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A RESOLVENT OPERATOR TECHNIQUE FOR SOLVING GENERALIZED SYSTEM OF NONLINEAR RELAXED COCOERCIVE MIXED VARIATIONAL INEQUALITIES

SHAMSHAD HUSAIN AND SANJEEV GUPTA*

Department of Applied Mathematics, Faculty of Engineering & Technology,

Aligarh Muslim University, Aligarh-202002, India

Abstract. The aim of this work is to use resolvent operator technique to find the common solutions for a system of generalized nonlinear relaxed cocoercive mixed variational inequalities and fixed point problems for Lipschitz mappings in Hilbert spaces. The results obtained in this work may be viewed as an extension, refinement and improvement of the previously known results.

Keywords: Generalized explicit iterative algorithm, System of generalized nonlinear relaxed cocoercive mixed variational inequalities, Resolvent operator, Relaxed (γ, r)-cocoercive mapping, Lipschitz continuity, Hilbert spaces.

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1.Introduction and Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a closed and convex set in H. Let $T_1, T_2 : H \times H \to H$ and $g, h : H \to H$ be four nonlinear different operators and $\phi : H \to R \cup \{+\infty\}$ be a

*Corresponding author

E-mail addresses: s_husain68@yahoo.com(S. Husain), guptasanmp@gmail.com(S. Gupta)

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continuous function. We consider the problem of finding $x^*, y^* \in H$ such that

$$\langle \rho T_1(y^*, x^*) + x^* - g(y^*), g(x) - x^* \rangle \geq \rho\phi(g(x^*)) - \rho\phi(g(x)),\ \forall x \in H : g(x) \in H,\ \rho > 0 \tag{1.1a}$$

$$\langle \eta T_2(x^*, y^*) + y^* - h(x^*), h(x) - y^* \rangle \geq \rho\phi(h(y^*)) - \rho\phi(g(x)),\ \forall x \in H : h(x) \in H,\ \eta > 0, \tag{1.1b}$$

which is called the system of generalized nonlinear relaxed cocoercive mixed variational inequalities (SGMVID). Here the parameters ρ and η are constants and play an important role in the study of the convergence analysis of proposed iterative methods. We now discuss some special cases.

(I) If $g = h = I$, then problem SGMVID reduces to finding $x^*, y^* \in H$ such that,

$$\langle \rho T_1(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq \rho\phi(x^*) - \rho\phi(x),\ \forall x \in H,\ \rho > 0 \tag{1.2a}$$

$$\langle \eta T_2(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq \rho\phi(y^*) - \rho\phi(x),\ \forall x \in H,\ \eta > 0, \tag{1.2b}$$

which is called the system of nonlinear mixed variational inequalities (SMVID) and has been considered and studied in [1].

(II) If $T_1,\ T_2 : H \to H$ are univariate mappings, then the SGMVID problem reduces to finding $x^*,\ y^* \in H$ such that

$$\langle \rho T_1(y^*) + x^* - g(y^*), g(x) - x^* \rangle \geq \rho\phi(g(x^*)) - \rho\phi(g(x)),\ \forall x \in H : g(x) \in H,\ \rho > 0 \tag{1.3a}$$

$$\langle \eta T_2(x^*) + y^* - h(x^*), h(x) - y^* \rangle \geq \rho\phi(h(y^*)) - \rho\phi(g(x)),\ \forall x \in H : h(x) \in H,\ \eta > 0, \tag{1.3b}$$

which appears to be a new system of generalized relaxed cocoercive mixed variational inequalities (SGMVI).

(III) If ϕ is an indicator function of a closed convex set K in H, then the SGMVID reduces to finding $x^*, y^* \in K$ such that

$$\langle \rho T_1(y^*, x^*) + x^* - g(y^*), g(x) - x^* \rangle \geq 0,\ \forall x \in K : g(x) \in K,\ \rho > 0 \tag{1.4a}$$

$$\langle \eta T_2(x^*, y^*) + y^* - h(x^*), h(x) - y^* \rangle \geq 0,\ \forall x \in K : h(x) \in K,\ \eta > 0, \tag{1.4b}$$

which is called the system of general variational inequalities (SGHVID) and has been considered and studied in [2].

In this paper, we generalize the results of Chang and Noor recent works [1,2,3], by using resolvent operator technique. We consider the convergence criteria of an algorithm under some mild conditions to find the common element of the solution of the system of generalized nonlinear relaxed cocoercive mixed variational inequalities problems (SGMVID) and fixed point problems of nonlinear Lipschitz mapping in Hilbert spaces. Results presented in this work also improve and generalize many known results of this field, see [1-13].

Definition 1.1. If ϕ is an indicator function of a closed convex set K in H, then $J_\phi = P_K$ i.e.

$$\phi(u) \;=\; \begin{cases} 0, & u \in K, \\ +\infty, & otherwise, \end{cases}$$

Definition 1.2. For any maximal operator T, the resolvent operator associated with T, for any $\rho > 0$, is defined as

$$J_T(u) = (I + \rho T)^{-1}(u) \;\forall\; u \in H.$$

It is well-known that an operator T is maximal monotone if and only if its resolvent operator J_T is defined everywhere. It is single-valued and non-expansive.

If $\phi(.)$ is a proper, convex and lower-semicontinuous function, then its subdifferential $\partial\phi(.)$ is a maximal monotone operator. In this case, we define the resolvent operator

$$J_\phi(u) = (I + \rho T)^{-1}(u),\; \forall\; u \in H,$$

associated with subdifferential $\partial\phi(.)$.

Definition 1.3. A mapping $T : H \to H$ is called r-strongly monotone if there exists a constant $r > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq r\|x - y\|^2,\; \forall x, y \in H.$$

Definition 1.4. A mapping $T : H \to H$ is called relaxed γ-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq -\gamma\|Tx - Ty\|^2,\; \forall x, y \in H.$$

Definition 1.5. A mapping $T : H \to H$ is called relaxed (γ, r)-cocoercive if there exist constants $\gamma > 0,\ r > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq -\gamma\|Tx - Ty\|^2 + r\|x - y\|^2,\ \forall x, y \in H.$$

The class of relaxed (γ, r)-cocoercive mappings is more general than the class of strongly monotone mappings.

Definition 1.6. A mapping $T : H \to H$ is called μ-Lipschitzian if there exists a constant $\mu > 0$ such that

$$\|Tx - Ty\| \leq \mu\|x - y\|,\ \forall x, y \in H.$$

2. Main Results

In this section, we suggest some explicit iterative algorithms for solving the system of generalized nonlinear relaxed cocoercive mixed variational inequalities (SGMVID). First of all, we establish the equivalence between the system of generalized nonlinear relaxed cocoercive mixed variational inequalities and fixed point problems with the resolvent operator technique. This alternative formulation enables us to suggest and analyze resolvent operator technique for solving (1.1). For this, we recall some well-known concepts and results.

Lemma 2.1. *For a given $z \in H, u \in H$ satisfies the inequality*

$$\langle u - z, v - u\rangle \ \geq \ \rho\phi(f(u)) - \rho\phi(f(v)),\ v \in H :\ f(v) \in H,$$

if and only if $u = J_\phi(z)$ where $J_\phi = (1 + \rho\partial\phi)^{-1}$ is the resolvent operator.

It is well known that the resolvent operator J_ϕ is nonexpansive, that is,

$$\|J_\phi u - J_\phi v\| \ \leq \ \|u - v\|,\ \forall u, v \in H.$$

Using Lemma 2.1, we can easily show that finding the solution $(x^*,\ y^*) \in H$ of SGMVID is equivalent to finding $x^*, y^* \in H$ such that

$$x^* \ = \ J_\phi[g(y^*) - \rho T_1(y^*, x^*)], \tag{2.1a}$$

$$y^* \ = \ J_\phi[h(x^*) - \eta \ T_2(x^*, y^*)]. \tag{2.1b}$$

We use this alternative equivalent formulation to suggest the following explicit iterative method for solving the system of generalized nonlinear relaxed cocoercive mixed variational inequalities (SGMVID).

Algorithm 2.1. For arbitrarily chosen initial points $x_0, y_0 \in H$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} \ = \ (1 - a_n)x_n \ + \ a_n J_\phi[g(y_n) - \rho T_1(y_n, x_n)] \tag{2.2a}$$

$$y_{n+1} \ = \ J_\phi[h(x_{n+1}) - \eta \ T_2(x_{n+1}, y_n)], \tag{2.2b}$$

where $a_n \in [0, 1]$ for all $n \geq 0$.

If $g = h = I$, then Algorithm 2.1 reduces to the following iterative method for solving the system of nonlinear mixed variational inequalities (SMVID).

Algorithm 2.2. For arbitrarily chosen initial points $x_0, y_0 \in H$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} \ = \ (1 - a_n)x_n \ + \ a_n J_\phi[y_n - \rho T_1(y_n, x_n)] \tag{2.3a}$$

$$y_{n+1} \ = \ J_\phi[x_{n+1} - \eta \ T_2(x_{n+1}, y_n)], \tag{2.3b}$$

where $a_n \in [0, 1]$ for all $n \geq 0$.

If $T_1, \ T_2 : H \to H$ are univariate mappings, then Algorithm 2.1 reduces to the following iterative method for solving the system of generalized nonlinear relaxed cocoercive mixed variational inequalities (SGMVI).

Algorithm 2.3. For arbitrarily chosen initial points $x_0, y_0 \in H$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} \ = \ (1 - a_n)x_n \ + \ a_n J_\phi[g(y_n) - \rho T_1(y_n)] \tag{2.4a}$$

$$y_{n+1} \ = \ J_\phi[h(x_{n+1}) - \eta \ T_2(x_{n+1})], \tag{2.4b}$$

where $a_n \in [0, 1]$ for all $n \geq 0$.

If ϕ is an indicator function of a closed set K in H, then $J_\phi = P_K$, the projection of H onto the closed convex set K. Then Algorithms 2.1 collapse to the following iterative projection method for solving a system of general variational inequalities (SGHVID).

Algorithm 2.4. For arbitrarily chosen initial points $x_0, y_0 \in H$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} = (1-a_n)x_n + a_n P_K[g(y_n) - \rho T_1(y_n, x_n)] \tag{2.5a}$$

$$y_{n+1} = P_K[h(x_{n+1}) - \eta\, T_2(x_{n+1}, y_n)], \tag{2.5b}$$

where $a_n \in [0,1]$ for all $n \geq 0$.

Lemma 2.2. *([13]) Suppose $\{\delta_\infty\}_{n=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality:*

$$\delta_{n+1} \leq (1-\lambda_n)\,\delta_n + \sigma_n,\ \forall\, n \geq 0,$$

with $\lambda_n \in [0,\ 1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$. Then $lim_{n\to\infty}\delta_n = 0$.

Theorem 2.1. *Let (x^*, y^*) be the solution of SGMVID. Suppose that $T_1 : H \times H \to H$ is relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian in the first variable, and $T_2 : H \times H \to H$ is relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian in the first variable. Let $g : H \to H$ be relaxed (γ_3, r_3)-cocoercive and μ_3-Lipschitz and let $h : H \to H$ be relaxed (γ_4, r_4)-cocoercive and μ_4-Lipschitz continuous. If*

$$\left|\rho - \frac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \frac{\sqrt{(r_1-\gamma_1\mu_1^2)^2 - \mu_1^2 k_1(2-k_1)}}{\mu_1^2},\ r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{k_1(2-k_1)},\ k_1 < 1, \tag{2.6}$$

$$\left|\eta - \frac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \frac{\sqrt{(r_2-\gamma_2\mu_2^2)^2 - \mu_2^2 k_2(2-k_2)}}{\mu_2^2},\ r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{k_2(2-k_2)},\ k_2 < 1, \tag{2.7}$$

where

$$k_1 = \sqrt{1 - 2(r_3 - \gamma_3\mu_3^2) + \mu_3^2}, \tag{2.8}$$

$$k_2 = \sqrt{1 - 2(r_4 - \gamma_4\mu_4^2) + \mu_4^2}, \tag{2.9}$$

and $a_n \in [0,1]$, $\sum_{n=0}^{\infty} a_n = \infty$, then for arbitrarily chosen initial points $x_0, y_0 \in H$, x_n and y_n obtained from Algorithm 2.1 converge strongly to x^* and y^* respectively.

Proof. It follows from (2.1a) and (2.2a), and the nonexpansive property of the resolvent operator J_ϕ that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|(1-a_n)x_n + a_n J_\phi[g(y_n)-\rho T_1(y_n,\ x_n)]-(1-a_n)x^*-a_n J_\phi[g(y^*)-\rho T_1(y^*,\ x^*)]\| \\ &\leq (1-a_n)\|x_n - x^*\| + a_n\|J_\phi[g(y_n)-\rho T_1(y_n,\ x_n)] - J_\phi[g(y^*)-\rho T_1(y^*,\ x^*)]\| \\ &\leq (1-a_n)\|x_n - x^*\| + a_n\|[g(y_n)-\rho T_1(y_n,\ x_n)] - [g(y^*)-\rho T_1(y^*,\ x^*)]\| \\ &= (1-a_n)\|x_n - x^*\| + a_n\|y_n - y^* - \rho[T_1(y_n,\ x_n) - T_1(y^*,\ x^*))\| \\ &\quad + a_n\|y_n - y^* - (g(y_n)-g(y^*))\|. \end{aligned} \tag{2.10}$$

Since T_1 is relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian in first variable , we have

$$\begin{aligned}\|y_n - y^* - \rho[T_1(y_n,\ x_n) - T_1(y^*,\ x^*)]\|^2 &= \|y_n - y^*\|^2 - 2\rho\langle T_1(y_n,\ x_n) - T_1(y^*,\ x^*),\ y_n - y^*\rangle \\ &\quad + \rho^2\ \|T_1(y_n,\ x_n) - T_1(y^*,\ x^*)\|^2 \\ &\leq \|y_n - y^*\|^2 - 2\rho\ [-\gamma_1\|T_1(y_n,\ x_n) - T_1(y^*,\ x^*)\|^2 + r_1\|y_n - y^*\|^2] \\ &\quad + \rho^2\ \|T_1(y_n,\ x_n) - T_1(y^*,\ x^*)\|^2 \\ &\leq \|y_n - y^*\|^2 + 2\rho\gamma_1\mu_1^2\|y_n - y^*\|^2 - 2\rho r_1\|y_n - y^*\|^2 + \rho^2\mu_1^2\|y_n - y^*\|^2 \\ &= [1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2]\ \|y_n - y^*\|^2. \end{aligned} \tag{2.11}$$

In a similar way, using the (γ_3, r_3)-cocoercivity and μ_3-Lipschitz continuity of the operator g, we have

$$\|y_n - y^* - (g(y_n) - g(y^*))\|^2 \leq k_1\|y_n - y^*\|, \tag{2.12}$$

where k_1 is defined by (2.8). Set

$$\theta_1 = k_1 + [1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2]^{1/2}.$$

It is clear from the condition (2.6) that $0 \leq \theta_1 < 1$. Hence from (2.10)-(2.12), it follows that

$$\|x_{n+1} - x^*\| \leq (1-a_n)\|x_n - x^*\| + a_n\theta_1\|y_n - y^*\|. \tag{2.13}$$

Similarly, T_2 is relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian in the first variable, we obtain

$$\begin{aligned}
\|x_{n+1} - x^* - \eta[T_2(x_{n+1},\ y_n) - T_2(x^*,\ y^*)]\|^2 \ &= \ \|x_{n+1} - x^*\|^2 \\
&\quad -2\eta\langle T_2(x_{n+1}, y_n) - T_2(x^*,\ y^*),\ x_{n+1} - x^*\rangle + \eta^2\|T_2(x_{n+1}, y_n) - T_2(x^*,\ y^*)\|^2 \\
&\leq \ \|x_{n+1} - x^*\|^2 - 2\eta\ [-\gamma_2\|T_2(x_{n+1}, y_n) - T_2(x^*,\ y^*)\|^2 \ + \ r_2\|x_{n+1} - x^*\|^2] \\
&\quad + \eta^2\|T_2(x_{n+1}, y_n) - T_2(x^*,\ y^*)\|^2 \\
&= \ \|x_{n+1} - x^*\|^2 \ + \ 2\eta\gamma_2\|T_2(x_{n+1}, y_n) - T_2(x^*,\ y^*)\|^2 - 2\eta r_2\|x_{n+1} - x^*\|^2 \\
&\quad + \eta^2\|T_2(x_{n+1}, y_n) - T_2(x^*,\ y^*)\|^2 \\
&\leq \ \|x_{n+1} - x^*\|^2 \ + \ 2\eta\gamma_2\mu_2^2\|x_{n+1} - x^*\|^2 - 2\eta r_2\|x_{n+1} - x^*\|^2 \\
&\quad + \eta^2\mu_2^2\|x_{n+1} - x^*\|^2 \\
&= \ [1 \ + \ 2\eta\gamma_2\mu_2^2 - 2\eta r_2 \ + \ \eta^2\mu_2^2]\|x_{n+1} - x^*\|^2. \qquad (2.14)
\end{aligned}$$

In a similar way, using the (γ_4, r_4)-cocoercivity and μ_4-Lipschitz continuity of the operator h, we have

$$\|y_n - y^* - (h(y_n) - h(x^*))\|^2 \ \leq \ k_2\|y_n - y^*\|, \qquad (2.15)$$

where k_2 is defined by (2.9).

Hence from (2.1b), (2.2b), (2.14) and (2.15) and the nonexpansive property of the resolvent operator J_ϕ , we have

$$\begin{aligned}
\|y_{n+1} - y^*\| \ &= \ \|J_\phi[h(x_{n+1}) - \eta T_2(x_{n+1},\ y_n)] - J_\phi[h(x^*) - \eta T_2(x^*,\ y^*)]\| \\
&\leq \ \|x_{n+1} - x^* - \eta(T_2(x_{n+1},\ y_n) - T_2(x^*,\ y^*))\| \ + \ \|x_{n+1} - x^* - (h(x_{n+1}) - h(x^*)\| \\
&\leq \ \{k_2 \ + \ [1 \ + \ 2\eta\gamma_2\mu_2^2 - 2\eta r_2 \ + \ \eta^2\mu_2^2]^{1/2}\}\|x_{n+1} - x^*\| \\
&= \ \theta_2\|x_{n+1} - x^*\|, \qquad (2.16)
\end{aligned}$$

where

$$\theta_2 \ = \ k_2 \ + \ [1 \ + \ 2\eta\gamma_2\mu_2^2 - 2\eta r_2 \ + \ \eta^2\mu_2^2]^{1/2}.$$

From (2.7), it follows that $\theta_2 < 1$. Combining (2.13) and (2.16), we obtain that

$$\|x_{n+1} - x^*\| \ \leq \ (1 - a_n)\|x_n - x^*\| \ + \ a_n\theta_1\|y_n - y^*\|$$

$$\leq \; (1-a_n)\|x_n - x^*\| \; + \; a_n\theta_1\|x_n - x^*\|$$

$$= \; [1 - a_n(1-\theta_1\theta_2)]\|x_n - x^*\|.$$

Since the constant $(1-\theta_1\theta_2) \in (0,\ 1]$ and $\sum\limits_{n=0}^{\infty}(1-\theta_1\theta_2) = \infty$, from Lemma 2.2, we have $\lim_{n\to\infty}\|x_n - x^*\| \; = 0$. Hence the result $\lim_{n\to\infty}\|y_n - y^*\| \; = 0$ is from (2.14). This completes the proof.

If $g \; = \; h \; = \; I$, the identity operator, then Theorem 2.1 reduces to the following result for solving a system of mixed variational inequalities SMVID, which is considered and introduced by Noor [7].

Theorem 2.2. *Let $(x^*, y^*) \in H$ be the solution of SMVID. If $T_1 : H \times H \to H$ is relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian in the first variable, and $T_2 : H \times H \to H$ is relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian in the first variable with conditions*

$$0 \; < \; \rho \; min\{2(r_1 - \gamma_1\mu_1^2)/\mu_1^2,\; 2(r_2 - \gamma_2\mu_2^2)/\mu_2^2\},\; r_1 > \gamma_1\mu_1^2,$$

$$0 \; < \; \eta \; min\{2(r_1 - \gamma_1\mu_1^2)/\mu_1^2,\; 2(r_2 - \gamma_2\mu_2^2)/\mu_2^2\},\; r_2 > \gamma_2\mu_2^2,$$

and $a_n \in [0,1]$, $\sum\limits_{n=0}^{\infty} a_n = \infty$, then for arbitrarily chosen initial points $x_0, y_0 \in H$, x_n and y_n obtained from Algorithm 2.2 converge strongly to x^* and y^* respectively.

If $T_1,\ T_2$ are univariate mappings, then the following result can be obtained from theorem 2.1

Theorem 2.3. *Let (x^*, y^*) be the solution of SGMVI. If $T_1 : H \times H \to H$ is relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian, and $T_2 : H \times H \to H$ is relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian. Let $g : H \to H$ be relaxed (γ_3, r_3)-cocoercive and μ_3-Lipschitz and let $h : H \to H$ be relaxed (γ_4, r_4)-cocoercive and μ_4-Lipschitz continuous. If*

$$\left|\rho - \frac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \frac{\sqrt{(r_1 - \gamma_1\mu_1^2)^2 - \mu_1^2 k_1(2-k_1)}}{\mu_1^2},\; r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{k_1(2-k_1)},\; k_1 < 1,$$

$$\left|\eta - \frac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \frac{\sqrt{(r_2 - \gamma_2\mu_2^2)^2 - \mu_2^2 k_2(2-k_2)}}{\mu_2^2},\; r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{k_2(2-k_2)},\; k_2 < 1,$$

where

$$k_1 = \sqrt{1 - 2(r_3 - \gamma_3\mu_3^2) + \mu_3^2},$$

$$k_2 = \sqrt{1 - 2(r_4 - \gamma_4\mu_4^2) + \mu_4^2},$$

and $a_n \in [0,1]$, $\sum_{n=0}^{\infty} a_n = \infty$, *then for arbitrarily chosen initial points* $x_0, y_0 \in H$, x_n *and* y_n *obtained from Algorithm 2.3 converge strongly to* x^* *and* y^* *respectively.*

If $\phi(.)$ is an indicator function of a closed convex set K in H, then $J_\phi = P_K$, the projection of H onto K. Consequently, Theorem 2.1 reduces to the following result for solving system of general variational inequalities SGHVID, which is mainly due to Noor[8].

Theorem 2.4. *Let* (x^*, y^*) *be the solution of SGHVID. It* $T_1 : H \times H \to H$ *is relaxed* (γ_1, r_1)*-cocoercive and* μ_1*-Lipschitzian in the first variable, and* $T_2 : H \times H \to H$ *is relaxed* (γ_2, r_2)*-cocoercive and* μ_2*-Lipschitzian in the first variable. Let* $g : H \to H$ *be relaxed* (γ_3, r_3)*-cocoercive and* μ_3*-Lipschitz and let* $h : H \to H$ *be relaxed* (γ_4, r_4)*-cocoercive and* μ_4*-Lipschitz continuous. If*

$$\left|\rho - \frac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \frac{\sqrt{(r_1 - \gamma_1\mu_1^2)^2 - \mu_1^2 k_1(2-k_1)}}{\mu_1^2}, \; r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{k_1(2-k_1)}, \; k_1 < 1,$$

$$\left|\eta - \frac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \frac{\sqrt{(r_2 - \gamma_2\mu_2^2)^2 - \mu_2^2 k_2(2-k_2)}}{\mu_2^2}, \; r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{k_2(2-k_2)}, \; k_2 < 1,$$

where

$$k_1 = \sqrt{1 - 2(r_3 - \gamma_3\mu_3^2) + \mu_3^2},$$

$$k_2 = \sqrt{1 - 2(r_4 - \gamma_4\mu_4^2) + \mu_4^2},$$

and $a_n \in [0,1]$, $\sum_{n=0}^{\infty} a_n = \infty$, *then for arbitrarily chosen initial points* $x_0, y_0 \in H$, x_n *and* y_n *obtained from Algorithm 2.4 converge strongly to* x^* *and* y^* *respectively.*

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