_i a_i<1$*) .*
Fourth, $g(X)=X$ *is complete,* $F(X^n)\subseteq g(X)$ *and* X *has the sequential monotone property. Thus all conditions of Theorem 2.1 hold and then* F *and* g *have one coincidence point* $(0,\dots,0)$*. Furthermore,* F *and* g *are weakly compatible (but not commuting) and the set* Φ *of all coincidence points is a lattice then Theorem 2.2 ensures that* $(0,\dots,0)$ *is the unique common fixed point of* F,g*.*

If we consider τ by the identity permutation on Λ_n, i.e., $\tau(i)=i\ \forall\ i$, we will state and prove the following theorem under more general contractive condition.

Theorem 2.3. *Let* $(X,\preceq,d)$ *be an ordered metric space and* $\phi=(\sigma_1,\dots,\sigma_n)$ *be a* (n)*-tuple of mappings from* Λ_n *into itself such that,* $\sigma_i\in\Omega_{A,B}$ *if* $i\in A$ *and* $\sigma_i\in\Omega'_{A,B}$ *if* $i\in B$*. Let* $F:X^n\to X$ *and* $g:X\to X$ *be two given mappings such that* F *has the mixed g-monotone property,* $F(X^n)\subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X*. Assume that there exist* $a_i,b_i,c_i\in R, i\in\Lambda_n$ *verifying* $\sum_{i=1}^n a_i+\sum_{i=1}^n b_i+\sum_{i=1}^n c_i<1$ *and*

$$\begin{aligned} d\big(F(x^1,\dots,x^n),F(y^1,\dots,y^n)\big) &\le \sum_{j=1}^n a_j d(gx^j,gy^j) \\ &+\sum_{j=1}^n b_j \frac{d\big(gx^j,F(x^{\sigma_j(1)},\dots,x^{\sigma_j(n)})\big)d\big(gy^j,F(y^{\sigma_j(1)},\dots,y^{\sigma_j(n)})\big)}{d(gx^j,gy^j)} \\ &+\sum_{j=1}^n c_j \frac{d\big(gx^j,F(y^{\sigma_j(1)},\dots,y^{\sigma_j(n)})\big)d\big(gy^j,F(x^{\sigma_j(1)},\dots,x^{\sigma_j(n)})\big)}{d(gx^j,gy^j)}, \end{aligned} \tag{2.14}$$

for which $gx^i\preceq_i gy^i$*. If there exist* $x_0^1,\dots x_0^n\in X$ *such that*

$$g(x_0^i)\preceq_i F(x_0^{\sigma_i(1)},\dots,x_0^{\sigma_i(n)}),\ \textit{for all } i\in\Lambda_n, \tag{2.15}$$

and X has the sequential g-monotone property. Then F and g have, at least, one $\phi-$ coincidence point.

Proof. Apply the same manner as in Theorem 2.1 we can easily construct sequences $\{x_m^1\}_{m\geq 0}$, $\ldots$, $\{x_m^n\}_{m\geq 0}$ such that

$$gx_{m+1}^i = F(x_m^{\sigma_i(1)},\ldots,x_m^{\sigma_i(n)}), \text{ for all } m\geq 0 \text{ and } i\in\Lambda_n.$$

The mathematical induction for $m\geq 0$ tends to

$$gx_m^i \preceq_i gx_{m+1}^i, \text{ for all } i\in\Lambda_n. \tag{2.16}$$

Then we use (2.16) and the contraction condition (2.14) to assert that the sequences $\{gx_m^i\}_{m\geq 0}$ are Cauchy for all $i\in\Lambda_n$ as follows

$$\begin{aligned}
d(gx_m^i, gx_{m+1}^i) =& d\big(F(x_{m-1}^{\sigma_i(1)},\ldots,x_{m-1}^{\sigma_i(n)}), F(x_m^{\sigma_i(1)},\ldots,x_m^{\sigma_i(n)})\big) \leq \sum_{j=1}^n a_j d(gx_{m-1}^{\sigma_i(j)}, gx_m^{\sigma_i(j)}) \\
&+ \sum_{j=1}^n b_j \frac{d\big(gx_{m-1}^{\sigma_i(j)}, F(x_{m-1}^{\sigma_{\sigma_i(j)}(1)},\ldots,x_{m-1}^{\sigma_{\sigma_i(j)}(n)})\big) d\big(gx_m^{\sigma_i(j)}, F(x_m^{\sigma_{\sigma_i(j)}(1)},\ldots,x_m^{\sigma_{\sigma_i(j)}(n)})\big)}{d(gx_{m-1}^{\sigma_i(j)}, gx_m^{\sigma_i(j)})} \\
&+ \sum_{j=1}^n c_j \frac{d\big(gx_{m-1}^{\sigma_i(j)}, F(x_m^{\sigma_{\sigma_i(j)}(1)},\ldots,x_m^{\sigma_{\sigma_i(j)}(n)})\big) d\big(gx_m^{\sigma_i(j)}, F(x_{m-1}^{\sigma_{\sigma_i(j)}(1)},\ldots,x_{m-1}^{\sigma_{\sigma_i(j)}(n)})\big)}{d(gx_{m-1}^{\sigma_i(j)}, gx_m^{\sigma_i(j)})} \\
&\leq \sum_{j=1}^n a_j d(gx_{m-1}^{\sigma_i(j)}, gx_m^{\sigma_i(j)}) + \sum_{j=1}^n b_j \frac{d(gx_{m-1}^{\sigma_i(j)}, gx_m^{\sigma_i(j)}) d(gx_m^{\sigma_i(j)}, gx_{m+1}^{\sigma_i(j)})}{d(gx_{m-1}^{\sigma_i(j)}, gx_m^{\sigma_i(j)})} \\
&+ \sum_{j=1}^n c_j \frac{d(gx_{m-1}^{\sigma_i(j)}, gx_{m+1}^{\sigma_i(j)}) d(gx_m^{\sigma_i(j)}, gx_m^{\sigma_i(j)})}{d(gx_{m-1}^{\sigma_i(j)}, gx_m^{\sigma_i(j)})} \\
&\leq \sum_{j=1}^n a_j d(gx_{m-1}^{\sigma_i(j)}, gx_m^{\sigma_i(j)}) + \sum_{j=1}^n b_j d(gx_m^{\sigma_i(j)}, gx_{m+1}^{\sigma_i(j)}) \\
&\leq \sum_{j=1}^n a_j \max_{1\leq i\leq n} d(gx_{m-1}^i, gx_m^i) + \sum_{j=1}^n b_j \max_{1\leq i\leq n} d(gx_m^i, gx_{m+1}^i).
\end{aligned}$$

Define $\delta_m = \max_{1\le i\le n} d(gx^i_m, gx^i_{m+1})$ and taking maximum above implies

$$
\begin{aligned}
&\delta_m \le \sum_{j=1}^{n} a_j \delta_{m-1} + \sum_{j=1}^{n} b_j \delta_m,\\
&(1-\sum_{j=1}^{n} b_j)\delta_m \le \sum_{j=1}^{n} a_j \delta_{m-1},\\
&\delta_m \le \lambda\,\delta_{m-1},\ \ \lambda = \frac{\sum_{j=1}^{n} a_j}{(1-\sum_{j=1}^{n} b_j)} < 1\\
&\Rightarrow d(gx^i_m, gx^i_{m+1}) \le \delta_m \le \lambda^m \delta_0.
\end{aligned}
\tag{2.17}
$$

For a fixed i we use the triangle inequality and (2.17) to obtain:

$$
\begin{aligned}
d(gx^i_m, gx^i_{m+p}) &\le d(gx^i_m, gx^i_{m+1}) + d(gx^i_{m+1}, gx^i_{m+2}) + \cdots + d(gx^i_{m+p-1}, gx^i_{m+p})\\
&\le (\lambda^m + \lambda^{m+1} + \cdots + \lambda^{m+p-1})\delta_0\\
&\le \lambda^m (1 + \lambda + \cdots + \lambda^{p-1})\delta_0\\
&\le \lambda^m \frac{1-\lambda^p}{1-\lambda}\delta_0 \to 0 \ \text{ as } \ n\to\infty.
\end{aligned}
$$

Therefore, $\{gx^i_m\}_{m\ge 0}$ are Cauchy sequences (for all $i \in \Lambda_n$) in $g(X)$. By the completeness of $g(X)$, there exist $\{gx^1, \dots, gx^n\} \in g(X)$, such that

$$
gx^i_m \to gx^i,\ \text{ as } \ n\to\infty \ \text{ for all } \ i\in\Lambda_n. \tag{2.18}
$$

Inequality (2.16), limit (2.18) and the sequential g- monotonicity of X yield $gx_m^i \preceq_i gx^i$.

Now consider

$$
\begin{aligned}
d\big(F(x^{\sigma_i(1)},\dots,x^{\sigma_i(n)}),gx^i\big) &\le d\big(F(x^{\sigma_i(1)},\dots,x^{\sigma_i(n)}),F(x_m^{\sigma_i(1)},\dots,x_m^{\sigma_i(n)})\big)+d(gx_{m+1}^i,gx^i)\\
&\le \sum_{j=1}^n a_j d(gx_m^{\sigma_i(j)},gx^{\sigma_i(j)})\\
&+\sum_{j=1}^n b_j \frac{d\big(gx_m^{\sigma_i(j)},F(x_m^{\sigma_{\sigma_i(j)}(1)},\dots,x_m^{\sigma_{\sigma_i(j)}(n)})\big)d\big(gx^{\sigma_i(j)},F(x^{\sigma_{\sigma_i(j)}(1)},\dots,x^{\sigma_{\sigma_i(j)}(n)})\big)}{d(gx_m^{\sigma_i(j)},gx^{\sigma_i(j)})}\\
&+\sum_{j=1}^n c_j \frac{d\big(gx_m^{\sigma_i(j)},F(x^{\sigma_{\sigma_i(j)}(1)},\dots,x^{\sigma_{\sigma_i(j)}(n)})\big)d\big(gx^{\sigma_i(j)},F(x_m^{\sigma_{\sigma_i(j)}(1)},\dots,x_m^{\sigma_{\sigma_i(j)}(n)})\big)}{d(gx_m^{\sigma_i(j)},gx^{\sigma_i(j)})}\\
&+d(gx_{m+1}^i,gx^i)\\
&\le \sum_{j=1}^n a_j d(gx_m^{\sigma_i(j)},gx^{\sigma_i(j)})+\sum_{j=1}^n b_j \frac{d(gx_m^{\sigma_i(j)},gx_{m+1}^{\sigma_i(j)})d\big(gx^{\sigma_i(j)},F(x^{\sigma_{\sigma_i(j)}(1)},\dots,x^{\sigma_{\sigma_i(j)}(n)})\big)}{d(gx_m^{\sigma_i(j)},gx^{\sigma_i(j)})}\\
&+\sum_{j=1}^n c_j \frac{d\big(gx_m^{\sigma_i(j)},F(x^{\sigma_{\sigma_i(j)}(1)},\dots,x^{\sigma_{\sigma_i(j)}(n)})\big)d(gx^{\sigma_i(j)},gx_{m+1}^{\sigma_i(j)})}{d(gx_m^{\sigma_i(j)},gx^{\sigma_i(j)})}+d(gx_{m+1}^i,gx^i)\\
&\to 0 \text{ as } m\to\infty.
\end{aligned}
$$

Thus, $F(x^{\sigma_i(1)},\dots,x^{\sigma_i(n)})=gx^i$ and $(x^1,\dots,x^n)$ is a $\phi-$ coincidence point of F and g.

Now, we shall prove the existence and uniqueness of the common ϕ-fixed point.

Theorem 2.4. *In addition to the the hypothesis of Theorem 2.3 suppose that for any non comparable elements $(x^1,\dots,x^n)$, $(y^1,\dots,y^n)\in\Phi$ there exists $(u^1,\dots,u^n)\in X^n$ such that*

$$(gu^1,gu^2,\dots,gu^n)$$

is comparable, at the same time, to

$$(gx^1,gx^2,\dots,gx^n) \text{ and } (gy^1,gy^2,\dots,gy^n).$$

Provided that F and g are weakly compatible. Then F and g have a unique common ϕ- fixed point.

Proof. According to Theorem 2.3 the set Φ is non-empty. Assume that $(x^1,\dots,x^n)$ and $(y^1,\dots,y^n)$ are two non comparable ϕ- coincidence points of F and g, that is

$$\begin{aligned} gx^i &= F(x^{\sigma_i(1)},\dots,x^{\sigma_i(n)}), \\ gy^i &= F(y^{\sigma_i(1)},\dots,y_m^{\sigma_i(n)}), \ \text{for all } \ i\in\Lambda_n. \end{aligned}$$

By the preceding assumption there exists $(u^1,\dots,u^n)$ such that $(gu^1,gu^2,\dots,gu^n)$ is comparable, at the same time, to $(gx^1,gx^2,\dots,gx^n)$ and $(gy^1,gy^2,\dots,gy^n)$. Put $u_0^i=u^i\ \forall\, i$ and apply the same argument in Theorem 2.3, one can determine sequences $\{u_m^1\}_{m\geq 0}$, $\{u_m^2\}_{m\geq 0}$, …, $\{u_m^n\}_{m\geq 0}$ such that $gu_{m+1}^i=F(u_m^{\sigma_i(1)},\dots,u_m^{\sigma_i(n)}),\ \forall\ m,i.$
Further, set $x_0^i=x^i$ and $y_0^i=y^i$, $\forall\, i$. By the same way we can define the sequences $\{x_m^i\}_{m\geq 0}$ and $\{y_m^i\}_{m\geq 0}\ \forall\, i$ with $gx_{m+1}^i=F(x_m^{\sigma_i(1)},\dots,x_m^{\sigma_i(n)})$ and $gy_{m+1}^i=F(y_m^{\sigma_i(1)},\dots,y_m^{\sigma_i(n)})\ \forall\, m,i$. Then

$$\begin{aligned} gx_m^i &= gx^i \ \texttt{and} \\ gy_m^i &= gy^i \ \text{for all } \ m\geq 0 \ \text{ and } \ i\in\Lambda_n. \end{aligned}$$

Using the assumption of the theorem we have

$$\begin{aligned} gu^i &\preceq_i gx^i \ \texttt{or} \\ gu^i &\succeq_i gx^i. \end{aligned}$$

Consider the first inequality, $gu_1^i\preceq_i gx^i$, then $gu_m^i\preceq_i gx^i$. The second inequality is same, then, gu_m^i is comparable to gx^i and to gy^i.

$$\begin{aligned} d(gu_{m+1}^i,gx^i) =& d(F(u_m^{\sigma_i(1)},\dots,u_m^{\sigma_i(n)}),F(x^{\sigma_i(1)},\dots,x^{\sigma_i(n)}))\leq\sum_{j=1}^n a_j d(gu_m^{\sigma_i(j)},gx^{\sigma_i(j)}) \\ &+\sum_{j=1}^n b_j\frac{d(gu_m^{\sigma_i(j)},F(u_m^{\sigma_{\sigma_i(j)}(1)},\dots,u_m^{\sigma_{\sigma_i(j)}(n)}))d(gx^{\sigma_i(1)},F(x^{\sigma_{\sigma_i(j)}(1)},\dots,x^{\sigma_{\sigma_i(j)}(n)}))}{d(gu_m^{\sigma_i(j)},gx^{\sigma_i(j)})} \\ &+\sum_{j=1}^n c_j\frac{d(gu_m^{\sigma_i(j)},F(x^{\sigma_{\sigma_i(j)}(1)},\dots,x^{\sigma_{\sigma_i(j)}(n)}))d(gx^{\sigma_i(1)},F(u_m^{\sigma_{\sigma_i(j)}(1)},\dots,u_m^{\sigma_{\sigma_i(j)}(n)}))}{d(gu_m^{\sigma_i(j)},gx^{\sigma_i(1)})} \\ &\leq\sum_{j=1}^n a_j\max_{1\leq i\leq n} d(gu_m^i,gx^i)+\sum_{j=1}^n c_j\max_{1\leq i\leq n} d(gx^i,gu_{m+1}^i). \end{aligned}$$

Define $\delta_m = \max_{1\leq i\leq n} d(gu_m^i, gx^i)$ and taking maximum above implies

$$\begin{aligned}
&\delta_{m+1} \leq \sum_{j=1}^{n} a_j \delta_m + \sum_{j=1}^{n} c_j \delta_{m+1},\\
&\delta_{m+1} \leq \lambda \delta_m,\ \ \lambda = \frac{\sum_{j=1}^{n} a_j}{1-\sum_{j=1}^{n} c_j} < 1,\\
&\delta_{m+1} \leq \cdots \leq \lambda^m \delta_0 \to 0 \ \text{ as } \ m \to \infty\\
&\Rightarrow gu_m^i \to gx^i \ \text{ for all } \ i \in \{1,\dots,n\}.
\end{aligned} \tag{2.19}$$

By a similar way we have

$$gu_m^i \to gy^i \ \text{ for all } \ i \in \{1,\dots,n\}. \tag{2.20}$$

By the uniqueness of the limit we get

$$gx^i = gy^i \ \text{ or } \ F(x^{\sigma_i(1)},\dots,x^{\sigma_i(n)}) = F(y^{\sigma_i(1)},\dots,y^{\sigma_i(n)}).$$

Applying the condition of weak compatibility yields to the existence of common ϕ-fixed point of F and g. Finally we can easily claim the uniqueness of this fixed point.

Let g be the identity mapping on X in Theorems 2.3 and 2.4.

Corollary 2.1. *Let $(X,\preceq,d)$ be a complete ordered metric space and $\phi = (\sigma_1,\dots,\sigma_n)$ be a (n)-tuple of mappings from Λ_n into itself such that, $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F : X^n \to X$ be a mapping having the mixed monotone property. Assume that there exist $a_i, b_i, c_i \in R, i \in \Lambda_n$ verifying $\sum_{i=1}^n a_i + \sum_{i=1}^n b_i + \sum_{i=1}^n c_i < 1$ and*

$$\begin{aligned}
d(F(x^1,\dots,x^n),F(y^1,\dots,y^n)) \leq &\sum_{j=1}^{n} a_j d(x^j,y^j)\\
&+ \sum_{j=1}^{n} b_j \frac{d(x^j,F(x^{\sigma_j(1)},\dots,x^{\sigma_j(n)}))d(y^j,F(y^{\sigma_j(1)},\dots,y^{\sigma_j(n)}))}{d(x^j,y^j)}\\
&+ \sum_{j=1}^{n} c_j \frac{d(x^j,F(y^{\sigma_j(1)},\dots,y^{\sigma_j(n)}))d(y^j,F(x^{\sigma_j(1)},\dots,x^{\sigma_j(n)}))}{d(x^j,y^j)},
\end{aligned}$$

for which $x^i \preceq_i y^i$. If there exist $x_0^1,\dots x_0^n \in X$ such that

$$x_0^i \preceq_i F(x_0^{\sigma_i(1)},\dots,x_0^{\sigma_i(n)}),\ \textit{for all}\ i \in \Lambda_n,$$

and X has the sequential monotone property. Then there exists $(z^1, z^2, \ldots, z^n) \in X^n$ such that

$$z^i = F(z^{\sigma_i(1)}, \ldots, z^{\sigma_i(n)}), \text{ for all } i \in \Lambda_n, \tag{2.21}$$

that is F has, at least, one $\phi-$ fixed point. Furthermore, let Δ be the set of all points of X^n verifying (2.21)and suppose that for all $(x^1, \ldots, x^n)$, $(y^1, \ldots, y^n) \in \Delta$ there exists $(u^1, \ldots, u^n) \in X^n$ such that $(u^1, u^2, \ldots, u^n)$ is comparable, at the same time, to $(x^1, x^2, \ldots, x^n)$ and $(y^1, y^2, \ldots, y^n)$. Then Δ is reduced to a single point

Example 2.2. *Let $X = \{(x, -x) : x \in R\} \subseteq R^2$ with the Euclidean metric and the usual order $(x, y) \leq (z, t) \Leftrightarrow x \leq z$ and $y \leq t$, contains a partially ordered complete metric space, whose different elements are not comparable. Define $F : X^n \to X$ as*

$$F((x_1, -x_1), (x_2, -x_2), \ldots, (x_n, -x_n)) = (x_1, -x_1)$$

Consider $\sigma_1, \sigma_2, \ldots, \sigma_n : \Lambda_n \to \Lambda_n$ defined as in Example 2.1. It is easy to show that all hypothesis of Corollary 2.1 are satisfied unless the condition which guaranties the uniqueness of ϕ-fixed point, for any two different fixed points we cannot find another point in X^n that comparable to them at the same time. So a greater number of fixed points of F can be found, because any point $((x_1, -x_1), (x_2, -x_2), \ldots, (x_n, -x_n)) \in X^n$ can be interpreted as a ϕ- fixed point for the mapping F as follows

$$\begin{aligned}
(x_1, -x_1) &= F((x_1, -x_1), (x_2, -x_2), \ldots, (x_n, -x_n)) \\
(x_2, -x_2) &= F((x_2, -x_2), (x_3, -x_3), \ldots, (x_n, -x_n), (x_1, -x_1)) \\
&\vdots \\
(x_i, -x_i) &= F((x_i, -x_i), (x_{i+1}, -x_{i+1}), \ldots, (x_n, -x_n), (x_1, -x_1), \ldots, (x_{i-1}, -x_{i-1})) \\
&\vdots \\
(x_n, -x_n) &= F((x_n, -x_n), (x_{n-1}, -x_{n-1}), \ldots, (x_1, -x_1)).
\end{aligned}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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