Available online at http://scik.org
Adv. Fixed Point Theory, 7 (2017), No. 2, 172-182
ISSN: 1927-6303

A CHARACTERIZATION BETWEEN FIXED POINT PROPERTIES OF WEAK NONEXPANSIVE SEMIGROUPS AND THE EXISTENCE OF A LEFT INVARIANT MEAN ON THE SPACE OF WEAKLY ALMOST PERIODIC FUNCTIONS

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Abstract. In this paper, we shall introduce a characterization between new fixed point theorems for weak nonexpansive semi-topological semigroups on a locally convex space and the presence of a left invariant mean on weakly almost periodic functions. Our outcomes expand the results of Lau and Zhang [13, 14] and Lau [15].

Keywords: fixed point property; locally convex space; weak nonexpansive mapping; weakly compact convex set; weakly almost periodic functions.

2010 AMS Subject Classification: Primary 47H10, 47H20, Secondary 43A07.

1. Introduction

Numerous research works are concerned with fixed point for nonexpansive mappings in Banach spaces such as: Browder [5] proved that every nonexpansive mapping on a closed bounded convex subset of a uniformly convex Banach space has a fp (fixed point). Since every uniformly convex Banach space has NS (normal structure)[16, Theorem 3.3.4, p. 148], then Kirk [11]

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Received October 25, 2015

extended the result Due to Browder [5] by showing that if C is WCS (a weakly compact subset) of E with N.S, then C has the fp property. For more information about fp for nonexpansive mappings (see [2, 3, 4, 6, 7, 8, 9, 10, 12, 17, 23]).

A semi-topological semi-group is a pair (S,H), where S is a nonempty set and H is a Hausdorff topology such that, for all $a \in S$, the two functions $s \mapsto sa$ and $s \mapsto as$ are continuous for each $s \in S$. We mean by (E,Q) the space of continuous semi-norms Q on a separated locally convex space E. The action of S on a subset K of E is called Q-non-expansive if it satisfy the following condition:

$$\rho(sx - sy) \leq \rho(x-y) \; \forall \, s \in S, \;\; x,y \in K \text{ and } \rho \in Q.$$

In 1973, Lau [15] proved AP(S) (the space of continuous almost periodic functions on S) has LIM (a left invariant mean) if and only if the following property are holds.

(E) If S is a $Q-$nonexpansive separately continuous action on a compact convex subset C of E, S has a common fp in C.

In 2008, Lau and Zhang [13] proved the following theorem which answered about the open question posed by Lau [21, 22].

Theorem 1.1 [13, Theorem 3.4]. Let S be a separable semitopological semigroup. Then WAP(S) (the space of continuous weakly almost periodic functions on S) has a LIM if and only if

(F) If S be a $Q-$nonexpansive action on WCCS (a weakly compact convex subset) C of (E,Q) and S is WSC (weakly separately continuous) and WQEQ (weakly quasi-equicontinuous), then S has a common fp in C.

The reason for this paper is to prove that if S is Q-weak nonexpansive (Definition 2.2 below) semi-topological semigroups of self-mappings and acts on WCCS of a locally convex space has a common fp if and only if WAP(S) on separable semitopological semigroups has a LIM.

2. Preliminaries

A semitopological semigroup S is said to be strongly left reversible if the set of countable subsemigroups $\{S_\alpha : \alpha \in I\}$ satiesfy: (i) $S = \bigcup_{\alpha \in I} S_\alpha$, (ii) $\overline{aS_\alpha} \bigcap \overline{bS_\alpha} \neq \emptyset$ for each $\alpha \in I$ and $a,b, \in S_\alpha$, and (iii) for each pair $\alpha_1, \alpha_2 \in I$, there is $\alpha_3 \in I$ such that $S_{\alpha_1} \bigcup S_{\alpha_2} \subset S_{\alpha_3}$ (see [13]).

We mean by $l^{\infty}(S)$ the commutative Banach algebra of all bounded complex-valued mappings on S with supremum norm and pointwise multiplication. For each $s \in S$ and $f \in l^{\infty}(S)$ let $l_a f$ and $r_a f$ are the left and right translates of f by a respectivelly, which are defined as: $l_a f(s) = f(as)$ and $r_a f(s) = f(sa)$. Let X ba a closed subalgebra of $l^{\infty}(S)$ containing 1_S. An element μ in X^* is said to be mean on X if $\|\mu\| = \mu(1_S) = 1$. As is well known μ is a mean on X if and only if $\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$. The mean μ is called left (resp. right) invariant, denoted by LIM (resp. RIM) , if $\mu(l_a f) = \mu(f)$ (reps. $\mu(r_a f) = \mu(f)$), for each $a \in S$ and $f \in X$. Assume that $C(S)$ a closed subalgebra of $l^{\infty}(S)$ including of all continuous bounded complex-valued mappings on S. We mean by $AP(S)$ the set of all $f \in C(S)$ such that: $LO(f) = \{l_s f : s \in S\}$ is relatively compact in the norm topology of $C(S)$, and mean by $WAP(S)$ the set of all $f \in C(S)$ such that $LO(f)$ is relatively compact in the weak topology of $C(S)$.

A mapping ψ defined on $S \times K$ into K , denoted by $\psi(s,x) = sx$ ($x \in K$ and $s \in S$), then we called the action S is joint continuous at $(s_0, x_0) \in S \times K$ if for neighbourhood W of $\psi(s_0, x_0)$ there exists a product of open $U \times V \subseteq S \times K$ containing (s_0, x_0) such that $\psi(U \times V) \subseteq W$, and we say that the action S is separately continuous if for each $s_0 \in S$ and $x_0 \in K$ the functions $x \to \psi(s_0, x)$ and $s \to \psi(s, x_0)$ are both continuous on K and S respectively. Thus it is clear that, joint continuity is a stronger condition then separate continuity.

The action S on a convex subset K of a linear topological space is said to be affine if for each $s \in S$ and $x, y \in K$ then $s(\alpha x + (1-\alpha)y) = \alpha sx + (1-\alpha)sy,\ \alpha \in [0,1]$ (see [13]).

Definition 2.1 [13]. Suppose the space (E, Q) be linear topological space with the topology by Q. For any $\rho \in Q$ and $A \subseteq E$, $\delta_\rho(A)$ will denote the ρ - diameter of A, which

$$\delta_\rho(A) = \sup\{\rho(x-y) : x, y \in A\}$$

A convex closed subset C of E has NS if for all closed bounded subset D of C which contains more than one point, and $\rho \in Q$ there is a point $x \in D$ satisfy the following condition

$$r_\rho(D, x) < \delta_\rho(D)$$

where

$$r_\rho(D, x) = \sup\{\rho(x-y) : y \in D\}$$

Lemma 2.1 [13, Lemma 3.1]. Suppose that S acts on a Hausdorff space X and S is quasi-equicontinuous. Then the following statements are holds:

(1) The action of S_0 on X is quasi-equicontinuous if S_0 is a subsemigroup of S,

(2) Every compact $S-$invariant subspace X_0 of compact X implies that the action S on X_0 is quasi-equicontinuous.

Lemma 2.2 [13, Lemma 3.2]. Let S as quasi-equicontinuouson and separately continuous and acts on a compact Hausdorff space X. Then for all $x \in X$ and all $f \in C(X)$, we have $f_x \in WAP(S)$, where f_x is defined by

$$f_x(s) = f(sx) \ \ (s \in S).$$

Lemma 2.3 [13, Lemma 3.3]. Let S Q-nonexpansive, weakly separately continuous and separable semitopological semigroup that acts on a WCCS K of (E,Q). Suppose that F is no-empty minimal weakly compact S-invariant subset of K satisfying $sF = F \ (s \in S)$. Then F is Q-compact.

Lemma 2.4 [13, Lemma 5.3]. Suppose that S acts on a compact Hausdorff space X and the action $S \times X \longrightarrow X$ is called jointly continuous. If there is a dense subset D in S such that $\overline{aS} \bigcap \overline{bS} \neq \emptyset$ for $a,b \in D$, then a non-empty compact subset K of X which is minimal S-invariant satisfies:

(1) $\bar{S}x = K \ \forall \ x \in K$

(2) $sK = K \ \forall \ s \in S$.

Lemma 2.5 [19, Lemma 2]. If C is a non-empty compact subset of separated locally convex (E,Q), and $\rho \in Q$ such that $\delta_\rho > 0$ then there exists an element $u \in \overline{co}(C)$ (depending on ρ) such that

$$\sup\{\rho(u-y) : y \in C\} < \delta_\rho(C),$$

where $\overline{co}(C)$ is the closed convex hull of C.

Lemma 2.6. Let S be a semi-topological Q-weak non-expansive semigroup and acts on WCCS K of a separated locally convex space (E,Q). Then for $a,b \in K$, the following hold:

(i) $\rho(s.a - s^2.b) \leq \rho(a - s.b)$

(ii) Either $\lambda\rho(a - s.a) \leq (a-b)$ or $\lambda\rho(s.a - s^2.a) \leq \rho(s.a - b)$ holds.

(iii) Either $\rho(s.a-s.b) \leq (a-b)$ or $\rho(s^2.a-s.b) \leq \rho(s.a-b)$ holds.

Proof. The proof similar to the proof of [18, Lemma 5].

Definition 2.2 [1]. Let S be a semitopological semigroup and action on a subset $K \subseteq E$. Then S is Q-weak non-expansive if it satisfy the following condition:

$$(1) \qquad \lambda\rho(x-s.x) \leq \rho(x-y) \text{ implies that } \rho(s.x-s.y) \leq \rho(x-y),\ \lambda \in (0,1),$$

for all $s \in S$, $x, y \in K$ and $\rho \in Q$.

Remark 2.1. If we put $\lambda = \frac{1}{2}$, then we get the Suzuki Q-non-expansive condition [18].

3. Main results

In this section, we prove that Q-weak non-expansive S has a common fixed point if and only if the existence of LIM on $WAP(S)$.

Lemma 3.1. Let S be a Q-weak non-expansive and separable continuous semitopological semigroup actions on WCCS K of (E,Q) as weakly separately continuous. Suppose that F is a minimal non-empty weakly compact S-invariant subset of K satisfying $sF \subseteq F$ $(s \in S)$. Then $\overline{co}^w(F)$ is closed and $Q-$separble in $Q-$topology.

Proof. Since F is nonempty minimal S-invariant subset of K, we have $Sa = F$, $a \in F$. From the weakly compactness of F we concludes $\overline{Sa}^W = F$. By the separability of S there exist S_c countable dense subsets of S such that $\overline{S_c x} = Sx$ which implies that $\overline{S_c a} = Sa = F = \overline{Sa}^W$ by the separate continuity of S. Moreover, $\overline{co}^W(Sa) = \overline{co}(S_c a)$. By using Mazur's theorem we have $\overline{co}(S_c a) = \overline{co}^W(S_c a)$ we conclude the desired result.

Lemma 3.2. Let S as in Lemma 3.1. Then F is Q-compact.

Proof. The idea of the proof is the same idea of proof [13, Lemma 3.3] which is based to show that F is $Q-$ totally bounded. Given a neighborhood N of 0 in (E,Q) , then there are finite seminorms $\{p_1,...,p_n\} \subset Q$ and $r, \varepsilon > 0$ such that $U = \{x \in E : p_i(x) < r+\varepsilon; i = 1,...,n\}$ is a neighborhood of 0 contained in N. Then the same conclusion as in the proof of [13, Lemma 3.3] leads to there is a weakly open neighborhood W of 0 and an element $w \in F$ such that $(w+W) \bigcap F \subset w+U$. Take another Q - open symmetrical neighborhood W_1 of 0 such that $W_1 + W_1 \subset W$, and finite seminorms $\{\rho_1,...,\rho_m\} \subset Q$ and $r_0 > 0$ such that $H = \{x \in E : \rho_j(x) <$

$r_0+\varepsilon, j=1,...,m\} \subset W_1$. Therefore, due to the separability property of F in Lemma 3.2, there is a sequence $\{y_n\} \subset F$ such that: $F \subset \cup_{n=1}^{\infty}\{y_n+H\}$. Since F is non-empty minimal, then for all $a \in F, \overline{Sa}^W = F$ and $w \in \overline{Sa}^W$, then there is a sequence $\{s_n\} \subset S$ such that: $s_1y_1 \in w+W_1, s_2s_1y_2 \in w+W_1, ..., s_ns_{n-1}...s_1y_n \in w+W_1 (n=1,2,...)$. If $x \in (s_ns_{n-1}...s_1)(y_n+H)\bigcap F$ therefore $x \in F$ and $x \in (s_ns_{n-1}...s_1)(y_n+H)$. Then x can be written as: $x=s(y_n+h)$, for some $h \in H$ and $s=s_ns_{n-1}...s_1$. By the density of SF in F, there exits elements $a_1,a_2,...,a_n$ in SF such that $\rho_j(y_n-a_n) < \frac{\varepsilon}{3}, \ \forall \ \varepsilon > 0, \forall \ \rho \in Q$. Let $z_n = y_n-a_n$ such that $F \subset \cup_{n=1}^{\infty}\{z_n+H\}$. Since F is non-empty minimal, then for all $a \in F, \overline{Sa}^W = F$ and $w \in \overline{Sa}^W$, then there is a sequence $\{s_n\} \subset S$ such that $s_1z_1 \in w+W_1, s_2s_1z_2 \in w+W_1, ..., s_ns_{n-1}...s_1z_n \in w+W_1 (n=1,2,...)$. If $x \in (s_ns_{n-1}...s_1)(z_n+H)\bigcap F$ therefore $x \in F$ and $x \in (s_ns_{n-1}...s_1)(z_n+H)$. Then x can be written as: $x=s(z_n+h)$, for some $h \in H$ and $s=s_ns_{n-1}...s_1$. Since $\rho_j(z_n) < \frac{\varepsilon}{3}$, then $\rho_j(sz_n - s(0)) < \frac{\varepsilon}{3}\ \forall\, \varepsilon > 0$ (by the continuity of s). Hence

$$
\begin{aligned}
\lambda\rho_j(z_n - sz_n) &\leq \lambda(\rho_j(z_n)+\rho_j(sz_n-s(0))+\rho_j(s(0))) < \lambda(\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\rho_j(s(0))) \\
&< \varepsilon+\rho_j(h),\ \forall\, \varepsilon>0, j=1,.....,m, \qquad (2)
\end{aligned}
$$

where $s(0) \in \cup_{n=1}^{\infty}\{z_n+H\}$, $s(0)=z_k+h$, $h \in H$ for some k.

Take $\varepsilon \to 0$ in (2) and by $Q-$weak nonexpansive, we obtain that

$$
\rho_j(s(z_n+h)-sz_n) \leq \rho_j(h) < r_0. \qquad (3)
$$

From (3) we get that $x \in (s_ns_{n-1},...,s_1)z_n+H$ and then

$$
(s_ns_{n-1}...s_1)(z_n+H)\bigcap F \subseteq (s_ns_{n-1},...,s_1)z_n+H \subset w+W_1+W_1 \subset w+W
$$

Therefore, $\{(s_n...s_1)^{-1}(w+W)\}_{n=1}^{\infty}$ is weakly open cover of F. Therefore $F \subset \bigcup_{k=1}^{n}(s_ks_{k-1}...s_1)^{-1}(w+W)$ for some integer n. According to $F=(s_n...s_1)F$ then

$$
F = \bigcup_{k=1}^{n}(s_n...s_{k+1})(w+W)\bigcap F \subseteq \bigcup_{k=1}^{n}(s_n...s_{k+1})(w+U)\bigcap F.
$$

Let $x \in \bigcup_{k=1}^{n}(s_n...s_{k+1})(w+U)\bigcap F$. Hence $x \in F$ and $x \in \bigcup_{k=1}^{n}(s_n...s_{k+1})(w+U)$, for some $k=1,...,n$. By the density again of SF in F there exist an element c in SF such that $\rho(w-c) < \varepsilon$ $\forall\ \varepsilon > 0$, $\rho \in Q$. Therefore x can be written as $x=\tilde{s}(z+u)$ such that $z=w-c$ for some

$\tilde{s} = s_n ... s_{k+1} \in S$ and $u \in U$. By conclusion of (2), (3) and from $Q-$weak nonexpansive one can get

$$p_i(\tilde{s}(z+u) - \tilde{s}z) < r,\ i = 1, ..., n. \tag{4}$$

Which implies that

$$F \subset \bigcup_{k=1}^{n} (s_n s_{n-1} ... s_{k+1})(w+U) \bigcap F \subset \bigcup_{k=1}^{n} (s_n ..... s_{k+1} z + U)$$

Thus F is $Q-$compact.

Remark 3.1. Whenever S acts on WCCS K of a separated locally convex (E,Q), then the weak continuity implies WQEQ if the action on K is affine and equicontinuous with respect to the topology determined by Q [13].

Consider the following generalized fixed point property.

(GF) Whenever the action S is Q-weak non-expansive, weakly separately continuous and weakly quasi-equicontinuous and acts on a weakly compact convex subset K of a separated locally convex space (E,Q), then S has a common fp in K.

Now, we are in the position to introduce our main theorem in this section.

Theorem 3.1. Let S be a separable semitopological semigroup. Then $WAP(S)$ has a LIM if and only if S has the generalized fixed point property (GF).

Proof. Suppose that (GF) holds and let S acts linearly on $WAP(S)^*$ such as $s(\psi) = l_s^* \psi$ for all $s \in S$ and $\psi \in WAP(S)^*$, where l_s^* is the dual of the translation operator l_s. Hence $(s(\psi))(f) = (l_s^* \psi)(f) = \psi(l_s f)$ for all $f \in WAP(S)$. Let K be the set of all means on $WAP(S)$, then if m_1 and $m_2 \in K$ and $\lambda \in [0,1]$, $(\lambda m_1 + (1-\lambda)m_2)(1_S) = \lambda m_1(1_S) + (1-\lambda)m_2(1_S) = 1$, hence K is convex subset of $AWP(S)^*$. Define $Q = \{\rho_f : f \in WAP(S)\}$ where $\rho_f(\psi) = \sup_{s\in S}\{|\psi(l_s f)|, |\psi(f)|\}, (\psi \in WAP(S)^*)$, then ρ_f is a seminorm on $WAP(S)^*$. One can note that $(WAP(S)^*, Q)$ is separated locally convex space and therefore K is WCCS of $(WAP(S)^*, Q)$. Also, this action on $WAP(S)^*$ (and therefore on K) is Q-weak nonexpansive because it is Q-nonexpansive. Since for all $m \in K$ and $f \in WAP(S)$, $LO(f)$ is relatively compact in the norm of weak topology of $C(S)$, and since the norm topology in $LO(f)$ is the same a the topology of point wise convergence. since the action $(sf)(t) = (l_s f)(t) = f(st)$ is continuous for each $s \in S$, the map $s(f) = l_s f$ is a continuous map $s \to (LO(f)$, weak norm). Hence the action $s(m) = l_s^*(m)$ is continuous on S into (K, weak *). Also it is clearly the action S on $WAP(S)^*$ (and therefore on K) is separately continuous and weakly separated continuous. Since for m_1 and $m_2 \in K$ and $\lambda \in [0,1]$ such that $s(\lambda m_1 + (1-\lambda)m_2) = l_s^*(\lambda m_1 + (1-\lambda)m_2) = \lambda(l_s^* m_1) + (1-\lambda)(l_s^* m_2) = \lambda(sm_1) + (1-\lambda)(sm_2)$, then the action S on K is affine and hence is weak quasi-equicontinuous on K. Since the property (GF) hold, then the action has a fixed point in K for S (let it is m) then $sm = l_s^* m = m$ and since $(l_S^*(m)f) = m(l_s) = m(f)$ for all $f \in WAP(S)$ then m is LIM of $WAP(S)$.

Conversely if $WAP(S)$ has a LIM. Let X be a non-empty minimal WCCS of K which is invariant under S and assume that $F \subset X$ be a non-empty minimal weakly compact subset of X that is invariant under S. By the first paragraph of the proof of [13, Theorem 3.4], F is $Q-$compact. We now follow an idea similar to that in [20, Lemma 2], we show that F contains only one point. Suppose, to the contrary, that F has x_1 and x_2, $x_1 \neq x_2$, (since otherwise F has a common fixed point of s and the proof is finished), there exists a continuous seminorm ρ in Q such that $\rho(x_1 - x_2) = \varepsilon > 0$. Let $\alpha = \lambda x_1 + (1-\lambda)x_2$, $\lambda \in [0,1)$. Then $\alpha \in co(F)$. Moreover $\rho(\alpha - x) \leq \varepsilon \; \forall\, x \in F$ such that $\varepsilon_0 = \sup\{\rho(\alpha - x); x \in F\} < \varepsilon$. Let $\Theta = \{x \in X : \rho(x-y) \leq \varepsilon_0, \forall\, y \in F\}$. Then $\alpha \in \Theta$ and Θ is a nonempty weakly closed convex proper subset of X. Furthermore, if $x \in \Theta$, then $\rho(x-y) \leq \varepsilon_0$, $y \in F$. Since S is $Q-$weak nonexpansive, then by Lemma 2.6 (iii) one can obtained that

$$\rho(sx - sy) \leq \varepsilon_0, \;\; or \;\; \rho(sx - s^2 y) \leq \varepsilon_0. \tag{5}$$

From (5), we get that $sx \in \Theta$ ($s \in S$, $x \in \Theta$), which implies that Θ is $S-$invariant. This implies to a contradiction to the minimality of X. then F must include a single common fp for S.

Remark 3.2. Theorem 3.1 extending result of Lemma 3.13 and Theorem 3.14 due to Lau and Zhang [14].

Theorem 3.2. Let S be a separable semitopological semigroup. If $AP(S)$ has a LIM , then the fixed point property (GE) holds.

(GE) Suppose that S acts on a WCCS K of a separated (E,Q) as Q-weak non-expansive self mappings and, the action is separately continuous and equicontinuous when K is equipped with the weak topology of (E,Q) then S has a common fixed point in K.

The proof is similar to Theorem 3.1 and [15, Theorem 3.2]

Theorem 3.3. A semitopological semigroup S has the following property $(G\acute{E})$ if and only if $AP(S)$ has *LIM*

$(G\acute{E})$ Whenever S acts on a weakly compact convex space (E,Q) as Q-weak non-expansive mappings, if K has Q - NS and the S-action is equicontinuous and separately continuous when K is equipped with the weak topology of (E,Q) , then K contains a common fp for S.

Proof. Let $AP(S)$ has a LIM ψ, and X be a set that is non-empty minimal WCCS of K that is invariant under S action. Consider $F \subset X$ be a non-empty minimal weakly compact subset of X that is invariant under S. Since the action on X is equicontinuous and separately continuous, f_y for each $f \in C(F)$ and $y \in F$. Hence μ defined by $\mu(f) = \psi(f_y)$ is a mean on $C(F)$. By the same steps in the proof of Theorem 3.1, we get F is Q-compact and Q-bounded. Let F has x_1,x_2 such that $x_1 \neq x_2$ and by taking $\alpha = \lambda x_1 + (1-\lambda)x_2$ where $\lambda \in [0,1]$ and $\rho \in Q$, by NS of K

$$r_0 = \sup\{\rho(\alpha - x) : x \in F\} < \delta_r(F)$$

Then by the same argument as in the proof of Theorem 3.1 lead to contradiction, consequently F must consist of single point and this point is a common fixed point for S. Conversely, let $(G\acute{E})$ holds. By replace E by $AP(S)^*$ with respect to the topology which determined by the family of continuous semi-norm $Q = \{\rho_f : f \in WAP(S)\}$ where

$$\rho_f(\psi) = \sup_{s\in S}\{|\psi(l_s f)|, |\psi(f)|\} \qquad (\psi \in AP(S)^*).$$

One can define S as a action on $AP(S)^*$ by $s(\psi) = l_s^*\psi$ for all $s \in S$ and $\psi \in AP(S)^*$. It is easy to see that, the semigroup S acts linearly on $AP(S)^*$ by $s \mapsto l_s^*$. Let K be the family of all means on $AP(S)$, therefore K is compact closed subset of $AP(S)^*$. Since from Lemma 2.5, a compact subset of separated locally convex space has normal structure, K has $Q-$ normal structure. By the same conclusion as in the proof of Theorem 3.1, it is clear the action of S on $AP(S)^*$ (and therefore on K) is equicontinuous and separately continuous with respect to the topology determined by Q, and Q-weak nonexpansive. Since property $(G\acute{E})$ hold. Then K has a common fp for S, which is a LIM on $AP(S)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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