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COMMON FIXED POINT THEOREMS SATISFYING A NEW TYPE OF WEAK CONTRACTION CONDITION ON A SAKS SPACE

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Abstract. The main purpose of this paper is to establish some common fixed point theorems in Saks spaces under C- class contraction condition for two pairs of discontinuous weak compatible maps. The proved results generalize and extend some of the known results in the literature.

Keywords: common fixed points; C-class contraction; altering distance function; weak compatible maps; Saks Space.

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1. Introduction

We need the following definitions and a lemma to establish some fixed points in Saks Spaces. Now we recall some definitions given by Orlize ([15]).

Definition 1.1. A real valued function N defined on a linear space X is called a B-norm if it satisfies the following conditions:

(1) $N(x) = 0$ if and only if $x = 0$,

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(2) $N(x+y) \leq N(x)+N(y)$,

(3) $N(ax) = |a|N(x)$, where a is any real numbers.

Definition 1.2. A real valued function N defined on a linear space X is called a F-norm if it satisfies the following conditions:

(1) $N(x) = 0$ if and only if $x = 0$,

(2) $N(x+y) \leq N(x)+N(y)$,

(3) if α_n be a sequence of real numbers converges to a real number α and $N(x_n - x) \to 0$ as $n \to \infty$ then $N(\alpha_n.x_n - \alpha x) \to 0$ as $n \to \infty$.

Definition 1.3. A two-norm space is a linear space X with two norms, a B-norm N_1 and F-norm N_2 and is denoted by (X,N_1,N_2).

Definition 1.4. Let N_1 and N_2 be two-norms defined on X, then N_1 is said to be non-weaker than N_2 in X (that is $N_2 \leq N_1$), if

$$N_1(x_n) \to 0 \text{ as } n \to \infty \Rightarrow N_2(x_n) \to 0 \text{ as } n \to \infty$$

where $\{x_n\}$ be a sequence in X.

We shall denote here that the two-norms N_1 and N_2 are equivalent if $(N_1 \leq N_2)$ as well as $(N_2 \leq N_1)$.

Definition 1.5. Let (X,N_1,N_2) be a two-norms space, then the sequence $\{x_n\}$ of X said to be γ- convergent to a point $x \in X$ if

$$\text{Sup } N_1(x_n) < \infty \text{ and } lim_{n\to\infty} N_2(x_n - x) = 0.$$

Definition 1.6. Let (X,N_1,N_2) be a two-norms space, then a sequence $\{x_n\}$ of X is a γ - cauchy sequence if

$$N_2(x_{p_n} - x_{q_n}) \to 0 \text{ as } p_n,\, q_n \to \infty$$

Definition 1.7. A two-norm space (X,N_1,N_2) is called γ - complete, if every γ - cauchy sequence $\{x_n\}$ in two -norm space, there exists a point $x \in X$ such that $x_n \to x$.

Let X be a linear set and suppose that N_1 and N_2 are B-norm and F-norm on X respectively. Let $X_s = (x \in X, N_1(x) < 1)$ and define $d(x,y) = N_2(x-y)$ for all $x,y \in X_s$. Then d is a metric on X_s and the metric space (X_s, d) will be called a Saks Set.

Definition 1.8. Let (X_s, d) be a Saks set, A Saks set is said to be Saks Spaces, if it is complete. We shall denoted this by, (X, N_1, N_2).

Now we recall the following lemma.

Lemma 1.9. Let (X_s, d) = (X, N_1, N_2) be a Saks Space. Then the following statements are equivalent:

(1) N_1 is equivalent to N_2 on X.
(2) (X, N_1) is a Banach Space and $N_1 \leq N_2$ on X .
(3) (X, N_2) is a Frechet Space and $N_2 \leq N_1$ on X.

Throughout this paper, $(X_s, d) = (X, N_1, N_2)$ denotes a Saks Space, in which N_1 is equivalent to N_2 on X.

2. Preliminaries

The weak contraction condition in Hilbert Space was introduced by Alber and Gurerre - Delabriere ([18]). Later Rhoades ([3]) has shown that the result of Alber and Gurerre - Delabriere ([18]) in Hilbert Spaces is also true in a complete metric space. Rhoades [3] established a fixed point theorem in a complete metric space by using the following contraction condition:

A weakly contractive mapping $T : X \to X$ which satisfies the condition

$$d(Tx, Ty) \leq d(x,y) - \varphi(d(x,y)),$$

where $x,y \in X$ and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

Remark: In the above result if $\varphi(t) = (1-k)t$ where $k \in (0,1)$, then we obtain the contraction condition of Banach.

Results on generalized (ϕ,ψ)- weak contractive condition in metric spaces were given by Rhoades ([3]), Dutta and Choudhury ([8]), Zhang and Song ([13]), Doric ([5]), Hosseini ([14]), Abkar and Choudhury ([1]),Murthy, Tas and Choudhary ([9]), Murthy and patel ([10])), Murthy, Tas and Patel ([11]) etc..

Now, we translate the weakly contractive condition in Saks Space from Murthy, Tas and Choudhary ([9]):

A mapping $T : X \to X$, where $(X_s,d) = (X,N_1,N_2)$ is a Saks space is said to be weakly contractive condition if

$$N_2(Tx-Ty) \le N_2(x-y) - \varphi(N_2(x-y))$$

where $x,y \in X$ and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

Definition 2.1.[7] (Altering distance function) A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(1) ψ is monotone increasing and continuous,

(2) $\psi(t) = 0$ if and only if $t = 0$.

In 2014 Ansari [2] introduced the concept of C-clas functions which cover a large class of contractive conditions.

Definition 2.2.[2] We say that $F : [0,\infty)^2 \to \mathbb{R}$ is called *C-class* function if it is continuous and satisfies following axioms:

(1) $F(s,t) \le s$,

(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$,

for all $s,t \in [0,\infty)$.

Note that for some Fwe have $F(0,0)=0$.

We denote the set of C-class functions by $\mathscr{C}$.

Example 2.3. The following functions $F:[0,\infty)^2 \to \mathbb{R}$ are elements of $\mathscr{C}$.

(1)$F(s,t)=ks$, $0<k<1$, $F(s,t)=s \Rightarrow s=0$;

(2) $F(s,t)=s-t$, $F(s,t)=s \Rightarrow t=0$;

(3) $F(s,t)=\frac{s}{(1+t)^r}$; $r\in(0,\infty)$, $F(s,t)=s \Rightarrow s=0$ or $t=0$;

(4) $F(s,t)=\log(t+a^s)/(1+t)$, $a>1$, $F(s,t)=s \Rightarrow s=0$ or $t=0$;

(5) $F(s,t)=\ln(1+a^s)/2$, $a>e$, $F(s,t)=s \Rightarrow s=0$;

(6) $F(s,t)=(s+l)^{(1/(1+t)^r)}-l$, $l>1, r\in(0,\infty)$, $F(s,t)=s \Rightarrow t=0$;

(7) $F(s,t)=s\log_{t+a}a$, $a>1$, $F(s,t)=s \Rightarrow s=0$ or $t=0$;

(8) $F(s,t)=s-(\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t)=s \Rightarrow t=0$;

(9) $F(s,t)=\ln(1+s)$, $F(s,t)=s \Rightarrow s=0$.

We recall the concept of weakly compatible mappings given by initially Gungck and Rhoades ([6]).

Definition 2.4.([6]) A pair of self mappings A and B of a metric space (X,d) is said to be weakly compatible, if they commute at their coincidence points. In other words, if $Ax=Bx$ for some $x\in X$, then $ABx=BAx$.

In this paper, we derive few common fixed point theorems for four maps by using C-class weak contractive condition using more than one control functions in Saks spaces for two pairs of weak compatible maps which is not necessarily continuous.

3. Main results

Theorem 3.1. Let (X_s, d)= (X, N_1, N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let A, B, S and $T : X \to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax - By)) \leq F(\psi(\alpha M(x,y)), \phi(N(x,y))) \tag{1}$$

where $x, y \in X$, $x \neq y$, $\alpha \in (0,1)$,

$$M(x,y) = max\{N_2(Sx - Ty), \tfrac{1}{2}(N_2(Sx - Ax) + N_2(Ty - By)), \tfrac{1}{2}(N_2(Sx - By) + N_2(Ty - Ax))\}$$

and

$$N(x,y) = min\{N_2(Sx - Ty), \tfrac{1}{2}(N_2(Sx - Ax) + N_2(Ty - By)), \tfrac{1}{2}(N_2(Sx - By) + N_2(Ty - Ax))\}$$

(1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
(2) (A, S) and (B, T) are weak compatible pairs,
(3) $\phi : [0, \infty) \to [0, \infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,
(4) $\psi : [0, \infty) \to [0, \infty)$ are altering distance function,
(5) F is element of $\mathscr{C}$.

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$ there exist $x_1 \in X$ such that $Ax_0 = Tx_1$ and for $x_1 \in X$ there exist $x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we construct a sequence

$$y_{2n+1} = A(x_{2n}) = T(x_{2n+1}),\ y_{2n+2} = B(x_{2n+1}) = S(x_{2n+2}).$$

We assume

$$y_{2n} \neq y_{2n+1}, \tag{2}$$

for all $n \in N \cup \{0\}$, where N is set of natural numbers.

First, we have to show that $N_2(y_{2n} - y_{2n+1}) \to 0$ as $n \to \infty$. For this, putting $x = x_{2n}$, $y = x_{2n+1}$

in (3.1), we have

$$\psi_1(N_2(y_{2n+1}-y_{2n+2})) \leq F(\psi(\alpha M(x_{2n},x_{2n+1})),\phi(N(x_{2n},x_{2n+1}))) \tag{3}$$

where

$$M(x_{2n},x_{2n+1}) = max\{N_2(y_{2n}-y_{2n+1}), \tfrac{1}{2}(N_2(y_{2n}-y_{2n+1})+N_2(y_{2n+1}-y_{2n+2})),$$
$$\tfrac{1}{2}(N_2(y_{2n}-y_{2n+2})+N_2(y_{2n+1}-y_{2n+1}))\}$$

and

$$N(x_{2n},x_{2n+1}) = min\{N_2(y_{2n}-y_{2n+1}), \tfrac{1}{2}(N_2(y_{2n}-y_{2n+1})+N_2(y_{2n+1}-y_{2n+2})),$$
$$\tfrac{1}{2}(N_2(y_{2n}-y_{2n+2})+N_2(y_{2n+1}-y_{2n+1}))\}.$$

Then by triangular inequality,

$$M(x_{2n},x_{2n+1}) \leq max\{N_2(y_{2n}-y_{2n+1}), \tfrac{1}{2}(N_2(y_{2n}-y_{2n+1})+N_2(y_{2n+1}-y_{2n+2})),$$
$$\tfrac{1}{2}(N_2(y_{2n}-y_{2n+1})+N_2(y_{2n+1}-y_{2n+2}))\}.$$

If

$$N_2(y_{2n}-y_{2n+1}) < N_2(y_{2n+1}-y_{2n+2}) \tag{4}$$

then, we get

$$M(x_{2n},x_{2n+1}) \leq N_2(y_{2n+1}-y_{2n+2}). \tag{5}$$

Using monotonic increasing property of ψ function, we have

$$\psi(M(x_{2n},x_{2n+1})) \leq \psi(N_2(y_{2n+1}-y_{2n+2})). \tag{6}$$

We have

$$\psi(N_2(y_{2n+1}-y_{2n+2})) \leq F(\psi(\alpha M(x_{2n},x_{2n+1})),\phi(N(x_{2n},x_{2n+1}))) \leq \psi(\alpha M(x_{2n},x_{2n+1}))$$
$$< \psi(M(x_{2n},x_{2n+1})) \leq \psi(N_2(y_{2n+1}-y_{2n+2})).$$

This implies that

$$\psi(N_2(y_{2n+1}-y_{2n+2}))<\psi(N_2(y_{2n+1}-y_{2n+2})).$$

which is a contradiction. Then we have

$$N_2(y_{2n+1}-y_{2n+2})\leq N_2(y_{2n}-y_{2n+1}). \tag{7}$$

Using (3.7), we have

$$M(x_{2n},x_{2n+1})=N_2(y_{2n}-y_{2n+1})\ and\ N(x_{2n},x_{2n+1})=\frac{1}{2}(N_2(y_{2n}-y_{2n+2})). \tag{8}$$

Now putting (3.8) in (3.3), we get

$$\psi(N_2(y_{2n+1}-y_{2n+2}))\leq F(\psi(\alpha N_2(y_{2n},y_{2n+1})),\phi(\frac{1}{2}(N_2(y_{2n},y_{2n+2})))). \tag{9}$$

Again (3.7) implies that $N_2(y_{2n}-y_{2n+1})$ is monotone decreasing sequence of non-negative real number there exist $r>0$ such that

$$lim_{n\to\infty}N_2(y_{2n}-y_{2n+1})=r>0. \tag{10}$$

By virtue of (3.2), we have $N(x_{2n},x_{2n+1})>0$. Taking $n\to\infty$ in (3.9), we get

$$lim_{n\to\infty}\psi(N_2(y_{2n+1}-y_{2n+2}))\leq F(lim_{n\to\infty}\psi(\alpha N_2(y_{2n},y_{2n+1})),lim_{n\to\infty}\phi(\tfrac{1}{2}(N_2(y_{2n},y_{2n+2})))).$$

Using (3.10), which implies that

$$\psi(r)\leq F(\psi(\alpha r),lim_{n\to\infty}\phi(\tfrac{1}{2}(N_2(y_{2n},y_{2n+2}))))\leq\psi(\alpha r)<\psi(r),$$

we observe that the term $lim_{n\to\infty}\phi(\frac{1}{2}(N_2(y_{2n},y_{2n+2}))$ of the above inequality is nonzero. We get a contradiction. Therefore, we have

$$lim_{n\to\infty}N_2(y_{2n}-y_{2n+1})=0. \tag{11}$$

Putting $x=x_{2n+1}$ and $y=x_{2n+2}$ in (3.1) and arguing as above, we have

$$lim_{n\to\infty}N_2(y_{2n+1}-y_{2n+2})=0. \tag{12}$$

Therefore for all $n\in N\cup\{0\}$,

$$lim_{n\to\infty}N_2(y_n-y_{n+1})=0. \tag{13}$$

Next, we prove that $\{y_n\}$ is a cauchy sequence. For this, it is enough to show that the subsequence $\{y_{2n}\}$ is a cauchy sequence. Suppose $\{y_{2n}\}$ is not a cauchy sequence then there exist an

$\varepsilon > 0$ and the sequence of natural number $\{2n(k)\}$ and $\{2m(k)\}$ such that, $2n(k) > 2m(k) > 2k$ for $k \in N$ and

$$N_2(y_{2m(k)} - y_{2n(k)}) \geq \varepsilon \tag{14}$$

corresponding to $2m(k)$, we can choose $2n(k)$ to be the smallest such that (3.14) is satisfied. Then we have

$$N_2(y_{2m(k)} - y_{2n(k)-1}) < \varepsilon. \tag{15}$$

Putting $x = x_{2m(k)-1}$ and $y = x_{2n(k)-1}$ in (3.1), where for all $k \in N$

$$\psi(N_2(y_{2m(k)} - y_{2n(k)})) \leq F(\psi(\alpha M(x_{2m(k)-1}, x_{2n(k)-1})), \phi(N(x_{2m(k)-1}, x_{2n(k)-1}))) \tag{16}$$

where

$M(x_{2m(k)-1}, x_{2n(k)-1}) = max\{N_2(y_{2m(k)-1} - y_{2n(k)-1}), \frac{1}{2}(N_2(y_{2m(k)-1} - y_{2m(k)}) + N_2(y_{2n(k)-1} - y_{2n(k)})),$

$$\frac{1}{2}(N_2(y_{2m(k)-1} - y_{2n(k)}) + N_2(y_{2n(k)-1} - y_{2m(k)}))\}$$

and

$N(x_{2m(k)-1}, x_{2n(k)-1}) = min\{N_2(y_{2m(k)-1} - y_{2n(k)-1}), \frac{1}{2}(N_2(y_{2m(k)-1} - y_{2m(k)}) + N_2(y_{2n(k)-1} - y_{2n(k)})),$

$$\frac{1}{2}(N_2(y_{2m(k)-1} - y_{2n(k)}) + N_2(y_{2n(k)-1} - y_{2m(k)}))\}.$$

Using triangle inequality,

$$N_2(y_{2m(k)} - y_{2n(k)}) \leq N_2(y_{2m(k)} - y_{2n(k)-1}) + N_2(y_{2n(k)-1} - y_{2n(k)}).$$

Letting limit $k \to \infty$,

$$lim_{k\to\infty} N_2(y_{2m(k)} - y_{2n(k)}) = \varepsilon. \tag{17}$$

Again for all k,

$$N_2(y_{2m(k)-1} - y_{2n(k)-1}) \leq N_2(y_{2m(k)} - y_{2m(k)-1}) + N_2(y_{2m(k)} - y_{2n(k)}) + N_2(y_{2n(k)-1} - y_{2n(k)}),$$

$$N_2(y_{2m(k)} - y_{2n(k)}) \leq N_2(y_{2m(k)} - y_{2m(k)-1}) + N_2(y_{2m(k)-1} - y_{2n(k)-1}) + N_2(y_{2n(k)-1} - y_{2n(k)}).$$

Letting limit $k \to \infty$, and using (3.13) and (3.17), we get

$$lim_{k\to\infty} N_2(y_{2m(k)-1} - y_{2n(k)-1}) = \varepsilon. \tag{18}$$

Again for all positive integer k,

$$N_2(y_{2m(k)-1} - y_{2n(k)}) \leq N_2(y_{2m(k)-1} - y_{2m(k)}) + N_2(y_{2m(k)} - y_{2n(k)}),$$

$$N_2(y_{2m(k)} - y_{2n(k)}) \leq N_2(y_{2m(k)} - y_{2m(k)-1}) + N_2(y_{2m(k)-1} - y_{2n(k)}).$$

Letting limit $k \to \infty$, and using (3.13) and (3.18), we get

$$lim_{k\to\infty} N_2(y_{2m(k)-1} - y_{2n(k)}) = \varepsilon. \tag{19}$$

Again for all positive integer k,

$$N_2(y_{2n(k)-1} - y_{2m(k)}) \leq N_2(y_{2n(k)-1} - y_{2n(k)}) + N_2(y_{2n(k)} - y_{2m(k)}),$$

$$N_2(y_{2n(k)} - y_{2m(k)}) \leq N_2(y_{2n(k)} - y_{2n(k)-1}) + N_2(y_{2n(k)-1} - y_{2m(k)}).$$

Letting $lim_k \to \infty$ and using (3.13) and (3.19), we get

$$lim_{k\to\infty} N_2(y_{2n(k)-1} - y_{2m(k)}) = \varepsilon. \tag{20}$$

Using (3.13)- (3.20), we get

$$lim_{k\to\infty} M(x_{2m(k)-1}, x_{2n(k)-1}) = \varepsilon.$$

$$lim_{k\to\infty} N(x_{2m(k)-1}, x_{2n(k)-1}) = 0.$$

Letting $k \to \infty$ in (3.16), we get

$$\psi(\varepsilon) \leq F(\psi(\alpha\varepsilon), lim_{k\to\infty}\phi(N(x_{2m(k)-1}, x_{2n(k)-1}))) \leq \psi(\alpha\varepsilon) < \psi(\varepsilon),$$

which is a contradiction. Hence $\{y_n\}$ is a Cauchy sequence with respect to N_1 by (Lemma 1.9) (X, N_1) is Banach Space. Therefore, the Cauchy sequence $\{y_n\}$ be a convergent sequence and converge to a point z(say) in X, Consequently, the subsequences of $\{y_n\}$ are also converges to z in X.

$$Ax_{2n} \to z\ ,\ Tx_{2n+1} \to z,\ Bx_{2n+1} \to z \text{ and } Sx_{2n} \to z.$$

Now we shall show that z is the common fixed point of A, B, S and T.

Since $B(X) \subset S(X)$, then $\exists\ v \in X$ such that $z = Sv$. Let $N_2(z - Av) \neq 0$. putting $x = v$ and $y = x_{2n+1}$ in (3.1), we get

$$\psi(N_2(Av - Bx_{2n+1})) \leq F(\psi(\alpha M(v, x_{2n+1})), \phi(N(v, x_{2n+1}))) \tag{21}$$

where

$$M(v, x_{2n+1}) = max\{N_2(Sv - Tx_{2n+1}), \tfrac{1}{2}(N_2(Sv - Av) + N_2(Tx_{2n+1} - Bx_{2n+1})),$$

$$\tfrac{1}{2}(N_2(Sv - Bx_{2n+1}) + N_2(Tx_{2n+1} - Av))\} \text{ and}$$

$$N(v, x_{2n+1}) = min\{N_2(Sv - Tx_{2n+1}), \tfrac{1}{2}(N_2(Sv - Av) + N_2(Tx_{2n+1} - Bx_{2n+1})),$$

$$\tfrac{1}{2}(N_2(Sv - Bx_{2n+1}) + N_2(Tx_{2n+1} - Av))\}$$

Letting $n \to \infty$ and using $z = Sv$, we get

$$M(v, z) = max\{N_2(Sv - z), \tfrac{1}{2}(N_2(Sv - Av) + N_2(z - z)), \tfrac{1}{2}(N_2(Sv - z) + N_2(z - Av))\} = \tfrac{1}{2}(N_2(z - Av)).$$

Letting $n \to \infty$ in (3.21)

$$\psi(N_2(Av - z)) \leq F(\psi(\alpha \tfrac{1}{2}(N_2(z - Av))), lim_{n\to\infty}\phi(N(v, x_{2n+1}))).$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$, we observe that the $lim_{n\to\infty}\phi(N(x_{2n+1}, v))$ term is non zero and F is an element of C, we obtain

$$\psi(N_2(Av - z)) \leq F(\psi(\alpha \tfrac{1}{2}(N_2(z - Av))), lim_{n\to\infty}\phi(N(v, x_{2n+1}))) \leq \psi(\alpha \tfrac{1}{2}(N_2(z - Av))).$$

Therefore we have

$$\psi(N_2(Av - z)) \leq \psi(\alpha \tfrac{1}{2}(N_2(z - Av))) < \psi(\tfrac{1}{2}(N_2(z - Av))),$$

a contradiction with the ψ function. Therefore $N_2(z - Av) = 0 \Rightarrow Av = z \Rightarrow Av = z = Sv$.

Since (A, S) is weakly compatible pair of maps, it commute at their coincidence point v i.e. $ASv = SAv \Rightarrow Az = Sz$.

Now we have to show that $Az = Sz = z$. For this,

putting $x = z$ and $y = x_{2n+1}$ in (3.1), we get

$$\psi(N_2(Az - Bx_{2n+1})) \leq F(\psi(\alpha M(z, x_{2n+1})), \phi(N(z, x_{2n+1}))) \tag{22}$$

where

$$M(z, x_{2n+1}) = max\{N_2(Sz - Tx_{2n+1}), \tfrac{1}{2}(N_2(Sz - Az) + N_2(Tx_{2n+1} - Bx_{2n+1})),$$

$$\tfrac{1}{2}(N_2(Sz - Bx_{2n+1}) + N_2(Tx_{2n+1} - Az))\} \text{ and}$$

$$N(z, x_{2n+1}) = min\{N_2(Sz - Tx_{2n+1}), \tfrac{1}{2}(N_2(Sz - Az) + N_2(Tx_{2n+1} - Bx_{2n+1})),$$

$$\tfrac{1}{2}(N_2(Sz - Bx_{2n+1}) + N_2(Tx_{2n+1} - Az))\}.$$

Taking $n \to \infty$ and using $Az = Sz$, we get

$$M(z, z) = N_2(Sz - z).$$

Letting $n \to \infty$ in (3.22)

$$\psi(N_2(Sz - z)) \leq F(\psi(\alpha N_2(Sz - z)), lim_{n\to\infty}\phi(N(z, x_{2n+1}))).$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$, we observe that the $lim_{n\to\infty}\phi(N(x_{2n+1}, z))$ term is non zero and F is an element of C, we get

$$\psi(N_2(Sz - z)) \leq F(\psi(\alpha N_2(Sz - z)), lim_{n\to\infty}\phi(N(z, x_{2n+1}))) \leq \psi(\alpha N_2(Sz - z)) < \psi(N_2(Sz - z)),$$

which is a contradiction. Therefore $N_2(Sz - z) = 0 \Rightarrow Sz = z \Rightarrow Sz = Az = z$.

Since $A(X) \subset T(X)$ then there exist $w \in X$ such that $z = Tw$. Let $N_2(z, Bw) \neq 0$. Putting $x = x_{2n}$ and $y = w$ in (3.1), we get

$$\psi(N_2(Ax_{2n}, Bw)) \leq F(\psi(\alpha M(x_{2n}, w)), \phi(N(x_{2n}, w))) \tag{23}$$

where

$M(x_{2n}, w) = max\{N_2(Sx_{2n}, Tw), \frac{1}{2}(N_2(Sx_{2n}, Ax_{2n}) + N_2(Tw, Bw)), \frac{1}{2}(N_2(Sx_{2n}, Bw) + N_2(Tw, Ax_{2n}))\}$

and

$$N(x_{2n},w)=min\{N_2(Sx_{2n},Tw),\tfrac{1}{2}(N_2(Sx_{2n},Ax_{2n})+N_2(Tw,Bw)),\tfrac{1}{2}(N_2(Sx_{2n},Bw)+N_2(Tw,Ax_{2n}))\}$$

Taking $n \to \infty$ and using $z = Tw$, we have

$$M(z,w) = max\{N_2(z,Tw),\tfrac{1}{2}(N_2(z,z)+N_2(Tw,Bw)),\tfrac{1}{2}(N_2(z,Bw)+N_2(Tw,z))\} = \tfrac{1}{2}(N_2(z,Bw)).$$

Also we have

$$\psi(N_2(z,Bw)) \leq F(\psi(\alpha\tfrac{1}{2}(N_2(z,Bw))),lim_{n\to\infty}\phi(N(z,x_{2n}))) \leq \psi(\alpha\tfrac{1}{2}(N_2(z,Bw)).$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$, we observe that the $lim_{n\to\infty}\phi(N(x_{2n},z))$ term is non zero. Therefore we obtain,

$$\psi(N_2(z,Bw)) \leq \psi(\alpha\tfrac{1}{2}(N_2(z,Bw)) < \psi(\tfrac{1}{2}(N_2(z,Bw)).$$

Hence we arrive at a contradiction with the ψ function.

Therefore $N_2(z,Bw) = 0 \Rightarrow Bw = z \Rightarrow Bw = z = Tw$.

Since (B,T) is the weakly compatible pair of the maps, it commute at their coincidence point w that is $BTw = TBw \Rightarrow Bz = Tz$.

Now we shall to show that $Bz = Tz = z$.

For this, Putting $x = x_{2n}$ and $y = z$ in (3.1), we get

$$\psi(N_2(Ax_{2n},Bz)) \leq F(\psi(\alpha M(x_{2n},z)),\phi(N(x_{2n},z))) \tag{24}$$

where

$M(x_{2n},z)=max\{N_2(Sx_{2n},Tz),\frac{1}{2}(N_2(Sx_{2n},Ax_{2n})+N_2(Tz,Bz)),\frac{1}{2}(N_2(Sx2n,Bz)+N_2(Tz,Ax_{2n}))\}$

and

$$N(x_{2n},z) = min\{N_2(Sx_{2n},Tz),\tfrac{1}{2}(N_2(Sx_{2n},Ax_{2n})+N_2(Tz,Bz)),\tfrac{1}{2}(N_2(Sx2n,Bz)+N_2(Tz,Ax_{2n}))\}$$

Taking $n \to \infty$ and using $Bz = Tz$, we have

$$M(z,z) = max\{N_2(z,Tz),\tfrac{1}{2}(N_2(z,z)+N_2(Tz,Bz)),\tfrac{1}{2}(N_2(z,Bz)+N_2(Tz,z))\} = N_2(z,Bz).$$

Also we have

$$\psi(N_2(z,Bz)) \leq F(\psi(\alpha N_2(z,Bz)), lim_{n\to\infty}\phi(N(x_{2n},z))).$$

Using discontinuity of ϕ at $t=0$ and $\phi(t)>0$ for $t>0$, we observe that the $lim_{n\to\infty}\phi(N(x_{2n},z))$ term is non zero and F is an element of class C. Therefore we have

$$\psi(N_2(z,Bz)) \leq F(\psi(\alpha N_2(z,Bz)), lim_{n\to\infty}\phi(N(x_{2n},z))) \leq \psi(\alpha N_2(z,Bz)),$$

This implies that

$$\psi(N_2(z,Bz)) \leq \psi(\alpha N_2(z,Bz)) < \psi(N_2(z,Bz)),$$

which is a contradiction. Therefore $N_2(z,Bz)=0 \Rightarrow Bz=z \Rightarrow Bz=z=Tz$.
Hence $Az=Bz=Tz=Sz=z$.

Now we shall show that z is the unique common fixed point of A,B,S and T.

Let z_1 is the another fixed point of A,B,S and T such that $z_1=Az_1=Sz_1=Bz_1=Tz_1$. Putting $x=z$ and $y=z_1$ in (3.1), we get

$$\psi(N_2(z-z_1)) \leq F(\psi(\alpha N_2(z-z_1)), \phi(N_2(z-z_1))) \leq \psi(\alpha N_2(z-z_1)) < \psi(N_2(z-z_1)),$$

which is a contradiction. Hence $N_2(z-z_1)=0 \Rightarrow z=z_1$. Hence A,B,S and T have a unique common fixed point in X. □

As an immediate consequence of the above theorem we have the following corollaries. When we take $S=T$ in Theorem 3.1 we have the following:

Corollary 3.2. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let A,B and $T: X\to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax-By)) \leq F(\psi(\alpha M(x,y)), \phi(N(x,y))) \tag{25}$$

where $x,y\in X$, $x\neq y$, $\alpha\in(0,1)$,

$M(x,y)=max\{N_2(Tx-Ty), \frac{1}{2}(N_2(Tx-Ax)+N_2(Ty-By)), \frac{1}{2}(N_2(Tx-By)+N_2(Ty-Ax))\}$

and

$N(x,y)=min\{N_2(Tx-Ty), \frac{1}{2}(N_2(Tx-Ax)+N_2(Ty-By)), \frac{1}{2}(N_2(Tx-By)+N_2(Ty-Ax))\}$

(1) $A(X) \subset T(X)$ and $B(X) \subset T(X)$,

(2) (A,T) and (B,T) are weak compatible pairs,

(3) $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,

(4) $\psi : [0,\infty) \to [0,\infty)$ are altering distance function,

(5) F is an element of $\mathscr{C}$.

Then A, B and T have a unique common fixed point in X.

When we take $A = B$ and $S = T$ in Theorem 3.1 we have the following theorem:

Corollary 3.3. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let $A,S : X \to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax - Ay)) \leq F(\psi(\alpha M(x,y)), \phi(N(x,y))) \tag{26}$$

where $x,y \in X$, $x \neq y$, $\alpha \in (0,1)$,

$$M(x,y) = max\{N_2(Sx - Sy), \tfrac{1}{2}(N_2(Sx - Ax) + N_2(Sy - Ay)), \tfrac{1}{2}(N_2(Sx - Ay) + N_2(Sy - Ax))\}$$

and

$$N(x,y) = min\{N_2(Sx - Sy), \tfrac{1}{2}(N_2(Sx - Ax) + N_2(Sy - Ay)), \tfrac{1}{2}(N_2(Sx - Ay) + N_2(Sy - Ax))\}$$

(1) $A(X) \subset S(X)$,

(2) (A,S) is weak compatible pairs,

(3) $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,

(4) $\psi : [0,\infty) \to [0,\infty)$ are altering distance function,

(5) F is an element of $\mathscr{C}$.

Then A and S have a unique common fixed point in X.

When we take $S = T = Identitymap$ in Theorem 3.1 we have the following:

Corollary 3.4. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on

X. Let $A,B : X \to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax - By)) \leq F(\psi(\alpha M(x,y)), \phi(N(x,y))) \tag{27}$$

where $x,y \in X$ with $x \neq y$, $\alpha \in (0,1)$,

$$M(x,y) = max\{N_2(x-y), \tfrac{1}{2}(N_2(x-Ax)+N_2(y-By)), \tfrac{1}{2}(N_2(x-By)+N_2(y-Ax))\}$$

and

$$N(x,y) = min\{N_2(x-y), \tfrac{1}{2}(N_2(x-Ax)+N_2(y-By)), \tfrac{1}{2}(N_2(x-By)+N_2(y-Ax))\}$$

(1) $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,

(2) $\psi : [0,\infty) \to [0,\infty)$ are altering distance function,

(3) F is an element of class C.

Then A and B have a unique common fixed point in X.

When we take $A = B$ and $S = T = identitymap$ in Theorem 3.1 we have the following:

Corollary 3.5. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let $A : X \to X$ be a self mapping which satisfies the following inequality:

$$\psi(N_2(Ax - Ay)) \leq F(\psi(\alpha M(x,y)), \phi(N(x,y))) \tag{28}$$

where $x,y \in X$ with $x \neq y$, $\alpha \in (0,1)$,

$$M(x,y) = max\{N_2(x-y), \tfrac{1}{2}(N_2(x-Ax)+N_2(y-Ay)), \tfrac{1}{2}(N_2(x-Ay)+N_2(y-Ax))\}$$

and

$$N(x,y) = min\{N_2(x-y), \tfrac{1}{2}(N_2(x-Ax)+N_2(y-Ay)), \tfrac{1}{2}(N_2(x-Ay)+N_2(y-Ax))\}$$

(1) $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,

(2) $\psi : [0,\infty) \to [0,\infty)$ are altering distance function.

(3) F is element of class C.

Then A has a unique fixed point in X.

Remark: When we take $\psi(t) = t$ in Theorem 3.1, Corollaries 3.2, 3.3, 3.4, 3.5 we have the following new corollaries:

Corollary 3.6. Let (X_s, d)= (X, N_1, N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let A, B, S and $T : X \to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax - By)) \leq F((\alpha M(x,y)), \phi(N(x,y))) \tag{29}$$

where $x, y \in X$, $x \neq y$, $\alpha \in (0,1)$,

$M(x,y) = max\{N_2(Sx - Ty), \frac{1}{2}(N_2(Sx - Ax) + N_2(Ty - By)), \frac{1}{2}(N_2(Sx - By) + N_2(Ty - Ax))\}$

and

$N(x,y) = min\{N_2(Sx - Ty), \frac{1}{2}(N_2(Sx - Ax) + N_2(Ty - By)), \frac{1}{2}(N_2(Sx - By) + N_2(Ty - Ax))\}$

(1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(2) (A, S) and (B, T) are weak compatible pairs,

(3) $\phi : [0, \infty) \to [0, \infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,

(4) F is an element of $\mathscr{C}$.

Then A, B, S and T have a unique common fixed point in X.

Corollary 3.7. Let (X_s, d)= (X, N_1, N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let A, B and $T : X \to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax - By)) \leq F((\alpha M(x,y)), \phi(N(x,y))) \tag{30}$$

where $x, y \in X$, $x \neq y$, $\alpha \in (0,1)$,

$M(x,y) = max\{N_2(Tx - Ty), \frac{1}{2}(N_2(Tx - Ax) + N_2(Ty - By)), \frac{1}{2}(N_2(Tx - By) + N_2(Ty - Ax))\}$

and

$N(x,y) = min\{N_2(Tx - Ty), \frac{1}{2}(N_2(Tx - Ax) + N_2(Ty - By)), \frac{1}{2}(N_2(Tx - By) + N_2(Ty - Ax))\}$

(1) $A(X) \subset T(X)$ and $B(X) \subset T(X)$,

(2) (A, T) and (B, T) are weak compatible pairs,

(3) $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,

(4) F is an element of $\mathscr{C}$.

Then A, B and T have a unique common fixed point in X.

Corollary 3.8. Let (X_s, d)= (X, N_1, N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let A and $S : X \to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax - Ay)) \le F((\alpha M(x,y)), \phi(N(x,y))) \tag{31}$$

where $x, y \in X$, $x \neq y$, $\alpha \in (0,1)$,

$$M(x,y) = max\{N_2(Sx - Sy), \tfrac{1}{2}(N_2(Sx - Ax) + N_2(Sy - Ay)), \tfrac{1}{2}(N_2(Sx - Ay) + N_2(Sy - Ax))\}$$

and

$$N(x,y) = min\{N_2(Sx - Sy), \tfrac{1}{2}(N_2(Sx - Ax) + N_2(Sy - Ay)), \tfrac{1}{2}(N_2(Sx - Ay) + N_2(Sy - Ax))\}$$

(1) $A(X) \subset S(X)$,

(2) (A, S) is weak compatible pairs,

(3) $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,

(4) F is an element of $\mathscr{C}$.

Then A and S have a unique common fixed point in X.

Corollary 3.9. Let (X_s, d)= (X, N_1, N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let $A, B : X \to X$ be self mappings which satisfies the following inequality:

$$N_2(Ax - By) \le F(\alpha M(x,y), \phi(N(x,y))) \tag{32}$$

where $x, y \in X$ with $x \neq y$, $\alpha \in (0,1)$,

$$M(x,y) = max\{N_2(x - y), \tfrac{1}{2}(N_2(x - Ax) + N_2(y - By)), \tfrac{1}{2}(N_2(x - By) + N_2(y - Ax))\}$$

and

$$N(x,y) = min\{N_2(x - y), \tfrac{1}{2}(N_2(x - Ax) + N_2(y - By)), \tfrac{1}{2}(N_2(x - By) + N_2(y - Ax))\}$$

(1) $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,

(2) F is an element of class C.

Then A and B have a unique common fixed point in X.

Corollary 3.10. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let $A : X \to X$ be self mapping which satisfies the following inequality:

$$N_2(Ax-Ay) \leq F(\alpha M(x,y), \phi(N(x,y))) \tag{33}$$

where $x,y \in X$ with $x \neq y$, $\alpha \in (0,1)$,

$$M(x,y) = max\{N_2(x-y), \tfrac{1}{2}(N_2(x-Ax)+N_2(y-Ay)), \tfrac{1}{2}(N_2(x-Ay)+N_2(y-Ax))\}$$

and

$$N(x,y) = min\{N_2(x-y), \tfrac{1}{2}(N_2(x-Ax)+N_2(y-Ay)), \tfrac{1}{2}(N_2(x-Ay)+N_2(y-Ax))\}$$

(1) $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,

(2) F is an element of class C.

Then A has a unique fixed point in X.

Similar manner of the Theorem 3.1, we can prove our another main result by replacing:

$$N(x,y) = min\{N_2(Sx,Ty), \tfrac{1}{2}(N_2(Sx,Ax)+N_2(Ty,By)), \tfrac{1}{2}(N_2(Sx,By)+N_2(Ty,Ax))\}$$

by

$$N(x,y) = min\{N_2(Sx,Ty), \tfrac{1}{2}(N_2(Sx,By)+N_2(Ty,Ax))\}$$

the theorem follows:

Theorem 3.11. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let A,B,S and $T : X \to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax-By)) \leq F(\psi(\alpha M(x,y)), \phi(N(x,y))) \tag{34}$$

where $x, y \in X$ with $x \neq y$, $\alpha \in (0,1)$,

$$M(x,y) = max\{N_2(Sx-Ty), \tfrac{1}{2}(N_2(Sx-Ax)+N_2(Ty-By)), \tfrac{1}{2}(N_2(Sx-By)+N_2(Ty-Ax))\}$$

$$N(x,y) = min\{N_2(Sx-Ty), \tfrac{1}{2}(N_2(Sx-By)+N_2(Ty-Ax))\}$$

(1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(2) (A,S) and (B,T) are weak compatible pairs,

(3) $\phi : [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\phi(t) > 0$ for all $t \in (0,\infty)$ and $\phi(0) = 0$,

(4) $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function which in addition is strictly monotone increasing.

(5) F is an element of C.

Then A, B, S and T have a unique common fixed point in X.

Similar manner of the Corollaries of the Theorem 3.1 we can find more corollaries of the Theorem 3.11.

When we take $S = T$ in the Theorem 3.11 we have the following:

Corollary 3.12. Let (X_s, d)= (X, N_1, N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let A, B and $T : X \to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax-By)) \leq F(\psi(\alpha M(x,y)), \phi(N(x,y))) \tag{35}$$

where $x, y \in X$, $x \neq y$, $\alpha \in (0,1)$,

$$M(x,y) = max\{N_2(Tx-Ty), \tfrac{1}{2}(N_2(Tx-Ax)+N_2(Ty-By)), \tfrac{1}{2}(N_2(Tx-By)+N_2(Ty-Ax))\}$$

and

$$N(x,y) = min\{N_2(Tx-Ty), \tfrac{1}{2}(N_2(Tx-By)+N_2(Ty-Ax))\}$$

(1) $A(X) \subset T(X)$ and $B(X) \subset T(X)$,

(2) (A,T) and (B,T) are weak compatible pairs,

(3) $\phi : [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\phi(t) > 0$ for all $t \in (0,\infty)$ and $\phi(0) = 0$,

(4) $\psi:[0,\infty)\to[0,\infty)$ is an altering distance function which in addition is strictly monotone increasing.

(5) F is an element of $\mathscr{C}$.

Then A,B and T have a unique common fixed point in X.

When we take $A=B$ and $S=T$ in Theorem 3.11 we have the following theorem:

Corollary 3.13. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let A and $S:X\to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax-Ay))\leq F(\psi(\alpha M(x,y)),\phi(N(x,y))) \tag{36}$$

where $x,y\in X$, $x\neq y$, $\alpha\in(0,1)$,

$$M(x,y)=max\{N_2(Sx-Sy),\tfrac{1}{2}(N_2(Sx-Ax)+N_2(Sy-Ay)),\tfrac{1}{2}(N_2(Sx-Ay)+N_2(Sy-Ax))\}$$

and

$$N(x,y)=min\{N_2(Sx-Sy),\tfrac{1}{2}(N_2(Sx-Ay)+N_2(Sy-Ax))\}$$

(1) $A(X)\subset S(X)$,

(2) (A,S) is weak compatible pairs,

(3) $\phi:[0,\infty)\to[0,\infty)$ is a lower semi continuous function with $\phi(t)>0$ for all $t\in(0,\infty)$ and $\phi(0)=0$,

(4) $\psi:[0,\infty)\to[0,\infty)$ is an altering distance function which in addition is strictly monotone increasing.

(5) F is an element of $\mathscr{C}$.

Then A and S have a unique common fixed point in X.

When we take $S=T=Identitymap$ in Theorem 3.11 we have the following:

Corollary 3.14. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let $A,B:X\to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax-By))\leq F(\psi(\alpha M(x,y)),\phi(N(x,y))) \tag{37}$$

where $x,y\in X$ with $x\neq y$, $\alpha\in(0,1)$,

$$M(x,y) = max\{N_2(x-y), \tfrac{1}{2}(N_2(x-Ax)+N_2(y-By)), \tfrac{1}{2}(N_2(x-By)+N_2(y-Ax))\}$$

and

$$N(x,y) = min\{N_2(x-y), \tfrac{1}{2}(N_2(x-By)+N_2(y-Ax))\}$$

(1) $\phi : [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\phi(t) > 0$ for all $t \in (0,\infty)$ and $\phi(0) = 0$,

(2) $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function which in addition is strictly monotone increasing.

(3) F is an element of class C.

Then A and B have a unique common fixed point in X.

When we take $A = B$ and $S = T = identitymap$ in Theorem 3.11, we have the following:

Corollary 3.15. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let $A : X \to X$ be self mapping which satisfies the following inequality:

$$\psi(N_2(Ax-Ay)) \leq F(\psi(\alpha M(x,y)), \phi(N(x,y))) \tag{38}$$

where $x,y \in X$ with $x \neq y$, $\alpha \in (0,1)$,

$$M(x,y) = max\{N_2(x-y), \tfrac{1}{2}(N_2(x-Ax)+N_2(y-Ay)), \tfrac{1}{2}(N_2(x-Ay)+N_2(y-Ax))\}$$

and

$$N(x,y) = min\{N_2(x-y), \tfrac{1}{2}(N_2(x-Ay)+N_2(y-Ax))\}$$

(1) $\phi : [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\phi(t) > 0$ for all $t \in (0,\infty)$ and $\phi(0) = 0$,

(2) $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function which in addition is strictly monotone increasing.

(3) F is an element of class C.

Then A has a unique fixed point in X.

Remark: When we take $\psi(t) = t$ in Theorem 3.11, Corollaries 3.12, 3.13, 3.14, 3.15 we have the following new corollaries:

Corollary 3.16. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let A,B,S and $T : X \to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax-By)) \leq F((\alpha M(x,y)),\phi(N(x,y))) \tag{39}$$

where $x,y \in X$, $x \neq y$, $\alpha \in (0,1)$,

$M(x,y) = max\{N_2(Sx-Ty), \frac{1}{2}(N_2(Sx-Ax)+N_2(Ty-By)), \frac{1}{2}(N_2(Sx-By)+N_2(Ty-Ax))\}$

and

$$N(x,y) = min\{N_2(Sx-Ty), \tfrac{1}{2}(N_2(Sx-By)+N_2(Ty-Ax))\}$$

(1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(2) (A,S) and (B,T) are weak compatible pairs,

(3) $\phi : [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\phi(t) > 0$ for all $t \in (0,\infty)$ and $\phi(0) = 0$,

(4) F is an element of $\mathscr{C}$.

Then A,B,S and T have a unique common fixed point in X.

Corollary 3.17. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let A,B and $T : X \to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax-By)) \leq F((\alpha M(x,y)),\phi(N(x,y))) \tag{40}$$

where $x,y \in X$, $x \neq y$, $\alpha \in (0,1)$,

$M(x,y) = max\{N_2(Tx-Ty), \frac{1}{2}(N_2(Tx-Ax)+N_2(Ty-By)), \frac{1}{2}(N_2(Tx-By)+N_2(Ty-Ax))\}$

and

$$N(x,y) = min\{N_2(Tx-Ty), \tfrac{1}{2}(N_2(Tx-By)+N_2(Ty-Ax))\}$$

(1) $A(X) \subset T(X)$ and $B(X) \subset T(X)$,

(2) (A,T) and (B,T) are weak compatible pairs,

(3) $\phi : [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\phi(t) > 0$ for all $t \in (0,\infty)$ and $\phi(0) = 0$,

(4) F is an element of $\mathscr{C}$.

Then A,B and T have a unique common fixed point in X.

Corollary 3.18. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let A and $S:X\to X$ be self mappings which satisfies the following inequality:

$$\psi(N_2(Ax-Ay))\le F((\alpha M(x,y)),\phi(N(x,y))) \tag{41}$$

where $x,y\in X$, $x\neq y$, $\alpha\in(0,1)$,

$$M(x,y)=max\{N_2(Sx-Sy),\tfrac{1}{2}(N_2(Sx-Ax)+N_2(Sy-Ay)),\tfrac{1}{2}(N_2(Sx-Ay)+N_2(Sy-Ax))\}$$

and

$$N(x,y)=min\{N_2(Sx-Sy),\tfrac{1}{2}(N_2(Sx-Ay)+N_2(Sy-Ax))\}$$

(1) $A(X)\subset S(X)$,

(2) (A,S) is weak compatible pairs,

(3) $\phi:[0,\infty)\to[0,\infty)$ is a lower semi continuous function with $\phi(t)>0$ for all $t\in(0,\infty)$ and $\phi(0)=0$,

(4) F is an element of $\mathscr{C}$.

Then A and S have a unique common fixed point in X.

Corollary 3.19. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let $A,B:X\to X$ be self mappings which satisfies the following inequality:

$$N_2(Ax-By)\le F(\alpha M(x,y),\phi(N(x,y))) \tag{42}$$

where $x,y\in X$ with $x\neq y$, $\alpha\in(0,1)$,

$$M(x,y)=max\{N_2(x-y),\tfrac{1}{2}(N_2(x-Ax)+N_2(y-By)),\tfrac{1}{2}(N_2(x-By)+N_2(y-Ax))\}$$

and

$$N(x,y)=min\{N_2(x-y),\tfrac{1}{2}(N_2(x-By)+N_2(y-Ax))\}$$

(1) $\phi:[0,\infty)\to[0,\infty)$ is a lower semi continuous function with $\phi(t)>0$ for all $t\in(0,\infty)$ and $\phi(0)=0$,

(2) F is an element of class C.

Then A and B have a unique common fixed point in X.

Corollary 3.20. Let (X_s,d)= (X,N_1,N_2) is a Saks Space in which N_1 is equivalent to N_2 on X. Let $A:X\to X$ be a self mappings which satisfies the following inequality:

$$N_2(Ax-Ay)\leq F(\alpha M(x,y),\phi(N(x,y))) \tag{43}$$

where $x,y\in X$ with $x\neq y$, $\alpha\in(0,1)$,

$$M(x,y)=max\{N_2(x-y),\tfrac{1}{2}(N_2(x-Ax)+\tfrac{1}{2}(N_2(x-Ay)+N_2(y-Ax))\}$$

and

$$N(x,y)=min\{N_2(x-y),\tfrac{1}{2}(N_2(x-Ay)+N_2(y-Ax))\}$$

(1) $\phi:[0,\infty)\to[0,\infty)$ is a lower semi continuous function with $\phi(t)>0$ for all $t\in(0,\infty)$ and $\phi(0)=0$,

(2) F is an element of class C.

Then A has a unique fixed point in X.

Conflict of Interests

The authors declare that there is no conflict of interests.

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