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AN ALGORITHM FOR APPROXIMATING A COMMON FIXED POINT OF A FINITE FAMILY OF LIPSCHITZ PSEUDOCONTRACTIVE MULTI-VALUED MAPPINGS

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Abstract. The purpose of this paper is twofold. We first give erratum to a proof given by Woldeamanuel *et al.* [Strong convergence theorems for a common fixed point of a finite family of Lipschitz hemicontractive-type multivaled mappings, Adv. Fixed Point Theory, 5 (2015), No. 2, 228-253]. In addition, we study an algorithm which approximates a common fixed point of a finite family of Lipschitz pseudocontractive multi-valued mappings under appropriate conditions.

Keywords: Demiclosed, Hausdorff metric; k-strictly pseudocontractive multi-valued mapping; Lipschitz pseudo-contractive multi-valued mapping; Monotone multi-valued mapping; Strong convergence.

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1. Introduction

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Let E be a nonempty real normed linear space. A subset K of E is called proximinal if for each $x \in E$ there exists $k \in K$ such that

$$\|x-k\| = \inf\{\|x-y\| : y \in K\} = d(x,K).$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. In fact, if K is a closed and convex subset of a uniformly convex Banach space E, then for any $x \in E$ there exists a unique point $u_x \in K$ such that (see, e.g., [26], [25], [15] and [8])

$$\|x-u_x\| = \inf\{\|x-y\| : y \in K\} = d(x,K).$$

Let E be a nonempty real normed space. We denote the family of all nonempty proximinal subsets of E by $P(E)$, the family of all nonempty closed, convex and bounded subsets of E by $CBC(E)$, the family of all nonempty closed and bounded subsets of E by $CB(E)$ and the family of all nonempty subsets of E by 2^E for a nonempty normed space E.

Let D be the Hausdorff metric induced by the metric d on E, that is, for every $A,B \in CB(E)$,

$$D(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}.$$

A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called L-Lipschitzian if there exists $L \geq 0$ such that,

$$\forall\, x,y \in D(T),\ D(Tx,Ty) \leq L\|x-y\|. \tag{1}$$

In (1), if $L \in [0,1)$, T is said to be a contraction, while T is nonexpansive if $L = 1$. A point $x \in C$ is a *fixed point of* T if $x \in Tx$ and we denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : x \in Tx\}$.

A mapping $T : D(T) \subset E \to CB(E)$ is said to be *hemicontractive-type* in the terminology of Hicks and Cubicek [17], if $F(T) \neq \emptyset$ and for all $p \in F(T), x \in D(T)$

$$D^2(Tx,Tp) \leq \|x-p\|^2 + \|x-u\|^2, \forall\, u \in Tx, \tag{2}$$

while, a mapping $T : D(T) \subset E \to CB(E)$ is said to be *demicontractive-type*, if $F(T) \neq \emptyset$ and for all $p \in F(T), x \in D(T)$ there exists $k \in [0,1)$ such that

$$D^2(Tx,Tp) \leq \|x-p\|^2 + k\|x-u\|^2, \forall\, u \in Tx. \tag{3}$$

For the definitions of k-strictly pseudocontractive-type, quasi-nonexpansive-type, pseudocontractive-type and nonexpansive-type multivalued mappings we refer the reader to the paper [31].

Recently, Woldeamanuel *et. al.* [31] introduced an iteration scheme $x_1 = w \in K$ by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n, \; u_n \in T_n x_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n, \; w_n \in T_n y_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n, \; n \geq 1, \end{cases} \tag{4}$$

where $T_n := T_n(mod\ N)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy some conditions.

They stated a theorem (Theorem 3.1 [31]) and proved strong convergence of the scheme to a common fixed point p, which is nearest to w, of $T_i, i = 1, \ldots, N$. The proof depends on the argument that $T : K \to CB(K)$ satisfies $\|u - v\| \leq 2D(Tx, Ty), \forall x, y \in K, u \in Tx, v \in Ty$.

Remark 1.1. A close look at the property of T shows that the argument considered may not be in general true. To see this, one may consider the following example.

Example 1.1. *Let* $T : \mathbb{R} \to 2^{\mathbb{R}}$ *be given by*

$$Tx = \begin{cases} [-\sqrt{2}x, 0] & x \in [0, \infty], \\ [0, -\sqrt{2}x], & x \in [-\infty, 0]. \end{cases}$$

It can be shown that T *is hemicontractive-type. Now, for* $x = 3$ *and* $y = 2$, *we have* $Tx = [-3\sqrt{2}, 0]$ *and* $Ty = [-5\sqrt{2}, 0]$, *so that*

$$D(tx, Ty) = D([-3\sqrt{2}, 0], [-5\sqrt{2}, 0]) = 2\sqrt{2}.$$

Now for $u = o \in Tx$ *and* $v = -5\sqrt{2} \in Ty$, *we have*

$$\|u - v\| = 5\sqrt{2} > 4\sqrt{2} = 2D(Tx, Ty).$$

A mapping $T : K \to CB(H)$ is said to be *pseudocontractive* (see [19, 20, 24]), if the inequality

$$\langle u - v, x - y \rangle \leq \|x - y\|^2, \tag{5}$$

holds for each $x,y \in K, u \in Tx,\ v \in Ty$. In this case,

$$\|x-y-(u-v)\|^2 + 2\langle u-v, x-y\rangle \leq 2\|x-y\|^2 + \|x-y-(u-v)\|^2,$$

which implies that

$$\|u-v\|^2 \leq \|x-y\|^2 + \|x-y-(u-v)\|^2.$$

Hence, $T : K \to CB(H)$ is said to be *pseudocontractive* multi-valued mapping, if $\forall\, x,y \in K$

$$\|u-v\|^2 \leq \|x-y\|^2 + \|x-y-(u-v)\|^2, \quad \forall u \in Tx,\ v \in Ty. \tag{6}$$

We observe that (6) implies that $\forall\, x,y \in K$,

$$D^2(Tx,Ty) \leq \|x-y\|^2 + \|x-y-(u-v)\|^2, \quad \forall u \in Tx,\ v \in Ty, \tag{7}$$

known as *pseudocontractive-type multi-valued* mapping (see, [31]).

For an example of pseudocontractive multi-valued mapping, see [32].

A mapping $T : K \to CB(H)$ is said to be *k-strongly pseudocontractive* (see [19, 20]), if there exists $k \in (0,1)$ such that the inequality

$$\langle u-v, x-y\rangle \leq k\|x-y\|^2, \tag{8}$$

holds for each $x,y \in K, u \in Tx,\ v \in Ty$.

Again we refer the reader to [32] for an example of k-strongly pseudocontractive multi-valued mapping.

Remark 1.2. Note that the class of pseudocontractive multi-valued mappings properly includes the class of k-strongly pseudocontractive multi-valued mappings.

Multi-valued pseudocontractive mappings are also related with the important class of nonlinear monotone mappings, where $A : K \to CB(H)$ is called *monotone*, if for any $x,y \in K$,

$$\langle u-v, x-y\rangle \geq 0, \quad \forall u \in Ax, v \in Ay. \tag{9}$$

A mapping $A : K \to CB(H)$ is said to be *k-strongly monotone* mapping if for all $x,y \in K$, there exists $k \in [0,1)$, such that

$$\langle u-v, x-y\rangle \geq k\|x-y\|^2, \quad \forall u \in Ax,\ v \in Ay. \tag{10}$$

We note that T is pseudocontractive if and only if $A := I - T$ is monotone and hence $x \in F(T)$ if and only if $x \in N(A) := \{x \in K : 0 \in Ax\}$.

Recently, Woldeamanuel *et al.* [32] introduced an iteration scheme $x_1 = w \in K$ by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n, \; n \geq 1, \end{cases} \tag{11}$$

where $u_n \in Tx_n, w_n \in Ty_n$ such that $||u_n - w_n|| \leq 2D(Tx_n, Ty_n)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy certain mild conditions.

They proved the strong convergence of the Scheme (11) to the fixed point of Lipschitz pseudocontractive multi-valued mapping T. This brings us to the next question.

Question: *Can we extend the results of* [32] *to a common fixed point of a finite family of Lipschitz pseudocontractive multi-valued mappings?*

The purpose of this paper is twofold. In section three, motivated by the result of Woldeamanuel *et al.* [31] and Remark 1.1, we consider the scheme studied in [31] with appropriate assumptions on T and give a modified proof which will enable us to correct the anomalies pointed out in Remark 1.1. In section four, we extend the work of Woldeamanuel *et al.* [32] to a finite family of Lipschitz pseudocontractive multi-valued mappings under appropriate conditions.

2. Preliminaries

Definition 2.1 Let E be a Banach space. Let $T : D(T) \subseteq E \to 2^E$ be a multi-valued mapping. $(I-T)$ is said to be demiclosed at zero, if for any sequence $\{x_n\} \subseteq D(T)$ such that $\{x_n\}$ converges weakly to p and $D(\{x_n\}, Tx_n) \to 0$, then $p \in Tp$.

Lemma 2.1. [30] *Let H be a real Hilbert space. Then, the following equations hold:*

(1) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall t \in [0,1]$,
(2) Given any x, y in H, $\|x-y\|^2 = \|x-z\|^2 + \|z-y\|^2 + 2\langle x-z, z-y\rangle$.

Lemma 2.2. [11] *Let H be a real Hilbert space. Then, the following equation holds: If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup z \in H$, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2, \forall y \in H.$$

Lemma 2.3. [1] *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $x \in H$. Then, $x_0 = P_K(x)$ if and only if $\langle z - x_0, x - x_0 \rangle \leq 0, \forall z \in K$.*

Lemma 2.4. [4] *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \to CBC(K)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$. Then, for any $x \in K, x_0 \in P_T(x)$ if and only if $\langle z - x_0, x - x_0 \rangle \leq 0, \forall z \in Tx$.*

Lemma 2.5. [17] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$: $a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$, In fact, $m_k := \max\{j \leq k : a_j < a_{j+1}\}$.*

Lemma 2.6. [28] *Let K be a metric space. Let $T : K \to P(K)$ be a multivalued mapping. Then, the following are equivalent:*
(1) $x \in Tx$, (2) $P_T x = \{x\}$, (3) $x \in F(P_T)$. Moreover, $F(T) = F(P_T)$.

Lemma 2.7. [33] *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ be a continuous pseudo-contractive mapping, then $(I - T)$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, as $n \to \infty$, then $x = Tx$.*

Lemma 2.8. *Let H be a real Hilbert space. Then,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

Proposition 2.1. [3] Let H be a Hilbert space. Let K be a nonempty closed and convex subset of H. Let $T : K \to CB(K)$ be k-strictly pseudocontractive-type multivalued mapping. Then T is L-Lipschitz mapping.

Lemma 2.9. [34] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation: $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, n \geq n_0$, where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions:*
$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\limsup_{n\to\infty} \delta_n \leq 0$. *Then,* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. [21] *Let E be a complete metric space. Let $A, B \in CB(E)$ and $a \in A$.*

(1) If $\gamma > 0$, then there exists $b \in B$ such that $d(a,b) \leq H(A,B) + \gamma$.
(2) If $x \in E$, then $d(x,A) \leq d(x,B) + D(A,B)$.

The proof of the following are given in [32].

Lemma 2.11. *Let H be a real Hilbert space. Suppose K is a closed, convex, nonempty subset of H. Assume that $T : K \to CB(K)$ is pseudocontractive multi-valued mapping with $F(T) \neq \emptyset$. Then, $F(T)$ is closed and convex.*

Lemma 2.12. *Let H be a real Hilbert space. Suppose K is a closed, convex, nonempty subset of H. Assume that $T : K \to CB(K)$ is Lipschitz pseudocontractive multi-valued mapping. Then, there is a single-valued nonexpansive mapping $S : K \to K$, such that for some $\lambda > 0$ and for any $y \in K$, $S(y)$ is a fixed point of $T_y(x) := (1-\lambda)y + \lambda Tx$.*

Lemma 2.13. *Let H be a real Hilbert space. Suppose K is a closed, convex, nonempty subset of H. Assume that $T : K \to CB(K)$ is Lipschitz pseudocontractive multi-valued mapping. Then $(I-T)$ is demiclosed at zero.*

3. Convergence results for a finite family of lipschitz hemicontractive-type mappings

Now, we give the modification of the statement and proof of Theorem 3.1 of [31].

Theorem 3.1. *Let K be a non-empty, closed and convex subset of a real Hilbert space H. Let $T_i : K \to CB(K), i = 1,2,\ldots,N$, be a finite family of Lipschitz hemicontractive-type mappings with Lipschitz constants $L_i, i = 1,2,\ldots,N$, respectively. Assume that $(I-T_i), i = 1,\ldots,N$ are demiclosed at zero and $\mathscr{F} = \cap_{i=1}^{N} F(T_i)$ is non-empty, closed and convex with $T_i(p) = \{p\}$, $\forall p \in F(T)$ and for each $i = 1,2,\ldots,N$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 =$*

$w \in K$ by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n,\ n \geq 1, \end{cases} \tag{12}$$

where, $u_n \in T_n x_n, w_n \in T_n y_n$ such that $||u_n - w_n|| \leq 2D(T_n x_n, T_n y_n)$ and $T_n := T_{n(mod\, N)+1}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \dfrac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$ *for* $L := \max\{L_i : 1, 2, \ldots, N\}$.

Then, $\{x_n\}$ converges strongly to some point $p \in \mathscr{F}$ nearest to w.

Proof. Let $p = P_{\mathscr{F}}(w)$. Now, using (1) of Lemma 2.1,

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(w-p) + (1-\alpha_n)(z_n - p)\|^2 \\ &\leq \alpha_n\|w-p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &= \alpha_n\|w-p\|^2 + (1-\alpha_n)\|\gamma_n(w_n - p) + (1-\gamma_n)(x_n - p)\|^2 \\ &= \alpha_n\|w-p\|^2 + (1-\alpha_n)\gamma_n\|w_n - p\|^2 + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 \\ &\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n - x_n\|^2, \\ &= \alpha_n\|w-p\|^2 + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 + (1-\alpha_n)\gamma_n\|w_n - p\|^2 \\ &\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n - x_n\|^2, \end{aligned} \tag{13}$$

which implies that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n\|w-p\|^2 + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 + (1-\alpha_n)\gamma_n D(T_n y_n, T_n p)^2 \\ &\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n - x_n\|^2 \\ &\leq \alpha_n\|w-p\|^2 + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 + (1-\alpha_n)\gamma_n \\ &\quad \left[\|y_n - p\|^2 + \|y_n - w_n\|^2\right] - (1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n - x_n\|^2. \end{aligned}$$

Thus,

$$\|x_{n+1}-p\|^2 \leq \alpha_n\|w-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2+(1-\alpha_n) \times \gamma_n\|y_n-p\|^2+(1-\alpha_n)\gamma_n\|y_n-w_n\|^2-(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2. \tag{14}$$

On the other hand, using (12) and using the assumption that $\|u_n-w_n\| \leq 2D(T_nx_n,T_ny_n)$ we have

$$\begin{aligned}
\|y_n-w_n\|^2 &= \|(1-\beta_n)(x_n-w_n)+\beta_n(u_n-w_n)\|^2\\
&= (1-\beta_n)\|x_n-w_n\|^2+\beta_n\|u_n-w_n\|^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2\\
&\leq (1-\beta_n)\|x_n-w_n\|^2+\beta_n 4D^2(T_nx_n,T_ny_n)-\beta_n(1-\beta_n)\|x_n-u_n\|^2\\
&\leq (1-\beta_n)\|x_n-w_n\|^2+\beta_n 4L^2\|x_n-y_n\|^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2\\
&\leq (1-\beta_n)\|x_n-w_n\|^2+4L^2\beta_n^3\|x_n-u_n\|^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2.
\end{aligned}$$

Hence,

$$\|y_n-w_n\|^2 \leq (1-\beta_n)\|x_n-w_n\|^2-\beta_n(1-\beta_n-4L^2\beta_n^2)\|x_n-u_n\|^2. \tag{15}$$

Again,

$$\begin{aligned}
\|y_n-p\|^2 &= \|(1-\beta_n)x_n+\beta_n u_n-p)\|^2\\
&= \|(1-\beta_n)(x_n-p)+\beta_n(u_n-p)\|^2\\
&= (1-\beta_n)\|x_n-p\|^2+\beta_n\|u_n-p\|^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2,
\end{aligned}$$

which gives that

$$\begin{aligned}
\|y_n-p\|^2 &\leq (1-\beta_n)\|x_n-p\|^2+\beta_n D^2(T_nx_n,T_np)-\beta_n(1-\beta_n)\|x_n-u_n\|^2\\
&\leq (1-\beta_n)\|x_n-p\|^2+\beta_n\left[\|x_n-p\|^2+\|x_n-u_n\|^2\right]\\
&\quad -\beta_n(1-\beta_n)\|x_n-u_n\|^2.
\end{aligned}$$

Thus,

$$\|y_n-p\|^2 \leq \|x_n-p\|^2+\beta^2\|x_n-u_n\|^2. \tag{16}$$

Now substituting (16), (15) into (14), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \alpha_n\|w-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2+(1-\alpha_n)\gamma_n||x_n-p||^2\\ &+ (1-\alpha_n)\gamma_n\beta_n^2\|x_n-u_n\|^2+(1-\alpha_n)\gamma_n(1-\beta_n)\|x_n-w_n\|^2\\ &- \beta_n(1-\alpha_n)\gamma_n(1-\beta_n-4L^2\beta_n^2)\|u_n-x_n\|^2\\ &\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2,\end{aligned}$$

which reduces to

$$\begin{aligned}(17)\quad \|x_{n+1}-p\|^2 \leq \alpha_n\|w-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\beta_n(1-\alpha_n)\\ \times\gamma_n(1-2\beta_n-4L^2\beta_n^2)\|u_n-x_n\|^2+(1-\alpha_n)\gamma_n(\gamma_n-\beta_n)\|x_n-w_n\|^2.\end{aligned}$$

From hypothesis (ii) in (12) we have that

$$(18)\qquad 1-2\beta_n-4L^2\beta_n^2 \geq 1-2\beta-4L^2\beta^2 \text{ and } \gamma_n \leq \beta_n.$$

Using (18) in (17), we get that

$$(19)\qquad \|x_{n+1}-p\|^2 \leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|w-p\|^2.$$

Thus, by induction

$$\|x_{n+1}-p\|^2 \leq \max\{\|x_1-p\|^2, \|w-p\|^2\},\ \forall n \geq 1.$$

This implies that $\{x_n\},\{y_n\}$ and $\{z_n\}$ are all bounded. Furthermore, from (12), Lemma 2.8 and (17), we get that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\alpha_n)(\gamma_n w_n+(1-\gamma_n)x_n)+\alpha_n w-p\|^2\\
&= \|(1-\alpha_n)((\gamma_n w_n+(1-\gamma_n)x_n)-p)+\alpha_n(w-p)\|^2\\
&\leq (1-\alpha_n)\|\gamma_n w_n+(1-\gamma_n)x_n-p\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle\\
&= (1-\alpha_n)\Big[\gamma_n\|w_n-p\|^2+(1-\gamma_n)\|x_n-p\|^2\\
&\quad -\gamma_n(1-\gamma_n)\|x_n-w_n\|^2\Big]+2\alpha_n\langle w-p,x_{n+1}-p\rangle\\
&\leq (1-\alpha_n)\Big[\gamma_n D(T_n y_n,T_n p)^2+(1-\gamma_n)\|x_n-p\|^2\\
&\quad -\gamma_n(1-\gamma_n)\|x_n-w_n\|^2\Big]+2\alpha_n\langle w-p,x_{n+1}-p\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\alpha_n)\big[\gamma_n(\|y_n-p\|^2+\|y_n-w_n\|^2)+(1-\gamma_n)\|x_n-p\|^2\\
&\quad -\gamma_n(1-\gamma_n)\|x_n-w_n\|^2\big]+2\alpha_n\langle w-p,x_{n+1}-p\rangle\\
&= (1-\alpha_n)\big[\gamma_n(\|y_n-p\|^2+\|y_n-w_n\|^2)+(1-\gamma_n)\|x_n-p\|^2\big]\\
&\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|x_n-w_n\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle\\
&\leq (1-\alpha_n)\gamma_n\|x_n-p\|^2+(1-\alpha_n)\gamma_n\beta_n^2\|x_n-u_n\|^2+(1-\alpha_n)\gamma_n\\
&\quad \times\big[(1-\beta_n)\|x_n-w_n\|^2-\beta_n(1-\beta_n-4L^2\beta_n^2)\|x_n-u_n\|^2\big]\\
&\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle\\
&= (1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n\beta_n(1-2\beta_n-4L^2\beta_n^2)\|x_n-u_n\|^2\\
&\quad +2\alpha_n\langle w-p,x_{n+1}-p\rangle+(1-\alpha_n)\gamma_n(\gamma_n-\beta_n)\|x_n-w_n\|^2.
\end{aligned}$$

That is, we get that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= (1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n\beta_n(1-2\beta_n-4L^2\beta_n^2)\|x_n-u_n\|^2\\
&\quad +2\alpha_n\langle w-p,x_{n+1}-p\rangle+(1-\alpha_n)\gamma_n(\gamma_n-\beta_n)\|x_n-w_n\|^2, \qquad (20)
\end{aligned}$$

which implies

$$\|x_{n+1}-p\|^2 \leq (1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n\beta_n(1-2\beta_n-4L^2\beta_n^2) \times\|x_n-u_n\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle, \tag{21}$$

and

$$\|x_{n+1}-p\|^2 \leq (1-\alpha_n)\|x_n-p\|^2-(1-c)\alpha^2(1-2\beta-4L^2\beta^2) \times\|x_n-u_n\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle. \tag{22}$$

Now we consider the following two cases:

Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\|x_n-p\|\}$ is non-increasing, $\forall n\geq n_0$. Then, we get that $\{\|x_n-p\|\}$ is convergent. So, from (22) and the fact that $\alpha_n\to 0$, we have that

$$(1-c)\alpha^2(1-2\beta-4L^2\beta^2)\|x_n-u_n\|^2\leq(1-\alpha_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2,$$

which gives that

$$x_n-u_n\to 0. \tag{23}$$

Now, from (12) and (23) we get

$$y_n-x_n=\beta_n(u_n-x_n)\to 0,$$

and hence we get that

$$\begin{aligned}\|z_n-x_n\| &= \gamma_n\|w_n-x_n\|=\gamma_n\|w_n-u_n+u_n-x_n\|\\ &\leq \gamma_n\|w_n-u_n\|+\gamma_n\|u_n-x_n\|\\ &\leq \gamma_n2D(T_ny_n,T_nx_n)+\gamma_n\|u_n-x_n\|\\ &\leq \gamma_n2L\|y_n-x_n\|+\gamma_n\|u_n-x_n\|\to 0. \quad (24)\end{aligned}$$

By (12), (24), the fact that $\|w-z_n\|$ is bounded and $\alpha_n\to 0$, we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|x_{n+1}-z_n+z_n-x_n\|\\ &\leq \|x_{n+1}-z_n\|+\|z_n-x_n\|\\ &= \alpha_n\|w-z_n\|+\|z_n-x_n\|\to 0. \quad (25)\end{aligned}$$

But then, since, $\|x_{n+i}-x_n\| \leq \|x_{n+i}-x_{n+i-1}\|+\ldots+\|x_{n+1}-x_n\|$, we get that

$$\|x_{n+i}-x_n\| \to 0,\ \forall i=1,2,\ldots,N. \tag{26}$$

Thus, from (23) and (26), we obtain that

$$\|u_{n+i}-x_n\| \leq ||u_{n+i}-x_{n+i}||+||x_{n+i}-x_n|| \to 0,\ \forall i=1,2,\ldots,N. \tag{27}$$

Now we show that for $i \in \{1,2,...,N\}$, $\lim_{n\to\infty} d(x_n, T_{n+i}x_n)=0$. But from Lemma 2.10, (23), (26) and Lipschitz property of T_i for each $i \in \{1,2,...,N\}$ we get that

$$\begin{aligned} d(x_n, T_{n+i}x_n) &= d(x_n, T_{n+i}x_{n+i})+D(T_{n+i}x_n, T_{n+i}x_{n+i}) \\ &\leq \|x_n-u_{n+i}\|+L\|x_n-x_{n+i}\| \to 0, \end{aligned} \tag{28}$$

which is the required result. The rest of the proof is the same as Theorem 3.1 of [31]

If, in Theorem 3.1, we consider a single hemicontractive-type mapping we get the following corollary.

Corollary 3.1. *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T: K \to CB(K)$, be Lipschitz hemicontractive-type mapping with Lipschitz constant L. Assume that $I-T$ is demiclosed at zero and $F(T)$ is non-empty with $T(p)=\{p\}$, $\forall p \in F(T)$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n+\beta_n u_n,\ u_n \in Tx_n, \\ z_n = \gamma_n w_n+(1-\gamma_n)x_n,\ w_n \in Ty_n, \\ x_{n+1} = \alpha_n w+(1-\alpha_n)z_n,\ n \geq 1, \end{cases} \tag{29}$$

where $u_n \in Tx_n, w_n \in Ty_n$ such that $||u_n-w_n|| \leq 2D(Tx_n, Ty_n)$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \dfrac{1}{\sqrt{4L^2+1}+1}$.

Then, $\{x_n\}$ converges strongly to some point $p \in \mathscr{F}$ nearest to w.

If, in Theorem 3.1 we assume that $P_{T_i}, i = 1, \ldots, N$ are Lipschitz hemicontractive-type mappings, then by Lemma 2.6 , the requirement that $T_i(p) = \{p\}$ may not be needed. Thus, we obtain the following corollary.

Corollary 3.2. *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T_i : K \to CB(K), i = 1,2,\ldots,N$, be a finite family of multivalued mappings. Let $P_{T_i}, i = 1,2,\ldots,N$, be Lipschitz hemicontractive-type mappings with Lipschitz constants $L_i, i = 1,2,\ldots,N$, respectively. Assume that $I - P_{T_i}, i = 1,\ldots,N$ are demiclosed and $\mathscr{F} = \cap_{i=1}^{N} F(T_i)$ is non-empty. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by*

$$(30) \qquad \begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n,\ u_n \in P_{T_n}x_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n,\ w_n \in P_{T_n}y_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n,\ n \geq 1, \end{cases}$$

where $u_n \in P_{T_n}x_n, w_n \in P_{T_n}y_n$ such that $||u_n - w_n|| \leq 2D(P_{T_n}x_n, P_{T_n}y_n)$ and $T_n := T_{n(mod\, N)+1}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$ *for* $L := \max\{L_i : 1,2,\ldots,N\}$.

Then, $\{x_n\}$ converges strongly to some point $p \in \mathscr{F}$ nearest to w.

If, in Theorem 3.1 , we assume that $T_i, i = 1,\ldots,N$, are k-strictly pseudocontractive-type mappings then by Proposition 2.1, T_i are Lipschitz with $L_i = \frac{1+\sqrt{k_i}}{1-\sqrt{k_i}}, i = 1,\ldots,N$. Hence, we have the following theorem.

Theorem 3.2. *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T_i : K \to CB(K), i = 1,2,\ldots,N$, be a finite family of k-strictly pseudocontractive-type mappings. Assume that $\mathscr{F} = \cap_{i=1}^{N} F(T_i)$ is non-empty with $T_i(p) = \{p\}$, $\forall p \in F(T)$ and for each $i = 1,2,\ldots,N$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by*

$$(31) \qquad \begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n,\ u_n \in T_n x_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n,\ w_n \in T_n y_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n,\ n \geq 1, \end{cases}$$

where $u_n \in T_nx_n, w_n \in T_ny_n$ *such that* $||u_n - w_n|| \leq 2D(T_nx_n, T_ny_n)$ *and* $T_n := T_{n(mod\, N)+1}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$ *for* $L := \max\{\frac{1+\sqrt{k_i}}{1-\sqrt{k_i}}, i = 1, \dots, N\}$.

Then, $\{x_n\}$ *converges strongly to some point* $p \in \mathscr{F}$ *nearest to* w.

The following follows from Theorem 3.2. For the detail we refer the reader to [31]

Corollary 3.3. *Let* H *be a real Hilbert space and* K *be a non-empty, closed and convex subset of* H. *Let* $T_i : K \to CB(K), i = 1, 2, \dots, N$, *be a finite family of nonexpansive-type mappings. Assume that* $F = \cap_{i=1}^{N} F(T_i)$ *is non-empty with* $T_i(p) = \{p\}$, $\forall p \in F(T)$ *and for each* $i = 1, 2, \dots, N$. *Let* $\{x_n\}$ *be the sequence generated from an arbitrary* $x_1 = w \in K$ *by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n,\ u_n \in T_nx_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n,\ w_n \in T_ny_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n,\ n \geq 1, \end{cases} \tag{32}$$

where $u_n \in T_nx_n, w_n \in T_ny_n$ *such that* $||u_n - w_n|| \leq 2D(T_nx_n, T_ny_n)$ *and* $T_n := T_{n(mod\, N)+1}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{5}+1}$, $\forall n \geq 1$.

Then, $\{x_n\}$ *converges strongly to some point* $p \in \mathscr{F}$ *nearest to* w.

4. Convergence results for finite family of lipschitz pseudocontractive multi-valued mappings

Theorem 4.1 *Let* H *be a real Hilbert space and* K *be a non-empty, closed and convex subset of* H. *Let* $T_i : K \to CB(K), i = 1, 2, \dots, N$, *be a finite family of Lipschitz pseudocontractive multi-valued mappings with Lipschitz constants* $L_i, i = 1, 2, \dots, N$, *respectively. Assume that* $\mathscr{F} = \cap_{i=1}^{N} F(T_i)$ *is non-empty. Let* $\{x_n\}$ *be the sequence generated from an arbitrary* $x_1 = w \in K$

by

$$
\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n, \; n \geq 1 \end{cases} \tag{33}
$$

where $u_n \in T_n x_n, w_n \in T_n y_n$ *such that* $||u_n - w_n|| \leq 2D(T_n x_n, T_n y_n)$ *and* $T_n := T_{n(mod\, N)+1}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

(i) $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$ *for* $L := \max\{L_i : i = 1,2,\ldots,N\}$.

Then, $\{x_n\}$ *converges strongly to some point* $p \in \mathscr{F}$ *nearest to w.*

Proof. Let $p = P_{\mathscr{F}}(w)$. Now, using Lemma 2.1 we get that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(w-p) + (1-\alpha_n)(z_n - p)\|^2 \\
&\leq \alpha_n\|w-p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&= \alpha_n\|w-p\|^2 + (1-\alpha_n)\|\gamma_n(w_n - p) + (1-\gamma_n)(x_n - p)\|^2 \\
&= \alpha_n\|w-p\|^2 + (1-\alpha_n)\gamma_n\|w_n - p\|^2 + (1-\alpha_n)(1-\gamma_n) \\
&\quad \times\|x_n - p\|^2 - (1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n - x_n\|^2 \\
&\leq \alpha_n\|w-p\|^2 + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 + (1-\alpha_n)\gamma_n \\
&\quad \left[\|y_n - p\|^2 + \|y_n - p - (w_n - p)\|^2\right] \\
&\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n - x_n\|^2 \\
&= \alpha_n\|w-p\|^2 + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 + (1-\alpha_n)\gamma_n \\
&\quad \left[\|y_n - p\|^2 + \|y_n - w_n\|^2\right] - (1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n - x_n\|^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \leq \alpha_n\|w-p\|^2 + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 + (1-\alpha_n) \\
\times \gamma_n\|y_n - p\|^2 + (1-\alpha_n)\gamma_n\|y_n - w_n\|^2 - (1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n - x_n\|^2.
\end{aligned} \tag{34}
$$

On the other hand, using (33), the assumption that $\|u_n - w_n\| \leq 2D(T_n x_n, T_n y_n)$, Lemma 2.1 and T_n is Lipschitz ,

$$\begin{aligned}
\|y_n - w_n\|^2 &= \|(1-\beta_n)(x_n - w_n) + \beta_n(u_n - w_n)\|^2 \\
&= (1-\beta_n)\|x_n - w_n\|^2 + \beta_n\|u_n - w_n\|^2 - \beta_n(1-\beta_n)\|x_n - u_n\|^2 \\
&\leq (1-\beta_n)\|x_n - w_n\|^2 + \beta_n 4D(T_n x_n, T_n y_n)^2 - \beta_n(1-\beta_n)\|x_n - u_n\|^2 \\
&\leq (1-\beta_n)\|x_n - w_n\|^2 + \beta_n 4L^2\|x_n - y_n\|^2 - \beta_n(1-\beta_n)\|x_n - u_n\|^2 \\
&= (1-\beta_n)\|x_n - w_n\|^2 + 4\beta_n^3 L^2\|x_n - u_n\|^2 - \beta_n(1-\beta_n)\|x_n - u_n\|^2.
\end{aligned}$$

Hence,

$$\|y_n - w_n\|^2 \leq (1-\beta_n)\|x_n - w_n\|^2 - \beta_n(1-\beta_n - 4L^2\beta_n^2)\|x_n - u_n\|^2 \tag{35}$$

Again, using the assumption that T_n is pseudocontractive,

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)x_n + \beta_n u_n - p)\|^2 \\
&= \|(1-\beta_n)(x_n - p) + \beta_n(u_n - p)\|^2 \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|u_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - u_n\|^2 \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\left[\|x_n - p\|^2 + \|x_n - u_n\|^2\right] \\
&\quad - \beta_n(1-\beta_n)\|x_n - u_n\|^2 \\
&= \|x_n - p\|^2 + \beta_n^2\|x_n - u_n\|^2.
\end{aligned}$$

Thus,

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 + \beta_n^2\|x_n - u_n\|^2. \tag{36}$$

Now, substituting (35), (36) into (34),

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n\|w - p\|^2 + (1-\alpha_n)(1-\gamma_n)\|x_n - p\|^2 + (1-\alpha_n)\gamma_n\|x_n - p\|^2 \\
&\quad + (1-\alpha_n)\gamma_n\beta_n^2\|x_n - u_n\|^2 + (1-\alpha_n)\gamma_n(1-\beta_n)\|x_n - w_n\|^2 \\
&\quad - \beta_n(1-\alpha_n)\gamma_n(1-\beta_n - 4L^2\beta_n^2)\|u_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n - x_n\|^2,
\end{aligned}$$

which reduces to

$$\|x_{n+1}-p\|^2 \leq \alpha_n\|w-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\beta_n(1-\alpha_n) \times \gamma_n(1-2\beta_n-4L^2\beta_n^2)\|u_n-x_n\|^2+(1-\alpha_n)\gamma_n(\gamma_n-\beta_n)\|x_n-w_n\|^2. \tag{37}$$

From hypothesis (ii) in (33) we have that

$$1-2\beta_n-4L^2\beta_n^2 \geq 1-2\beta-4L^2\beta^2 \tag{38}$$

$$\gamma_n \leq \beta_n. \tag{39}$$

Using (38) and (39) in (37) we get that

$$\|x_{n+1}-p\|^2 \leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|w-p\|^2. \tag{40}$$

Thus, by induction, we have

$$\|x_{n+1}-p\|^2 \leq \max\{\|x_1-p\|^2, \|w-p\|^2\},\ \forall n \geq 1.$$

This implies that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are all bounded. Furthermore, from (33), Lemma 2.8 and (37) we get that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\alpha_n)\left(\gamma_n w_n+(1-\gamma_n)x_n\right)+\alpha_n w-p)\|^2 \\
&= \|(1-\alpha_n)\left((\gamma_n w_n+(1-\gamma_n)x_n)-p\right)+\alpha_n(w-p)\|^2 \\
&\leq (1-\alpha_n)\|\gamma_n w_n+(1-\gamma_n)x_n-p\|^2+2\alpha_n\langle w-p, x_{n+1}-p\rangle \\
&= (1-\alpha_n)\Big[\gamma_n\|w_n-p\|^2+(1-\gamma_n)\|x_n-p\|^2 \\
&\quad -\gamma_n(1-\gamma_n)\|x_n-w_n\|^2\Big]+2\alpha_n\langle w-p, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)\left[\gamma_n(\|y_n-p\|^2+\|y_n-w_n\|^2)+(1-\gamma_n)\|x_n-p\|^2\right. \\
&\quad \left.-\gamma_n(1-\gamma_n)\|x_n-w_n\|^2\right]+2\alpha_n\langle w-p, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)\gamma_n\|y_n-p\|^2+(1-\alpha_n)\gamma_n\|y_n-w_n\|^2 \\
&\quad +(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n(1-\gamma_n)\|x_n-w_n\|^2 \\
&\quad +2\alpha_n\langle w-p, x_{n+1}-p\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n)\gamma_n\|x_n-p\|^2+(1-\alpha_n)\gamma_n\beta_n^2\|x_n-u_n\|^2+(1-\alpha_n)\gamma_n\\&\quad\times\left[(1-\beta_n)\|x_n-w_n\|^2-\beta_n(1-\beta_n-4L^2\beta_n^2)\|x_n-u_n\|^2\right]\\&\quad+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2\\&\quad+2\alpha_n\langle w-p,x_{n+1}-p\rangle\\&\le (1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n\beta_n(1-2\beta_n-4L^2\beta_n^2)\|x_n-u_n\|^2\\&\quad+2\alpha_n\langle w-p,x_{n+1}-p\rangle+(1-\alpha_n)\gamma_n(\gamma_n-\beta_n)\|x_n-w_n\|^2.\end{aligned}$$

This implies that,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n\beta_n(1-2\beta_n-4L^2\beta_n^2)\\&\quad\times\|x_n-u_n\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle,\end{aligned}\tag{41}$$

and hence by (i) and (ii) we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n)\|x_n-p\|^2-(1-c)\alpha^2(1-2\beta-4L^2\beta^2)\|x_n-u_n\|^2\\&\quad+2\alpha_n\langle w-p,x_{n+1}-p\rangle.\end{aligned}\tag{42}$$

Now we consider the following two cases:

Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\|x_n-p\|\}$ is non-increasing, $\forall n\ge n_0$. Then, we get that $\{\|x_n-p\|\}$ is convergent. So, from (42) we have that

$$\begin{aligned}(1-c)\alpha^2(1-2\beta-4L^2\beta^2)\|x_n-u_n\|^2 &\le (1-\alpha_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2\\&\quad+2\alpha_n\langle w-p,x_{n+1}-p\rangle.\end{aligned}$$

Thus, from the fact that $\alpha_n\to 0$, we get

$$\lim_{n\to\infty}\|x_n-u_n\|=0.\tag{43}$$

Now, from (33) we obtain that

$$y_n-x_n=\beta_n(u_n-x_n)\to 0,$$

and hence we get that

$$
\begin{aligned}
\|z_n - x_n\| &= \gamma_n\|w_n - x_n\| = \gamma_n\|w_n - u_n + u_n - x_n\| \\
&\leq \gamma_n\|w_n - u_n\| + \gamma_n\|u_n - x_n\| \\
&\leq 2\gamma_n D(T_n y_n, T_n x_n) + \gamma_n\|u_n - x_n\| \\
&\leq 2\gamma_n L\|y_n - x_n\| + \gamma_n\|u_n - x_n\| \to 0. \qquad (44)
\end{aligned}
$$

Furthermore, from (33), (44), the fact that $\|w - z_n\|$ is bounded and $\alpha_n \to 0$, we obtain

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|x_{n+1} - z_n + z_n - x_n\| \\
&\leq \|x_{n+1} - z_n\| + \|z_n - x_n\| \\
&= \alpha_n\|w - z_n\| + \|z_n - x_n\| \to 0. \qquad (45)
\end{aligned}
$$

But then, since, $\|x_{n+i} - x_n\| \leq \|x_{n+i} - x_{n+i-1}\| + \ldots + \|x_{n+1} - x_n\|$, we get that

$$\|x_{n+i} - x_n\| \to 0, \ \forall i = 1, 2, \ldots, N. \qquad (46)$$

Thus, from (43) and (46) we obtain that

$$\|u_{n+i} - x_n\| \leq ||u_{n+i} - x_{n+i}|| + ||x_{n+i} - x_n|| \to 0, \ \forall i = 1, 2, \ldots, N. \qquad (47)$$

Now we show that for $i \in \{1, 2, ..., N\}$, $\lim_{n\to\infty} d(x_n, T_{n+i}x_n) = 0$. But from (46), Lemma 2.10, (47) and Lipschitz property of T_i for each $i \in \{1, 2, ..., N\}$ we get that

$$
\begin{aligned}
d(x_n, T_{n+i}x_n) &= d(x_n, T_{n+i}x_{n+i}) + D(T_{n+i}x_n, T_{n+i}x_{n+i}) \\
&\leq \|x_n - u_{n+i}\| + L\|x_n - x_{n+i}\| \to 0, \qquad (48)
\end{aligned}
$$

which is the required result. Now, since $\{\|x_n - p\|\}$ converges, there exists a subsequence $\{x_{n_j+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n\to\infty}\langle w - p, x_{n+1} - p\rangle = \lim_{j\to\infty}\langle w - p, x_{n_j+1} - p\rangle,$$

and $x_{n_j+1} \rightharpoonup z$, for some $z \in K$. Now, from (45) we get $x_{n_j} \rightharpoonup z$. Hence, from (48) and the fact that T_i, $\forall i = 1, \ldots, N$ are demiclosed by Lemma 2.13 , we get that $z \in F(T_i)$, $\forall i = 1, \ldots, N$. i.e.,

$z \in \mathscr{F}$. Therefore, by Lemma 2.4 we obtain that

$$\begin{aligned} \limsup_{n\to\infty}\langle w-p, x_{n+1}-p\rangle &= \lim_{j\to\infty}\langle w-p, x_{n_j+1}-p\rangle \\ &= \langle w-p, z-p\rangle \leq 0. \end{aligned} \tag{49}$$

Now, from (42) we have that

$$\|x_{n+1}-p\|^2 \leq (1-\alpha_n)\|x_n-p\|^2 + 2\alpha_n\langle w-p, x_{n+1}-p\rangle. \tag{50}$$

It then follows from (50), (49) and Lemma 2.9 that $\|x_n - p\| \to 0$ i.e., $x_n \to p$.

Case 2. Suppose there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\|x_{n_k}-p\| < \|x_{n_k+1}-p\|,\ \forall k \in \mathbb{N}.$$

Thus, by Lemma 2.5, there is a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $\|x_{m_k} - p\| \leq \|x_{m_k+1}-p\|$ and $\|x_k - p\| \leq \|x_{m_k+1}-p\|$, $\forall k \in \mathbb{N}$. Now, from (42) and the fact that $\alpha_n \to 0$ we get that $x_{m_k} - u_{m_k} \to 0$, when $u_{m_k} \in T_i x_{m_k}$, $\forall i = 1, \ldots, N$. Hence as in Case 1, $x_{m_k+1} - x_{m_k} \to 0$ and that

$$\limsup_{k\to\infty}\langle w-p, x_{m_k+1}-p\rangle \leq 0. \tag{51}$$

From (42) we have that

$$\|x_{m_k+1}-p\|^2 \leq (1-\alpha_{m_k})\|x_{m_k}-p\|^2 + 2\alpha_{m_k}\langle w-p, x_{m_k+1}-p\rangle \tag{52}$$

and since $\|x_{m_k}-p\| \leq \|x_{m_k+1}-p\|$, (52) implies that

$$\begin{aligned} \alpha_{m_k}\|x_{m_k}-p\|^2 &\leq \|x_{m_k}-p\|^2 - \|x_{m_k+1}-p\|^2 + 2\alpha_{m_k}\langle w-p, x_{m_k+1}-p\rangle \\ &\leq 2\alpha_{m_k}\langle w-p, x_{m_k+1}-p\rangle, \end{aligned}$$

which implies that

$$\|x_{m_k}-p\|^2 \leq 2\langle w-p, x_{m_k+1}-p\rangle.$$

So, from (51) we get that $\|x_{m_k}-p\|^2 \to 0$ and hence this with (52) give that $\|x_{m_k+1}-p\| \to 0$. But, $\|x_k - p\| \leq \|x_{m_k+1}-p\|$, $\forall k \in \mathbb{N}$. Thus, $x_k \to p$. Therefore, $\{x_n\}$ converges strongly to some point $p \in \mathscr{F}$ nearest to w.

Remark 4.1. We note that, since every Lipschitz k-strongly pseudocontractive mapping is Lipschitz pseudocontractive mapping the above theorem holds for a finite family of Lipschitz k-strongly pseudocontractive mappings.

If, in Theorem 4.1 we consider a single Lipschitz pseudocontractive mapping we get the following corollary.

Corollary 4.1. *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T : K \to CB(K)$, be Lipschitz pseudocontractive multi-valued mapping with Lipschitz constant L. Assume that $F(T)$ is non-empty and that $T(p) = \{p\}$, $\forall p \in F(T)$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n, \ n \geq 1, \end{cases} \tag{53}$$

where $u_n \in Tx_n, w_n \in Ty_n$ such that $||u_n - w_n|| \leq 2D(Tx_n, Ty_n)$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \dfrac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$.

Then, $\{x_n\}$ converges strongly to some point $p \in \mathscr{F}$ nearest to w.

Proof. Put $T_i := T$, $\forall i = 1, \ldots, N$ in (33) and the scheme reduces to (53). Now, as in (41) and (42),

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)\gamma_n\beta_n(1 - 2\beta_n - 4L^2\beta_n^2) \\ &\quad \times \|x_n - u_n\|^2 + 2\alpha_n\langle w - p, x_{n+1} - p\rangle, \ u_n \in Tx_n \\ &\leq (1-\alpha_n)\|x_n - p\|^2 - (1-c)\alpha^2(1 - 2\beta - 4L^2\beta^2)\|x_n - u_n\|^2 \\ &\quad + 2\alpha_n\langle w - p, x_{n+1} - p\rangle \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle w - p, x_{n+1} - p\rangle \end{aligned}$$

The rest of the proof is as in Theorem 4.1.

If, in Theorem 4.1 we assume that $P_{T_i}, i = 1,\dots,N$ are Lipschitz pseudocontractive mappings, then by Lemma 2.6 , the requirement that $T_i(p) = \{p\}$ may not be needed. Thus, we get the following Corollary.

Corollary 4.2. *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T_i : K \to CB(K), i = 1,2,\dots,N$, be a finite family of multi-valued mappings. Let $P_{T_i}, i = 1,2,\dots,N$, be Lipschitz pseudocontractive mappings with Lipschitz constants $L_i, i = 1,2,\dots,N$, respectively. Suppose also that $\mathscr{F} = \cap_{i=1}^{N} F(T_i)$ is non-empty. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n,\ n \geq 1, \end{cases} \tag{54}$$

where $u_n \in P_{T_n}x_n, w_n \in P_{T_n}y_n$ such that $||u_n - w_n|| \leq 2D(P_{T_n}x_n, P_{T_n}y_n)$, and $T_n := T_{n(mod\, N)+1}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \dfrac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$ *for* $L := \max\{L_i : 1,2,\dots,N\}$.

Then, $\{x_n\}$ converges strongly to some point $p \in \mathscr{F}$ nearest to w.

If, in Theorem 4.1 we assume that $P_{T_i} : K \to CBC(K), i = 1,\dots,N$ are Lipschitz pseudocontractive mappings, then $P_{T_i}(x)$ is singleton and hence the following corollary follows.

Corollary 4.3 *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T_i : K \to CBC(K), i = 1,2,\dots,N$, be a finite family of multi-valued mappings. Let $P_{T_i}, i = 1,2,\dots,N$, be Lipschitz pseudocontractive mappings with Lipschitz constants $L_i, i = 1,2,\dots,N$, respectively. Suppose also that $\mathscr{F} = \cap_{i=1}^{N} F(P_{T_i})$ is non-empty. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n P_{T_n}x_n, \\ z_n = \gamma_n P_{T_n}y_n + (1-\gamma_n)x_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n,\ n \geq 1, \end{cases} \tag{55}$$

where $T_n := T_{n(mod\, N)+1}$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$ *for* $L := \max\{L_i : 1, 2, \ldots, N\}$.

Then, $\{x_n\}$ *converges strongly to some point* $p \in \mathscr{F}$ *nearest to w.*

Next, we state and prove a convergence theorem for a common zero of a finite family of monotone mappings.

Theorem 4.2 *Let H be a real Hilbert space. Let* $A_i : H \to CB(H), i = 1, 2, \ldots, N$ *be a family of Lipschitz monotone mappings with Lipschitz constants,* $1 + L_i, i = 1, 2, \ldots, N$, *respectively. Assume* $\mathscr{F} := \cap_{i=1}^{N} N(A_i) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence generated from an arbitrary* $x_1 = w \in H$ *by*

$$\begin{cases} y_n = x_n - \beta_n u_n, \\ z_n = x_n - \gamma_n w_n, \\ x_{n+1} = \alpha_n w + (1 - \alpha_n) z_n, \ n \geq 1, \end{cases} \tag{56}$$

where $u_n \in A_n x_n, w_n \in A_n y_n$ *such that* $||u_n - w_n|| \leq 2D(x_n - A_n x_n, y_n - A_n y_n) + ||x_n - y_n||$, *and* $A_n := A_{n(mod\,N)+1}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$ *for* $L := \max\{L_i, i = 1, \ldots, N\}$.

Then, $\{x_n\}$ *converges strongly to a common zero point of* $A_1, A_2, \ldots, A_n$ *nearest to w.*

Proof. Let $T_i x := (I - A_i)x, i = 1, 2, \ldots, N$. Then $T_i, i = 1, 2, \ldots, N$ are Lipschitz pseudocontractive mappings with Lipschitz constants $L_i := (1 + L_i)$ and

$$\cap_{i=1}^{N} F(T_i) = \cap_{i=1}^{N} N(A_i) \neq \emptyset.$$

Now replacing A_i with $(I - T_i)$ for each $i = 1, 2, \ldots, N$ in (56) we get the Scheme (33). Hence the result follows from Theorem 4.1 .

In Theorem 4.2 , if we consider a single Lipschitz monotone mapping we obtain,

Corollary 4.4. *Let H be a real Hilbert space. Let* $A : H \to CB(H)$ *be a Lipschitz monotone mapping with Lipschitz constant, L. Assume* $N(A) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence generated* *from an arbitrary $x_1 = w \in H$ by*

$$\begin{cases} y_n = x_n - \beta_n u_n, \\ z_n = x_n - \gamma_n w_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n) z_n, \ n \geq 1, \end{cases} \tag{57}$$

where, $u_n \in Ax_n, w_n \in Ay_n$ such that $||u_n - w_n|| \leq 2D(x_n - Ax_n, y_n - Ay_n) + ||x_n - y_n||$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L'^2+1}+1}$, $\forall n \geq 1$ *for* $L' := 1 + L$.

Then, $\{x_n\}$ converges strongly to a zero of A, nearest to w.

Remark 4.2. Our work improves Theorem 1 and Theorem 2 of Song and Wang [29] and Theorem 2.7 of Shahzad and Zegeye [27] and extends the work of Woldeamanuel *et. al.* [32] for Lipschitz pseudocontractive multi-valued case. It also extends the work of Daman and Zegeye [6] for the multivalued case.

Conflict of Interests

The authors declare that there is no conflict of interests.

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