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ON ABSTRACT FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS VIA MEASURE OF NONCOMPACTNESS

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Abstract. This paper deals with the study of existence, uniqueness and stability of solutions of fractional integro-differential equations involving Caputo derivative in Banach spaces. Our tools employed are fractional calculus, Hölder inequality and Darbo-Sadovskii fixed point theorem via Hausdorff's measure of noncompactness.

Keywords: Fractional integro-differential equations; Boundary value problems; Darbo-Sadovskii fixed point theorem; Measure of noncompactness; Uniqueness and stability.

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1. Introduction

The ordinary differential equations is considered the basis of the fractional differential e-quations. Comparing with integer derivatives, the most important advantage of fractional

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derivatives that it could describes the property of memory and heredity of diverse materials and processes. For more details about fractional calculus and its applications we refer the reader to Hilfer [10], Kilbas et al. [13], Miller and Ross [16], Podlubny [17], Samko et al. [18], and the references given therein. In last decades, measures of noncompactness became inevitable procedure in nonlinear analysis. They are used in different fields such as fixed point theory, linear operator theory, theory of differential and integral equations. Among the theory of measures of noncompactness, the two ones: the Kuratowski measure of noncompactness and the Hausdorff measure of noncompactness are considered the most important. Hausdorff measure of noncompactness, in particular, is often used in many branches of nonlinear analysis and its applications. See for examples [2, 3, 4, 6, 8, 11, 14, 15, 19, 21, 22], and the references given therein.

In recent years, sufficient conditions for the existence and uniqueness of solutions have been established by Agarwal et al. [1] in the space $\mathbb{R}$ for the following fractional boundary value problems (for short BVP)

$$\begin{cases} {}^cD^{\alpha}y(t) = f\big(t, y(t)\big), t \in J = [0,T], \alpha \in (0,1], \\ ay(0) + by(T) = c. \end{cases}$$

Wang et al. [20] extended the work [1], in abstract spaces X by use more general conditions on nonlinear function f . Karthikeyan and Trujillo [12] used Banach contraction principle and Schaefer's fixed point theorem to study the existence and uniqueness of fractional integrodifferential equations with boundary condition

$$\begin{cases} {}^cD^{\alpha}y(t) = f\big(t, y(t), \int_0^t k(t,s)y(s)ds\big), t \in J = [0,T], \alpha \in (0,1), \\ ay(0) + by(T) = c. \end{cases}$$

In this paper, we develop the works in [1, 20, 12] by studying the following more general boundary value problem for fractional integro-differential equation

$$\begin{cases} {}^cD^{\alpha}x(t) = f\big(t, x(t), (Sx)(t)\big), t \in J = [0,T], \alpha \in (0,1], \\ ax(0) + bx(T) = c, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha, f : J \times X \times X \to X$ is a given function and a, b, c are real numbers with $a + b \neq 0$ and S is a nonlinear integral operator given by $(Sx)(t) = \int_0^t k\big(t, s, x(s)\big) ds$, where $k \in C(J \times J \times X, X)$.

This paper is organized as follows. In Section 2, we recall some preliminaries. Section 3, proves main results by using Hausdorff's measure of noncompactness and Darbo-Sadovskii fixed point theorem. In Section 4, we establish sufficient conditions for the stability and uniqueness of solutions of the fractional BVP.

2. **Preliminaries**

Here in this section, we shall give the following preliminaries that will be used in our future discussion.

Let X be a Banach space with the norm $\|\cdot\|$. We denote by $C(J, X)$ the space of X-valued continuous functions on J with the supremum norm

$$\|x\|_\infty := \sup\{\|x(t)\| : t \in J\}.$$

For measurable functions $m : J \to \mathbb{R}$, define the norm $\|m\|_{L^p(J,\mathbb{R})} = (\int_J |m(t)|^p dt)^{\frac{1}{p}}, 1 \leq p < \infty$, where $L^p(J, \mathbb{R})$ the Banach space of all Lebesgue measurable functions m with $\|m\|_{L^p(J,\mathbb{R})} < \infty$. Throughout this paper, we denote $\mathbb{R}_+ = [0, \infty)$.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a suitable function h is defined by*

$$I^\alpha_{a+} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where $a \in \mathbb{R}$ and Γ is the Gamma function.

Definition 2.2. *For a suitable function h given on the interval $[a,b]$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ of h, is defined by*

$$(D_{a+}^{\alpha}h)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}(t-s)^{n-\alpha-1}h(s)ds,$$

where $n = [\alpha]+1, [\alpha]$ denotes the integer part of α.

Definition 2.3. *For a suitable function h given on the interval $[a,b]$, the Caputo fractional order derivative of order $\alpha > 0$ of h, is defined by*

$$({}^{c}D_{a+}^{\alpha}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}h^{(n)}(s)ds,$$

where $n = [\alpha]+1, [\alpha]$ denotes the integer part of α.

Remark 2.1. *We note that, if h is an abstract function with values in X, then integrals which appear in Definitions 2.1, 2.2 and 2.3 are taken in Bochner's sense.*

Definition 2.4. *A function $x \in C^{1}(J,X)$ is said to be a solution of the fractional BVP* (1.1) *if x satisfies the equation ${}^{c}D^{\alpha}x(t) = f\big(t,x(t),(Sx)(t)\big)$ a.e. on J, and the condition $ax(0)+bx(T) = c$.*

For the existence of solutions for the fractional BVP (1.1), we need the following auxiliary lemma.

Lemma 2.1. *(Lemma 3.2,*[1]*) A function $x \in C(J,X)$ is solution of the fractional integral equation*

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}\overline{f}(s)ds - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_{0}^{T}(T-s)^{\alpha-1}\overline{f}(s)ds - c\right], \tag{2.1}$$

if and only if, x is a solution of the following fractional BVP

$$\begin{cases} {}^{c}D^{\alpha}x(t) = \overline{f}(t), t \in J = [0,T], \alpha \in (0,1], \\ ax(0)+bx(T) = c. \end{cases} \tag{2.2}$$

Consequently, we can prove the following result which is useful in further discussion.

Lemma 2.2. *Let $f : J\times X\times X \to X$ be continuous function. Then, $x \in C(J,X)$ is a solution of the fractional integral equation*

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s,x(s),(Sx)(s)\big)ds - \frac{1}{a+b}\Big[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f\big(s,x(s),(Sx)(s)\big)ds - c\Big], \tag{2.3}$$

if and only if x is solution of the fractional BVP (1.1).

Now, we recall the Hausdorff's measure of noncompactness $\Psi_Y(\cdot)$ defined by

$$\Psi_Y(B) = \inf\{r > 0 |\ B \text{ can be covered by finite union of balls with radii } r\}.$$

for bounded set B in a Banach space Y.

Some basic properties of $\Psi_Y(\cdot)$ are presented in the following lemma.

Lemma 2.3. *(*[2]*) Let Y be a real Banach space and $B,C \subseteq Y$ be bounded, then the following properties are satisfied:*

(1) *B is precompact if and only if $\Psi_Y(B) = 0$;*
(2) *$\Psi_Y(B) = \Psi_Y(\overline{B}) = \Psi_Y(convB)$ where $\overline{B}$ and $convB$ mean the closure and convex hull of B respectively;*
(3) *$\Psi_Y(B) \leq \Psi_Y(C)$ when $B \subseteq C$;*
(4) *$\Psi_Y(B+C) \leq \Psi_Y(B)+\Psi_Y(C)$ where $B+C = \{x+y; x\in B, y\in C\}$;*
(5) *$\Psi_Y(B\cup C) \leq \max\{\Psi_Y(B),\Psi_Y(C)\}$;*
(6) *$\Psi_Y(\lambda B) = |\lambda|\Psi_Y(B)$ for any $\lambda \in \mathbb{R}$;*
(7) *If the map $Q : D(Q) \subseteq Y \to Z$ is Lipschitz continuous with constant k then $\Psi_Z(Q(B)) \leq k\Psi_Y(B)$ for any bounded set $B \subseteq D(Q)$, where Z is Banach space;*
(8) *$\Psi_Y(B) = \inf\{d_Y(B,C); C \subseteq Y$ be precompact $\} = \inf\{d_Y(B,C); C \subseteq Y$ be finite valued $\}$, where $d_Y(B,C)$ means the nonsymmetric (or symmetric) Hausdroff distance between B and C in Y;*
(9) *If $\{\mathbb{W}_n\}_{n=1}^{\infty}$ is decreasing sequence of bounded, closed nonempty subsets of Y and $\lim_{n\to+\infty}\Psi_Y(\mathbb{W}_n) = 0$, then $\cap_{n=1}^{+\infty}\mathbb{W}_n$ is nonempty and compact in Y.*

Definition 2.5. *The map $Q : \mathbb{W} \subseteq Y \to Y$ is said to be a Ψ_Y-contraction if there exists a positive constant $k < 1$ such that $\Psi_Y(Q(S)) \leq k\Psi_Y(S)$ for any bounded closed subset $S \subseteq \mathbb{W}$ where Y is a Banach space.*

Lemma 2.4. *([2], Darbo-Sadovskii theorem) If $\mathbb{W} \subseteq Y$ is bounded, closed and convex, the continuous map $Q : \mathbb{W} \to \mathbb{W}$ is a Ψ_Y-contraction, then the map Q has at least one fixed point in $\mathbb{W}$.*

We denote Ψ by the Hausdorff's measure of noncompactness of X and also denote Ψ_c by the Hausdorff's measure of noncompactness of $C(J,X)$.

Lemma 2.5. *([2]) If $\mathbb{W} \subseteq C([a,b],X)$ is bounded, then*

$$\Psi(\mathbb{W}(t)) \leq \Psi_c(\mathbb{W}),$$

for all $t \in [a,b]$, where $\mathbb{W}(t) = \{u(t); u \in \mathbb{W}\} \subseteq X$. Furthermore if $\mathbb{W}$ is equicontinuous on $[a,b]$, then $\Psi(\mathbb{W}(t))$ is continuous on $[a,b]$ and

$$\Psi_c(\mathbb{W}) = \sup\{\Psi(\mathbb{W}(t)), t \in [a,b]\}.$$

Lemma 2.6. *([11]) If $\{u_n\}_{n=1}^{\infty} \subset L^1([a,b],X)$ is uniformly integrable, then the function $\Psi(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and*

$$\Psi\Big(\Big\{\int_0^t u_n(s)ds\Big\}_{n=1}^{\infty}\Big) \leq 2\int_0^t \Psi(\{u_n(s)\}_{n=1}^{\infty})ds.$$

Lemma 2.7. *([2]) If $\mathbb{W} \subseteq C([a,b],X)$ is bounded and equicontinuous, then $\Psi(\mathbb{W}(t))$ is continuous and*

$$\Psi\Big(\int_0^t \mathbb{W}(s)ds\Big) \leq \int_0^t \Psi(\mathbb{W}(s))ds,$$

for all $t \in [a,b]$, where $\int_0^t \mathbb{W}(s)ds = \{\int_0^t u(s)ds; u \in \mathbb{W}\}$.

Lemma 2.8. *([3]) If $\mathbb{W}$ is bounded, then for each $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subseteq \mathbb{W}$, such that*

$$\Psi(\mathbb{W}) \leq 2\Psi(\{u_n\}_{n=1}^{\infty}) + \varepsilon.$$

3. **Existence of Solutions**

To study the existence of solutions, we list the following hypotheses:

- (H1) The function $f : J \times X \times X \to X$ satisfies the Carathéodory-type conditions; i.e., $f(\cdot,x,y)$ is measurable for all $x,y \in X$ and $f(t,\cdot,\cdot)$ is continuous for a.e $t \in [0,T]$.
- (H2) The function $k : J \times J \times X \to X$ satisfies the Carathéodory-type conditions; i.e., $k(\cdot,\cdot,x)$ is measurable for all $x \in X$ and $k(t,s,\cdot)$ is continuous for a.e $s \in [0,t], t \in J$.
- (H3) There exist constants $N_f, N_k > 0$, such that

$$\|f\big(t,x(t),(Sx)(t)\big)\| \leq N_f\big(1+\|x(t)\|+\|(Sx)(t)\|\big),$$

$$\|k\big(t,s,x(s)\big)\| \leq N_k\big(1+\|x(s)\|\big),$$

for each $s \in [0,t]$, $t \in J$ and all $x \in X$.
- (H4) There exists a constant $\alpha_1 \in (0,\alpha)$ and real-valued functions $m_1(t), m_2(t) \in L^{\frac{1}{\alpha_1}}(J,\mathbb{R})$, such that

$$\Psi\left(f\big(t,B,C\big)\right) \leq m_1(t)\big(\Psi(B)+\Psi(C)\big),$$

$$\Psi\left(k\big(t,s,C\big)\right) \leq m_2(t)\Psi(C),$$

for bounded sets $B,C \subset X$, a.e $t \in J$.

For brevity, let $M = \|m_1 + 2Tm_1m_2\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R})}$.

- (H5) Suppose that

$$\Big(1+\frac{|b|}{|a+b|}\Big)\frac{N_f(1+N_kT)T^{\alpha}}{\Gamma(\alpha+1)} \leq 1.$$

Now, we are in position to state and prove our main results.

Theorem 3.1. *Assume that (H1)-(H5) hold. If*

$$\left[1+\frac{|b|}{|a+b|}\right]\frac{4M}{\Gamma(\alpha)}\left(\frac{T^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}}\right)^{1-\alpha_1} < 1, \tag{3.1}$$

then, the fractional BVP (1.1) *has at least one solution on J.*

Proof. Define the mapping $F : C(J,X) \to C(J,X)$ as follows:

$$(3.2)\quad \begin{aligned}(F(x))(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s,x(s),(Sx)(s)\big)ds \\ &\quad - \frac{1}{a+b}\Big[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f\big(s,x(s),(Sx)(s)\big)ds - c\Big], t \in J.\end{aligned}$$

From Lemma 2.2, it is clear that the fixed point of the operator F is the solution of the boundary value problem problem (1.1).

First, we show that F is continuous operator on $C(J,X)$. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C(J,X)$ as $n \to \infty$. For every $t \in J$ and by hypotheses (H1) and (H2), we have

$$f\big(t,x_n(t),(Sx_n)(t)\big) \to f\big(t,x(t),(Sx)(t)\big).$$

Using the dominated convergence theorem, we get

$$\begin{aligned}&\|(F(x_n))(t) - (F(x))(t)\| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \big\|f\big(s,x_n(s),(Sx_n)(s)\big) - f\big(s,x(s),(Sx)(s)\big)\big\| ds \\ &\quad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} \big\|f\big(s,x_n(s),(Sx_n)(s)\big) - f\big(s,x(s),(Sx)(s)\big)\big\| ds \\ &\le \frac{\big\|f\big(\cdot,x_n(\cdot),(Sx_n)(\cdot)\big) - f\big(\cdot,x(\cdot),(Sx)(\cdot)\big)\big\|_\infty}{\Gamma(\alpha)} \\ &\quad \times \Big[\int_0^t (t-s)^{\alpha-1}ds + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds\Big] \\ &\le \frac{T^\alpha}{\Gamma(\alpha+1)}\Big(1 + \frac{|b|}{|a+b|}\Big)\big\|f\big(\cdot,x_n(\cdot),(Sx_n)(\cdot)\big) - f\big(\cdot,x(\cdot),(Sx)(\cdot)\big)\big\|_\infty.\end{aligned}$$

Taking supremum, we get

$$\begin{aligned}&\|Fx_n - Fx\|_\infty \\ &\le \frac{T^\alpha}{\Gamma(\alpha+1)}\Big(1 + \frac{|b|}{|a+b|}\Big)\big\|f\big(\cdot,x_n(\cdot),(Sx_n)(\cdot)\big) - f\big(\cdot,x(\cdot),(Sx)(\cdot)\big)\big\|_\infty.\end{aligned}$$

Hence,

$$\|Fx_n - Fx\|_\infty \to 0 \text{ as } n \to \infty.$$

Therefore, F is continuous operator.

Set

$$\mathbb{W}_r = \{x \in C(J,X), \|x\|_\infty \le r\}$$

for each $r \in N$ (the set of all positive integers). Then $\mathbb{W}_r \subset C(J,X)$ is bounded and convex.

Now, suppose that for each $r \in N$ there is $x \in \mathbb{W}_r$ and $t \in J$ such that $\|(F(x))(t)\| > r$. Using (H3), we have

$$\begin{aligned}
r &< \|(F(x))(t)\| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f\big(s, x(s), (Sx)(s)\big)\| ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|f\big(s, x(s), (Sx)(s)\big)\| ds + \frac{|c|}{|a+b|} \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f \big(1 + \|x(s)\| + \|(Sx)(s)\|\big) ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} N_f \big(1 + \|x(s)\| + \|(Sx)(s)\|\big) ds + \frac{|c|}{|a+b|} \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f \big(1 + \|x\|_\infty + N_k(1+\|x\|_\infty)T\big) ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} N_f \big(1 + \|x\|_\infty + N_k(1+\|x\|_\infty)T\big) ds + \frac{|c|}{|a+b|} \\
&\le \frac{N_f(1+r)(1+N_kT)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&\quad + \frac{|b| N_f(1+r)(1+N_kT)}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds + \frac{|c|}{|a+b|} \\
&\le \frac{N_f(1+r)(1+N_kT)}{\Gamma(\alpha+1)} T^\alpha + \frac{|b| N_f(1+r)(1+N_kT)}{|a+b|\Gamma(\alpha+1)} T^\alpha + \frac{|c|}{|a+b|}.
\end{aligned}$$

Then, we get

$$r < \Big(1 + \frac{|b|}{|a+b|}\Big) \frac{T^\alpha}{\Gamma(\alpha+1)} N_f (1+r)(1+N_kT) + \frac{|c|}{|a+b|}. \tag{3.3}$$

Dividing by r on both sides of (3.3), we obtain

$$1 < \Big(1 + \frac{|b|}{|a+b|}\Big) \frac{T^\alpha}{\Gamma(\alpha+1)} N_f \Big(\frac{1}{r} + 1\Big)(1 + N_k T) + \frac{|c|}{r|a+b|}. \tag{3.4}$$

Now taking lim as $r \to \infty$ on both sides of (3.4), we get

$$1 < \Big(1 + \frac{|b|}{|a+b|}\Big) \frac{T^\alpha}{\Gamma(\alpha+1)} N_f (1 + N_k T),$$

which is contradiction to the hypothesis (H5). Thus there is some $r \in N$ such that $F(\mathbb{W}_r) \subseteq \mathbb{W}_r$, that is, $F : \mathbb{W}_r \to \mathbb{W}_r$.

Next, let B bounded subset of $\mathbb{W}_r$, $x \in B$ and $0 \le t_1 \le t_2 \le T$; then by using (H3), we have

$$\begin{aligned}
&\|(F(x))(t_2) - (F(x))(t_1)\| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \big[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\big] \big\| f\big(s, x(s), (Sx)(s)\big)\big\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \big\| f\big(s, x(s), (Sx)(s)\big)\big\| ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \big[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\big] N_f \big(1 + \|x(s)\| + \|(Sx)(s)\|\big) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} N_f \big(1 + \|x(s)\| + \|(Sx)(s)\|\big) ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \big[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\big] N_f \big(1 + \|x\|_\infty + N_k(1 + \|x\|_\infty) T\big) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} N_f \big(1 + \|x\|_\infty + N_k(1 + \|x\|_\infty) T\big) ds \\
&\le \frac{N_f(1+r)(1+N_k T)}{\Gamma(\alpha)} \int_0^{t_1} \big[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\big] ds \\
&\quad + \frac{N_f(1+r)(1+N_k T)}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \\
&\le \frac{N_f(1+r)(1+N_k T)}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha).
\end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero and since x is an arbitrary in B, we conclude that $F(B) \subseteq C(J, X)$ is bounded and equicontinuous.

Finally, we shall prove that F is Ψ_c-contraction on $\mathbb{W}_r$. For every bounded subset $B \subset \mathbb{W}_r, t \in J$ and every $\varepsilon > 0$, there is a sequence $\{u_k\}_{k=1}^\infty \subset B$, by using Lemma 2.3, Lemma 2.5, Lemma 2.6, Lemma 2.7, Lemma 2.8, (H4) and Hölder inequality, we can obtain

$$\begin{aligned}
&\Psi(F(B(t)))\\
&=\Psi\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\big(s,B(s),(SB)(s)\big)ds\\
&\quad-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f\big(s,B(s),(SB)(s)\big)ds+\frac{c}{a+b}\Big)\\
&\leq 2\Psi\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\big(s,\{u_k(s)\}_{k=1}^{\infty},\big(S\{u_k(s)\}_{k=1}^{\infty}\big)\big)ds\\
&\quad-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f\big(s,\{u_k(s)\}_{k=1}^{\infty},\big(S\{u_k(s)\}_{k=1}^{\infty}\big)\big)ds+\frac{c}{a+b}\Big)+\varepsilon\\
&\leq\frac{4}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Psi\Big(f\big(s,\{u_k(s)\}_{k=1}^{\infty},\big(S\{u_k(s)\}_{k=1}^{\infty}\big)\big)\Big)ds\\
&\quad+\frac{4|b|}{|a+b|\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\Psi\Big(f\big(s,\{u_k(s)\}_{k=1}^{\infty},\big(S\{u_k(s)\}_{k=1}^{\infty}\big)\big)\Big)ds+\varepsilon\\
&\leq\frac{4}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}m_1(s)\Big(\Psi\big(\{u_k(s)\}_{k=1}^{\infty}\big)+\Psi\big(S\{u_k(s)\}_{k=1}^{\infty}\big)\big)\Big)ds\\
&\quad+\frac{4|b|}{|a+b|\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}m_1(s)\Big(\Psi\big(\{u_k(s)\}_{k=1}^{\infty}\big)+\Psi\big(S\{u_k(s)\}_{k=1}^{\infty}\big)\big)\Big)ds+\varepsilon\\
&\leq\frac{4}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}m_1(s)\Big(\Psi\big(\{u_k(s)\}_{k=1}^{\infty}\big)+\Psi\Big(\int_0^s k\big(s,\tau,\{u_k(\tau)\}_{k=1}^{\infty}\big)d\tau\Big)\Big)ds\\
&\quad+\frac{4|b|}{|a+b|\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\\
&\quad\times m_1(s)\Big(\Psi\big(\{u_k(s)\}_{k=1}^{\infty}\big)+\Psi\Big(\int_0^s k\big(s,\tau,\{u_k(\tau)\}_{k=1}^{\infty}\big)d\tau\Big)\Big)ds+\varepsilon\\
&\leq\frac{4}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}m_1(s)\Big(\Psi\big(\{u_k(s)\}_{k=1}^{\infty}\big)+2\int_0^s\Psi\Big(k\big(s,\tau,\{u_k(\tau)\}_{k=1}^{\infty}\big)d\tau\Big)\Big)ds\\
&\quad+\frac{4|b|}{|a+b|\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\\
&\quad\times m_1(s)\Big(\Psi\big(\{u_k(s)\}_{k=1}^{\infty}\big)+2\int_0^s\Psi\Big(k\big(s,\tau,\{u_k(\tau)\}_{k=1}^{\infty}\big)d\tau\Big)\Big)ds+\varepsilon\\
&\leq\frac{4}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}m_1(s)\Big(\Psi\big(\{u_k(s)\}_{k=1}^{\infty}\big)+2\int_0^s m_2(s)\Psi\big(\{u_k(\tau)\}_{k=1}^{\infty}\big)d\tau\Big)ds\\
&\quad+\frac{4|b|}{|a+b|\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\\
&\quad\times m_1(s)\Big(\Psi\big(\{u_k(s)\}_{k=1}^{\infty}\big)+2\int_0^s m_2(s)\Psi\big(\{u_k(\tau)\}_{k=1}^{\infty}\big)d\tau\Big)ds+\varepsilon
\end{aligned}$$

$$\leq \frac{4}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} m_1(s)\Big(\Psi_c\big(B\big)+2Tm_2(s)\Psi_c\big(B\big)\Big)ds$$
$$+\frac{4|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} m_1(s)\Big(\Psi_c\big(B\big)+2Tm_2(s)\Psi_c\big(B\big)\Big)ds+\varepsilon$$
$$\leq \frac{4\Psi_c\big(B\big)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(m_1(s)+2Tm_1(s)m_2(s)\Big)ds$$
$$+\frac{4|b|\Psi_c\big(B\big)}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\Big(m_1(s)+2Tm_1(s)m_2(s)\Big)ds+\varepsilon.$$

Since ε is arbitrary, it follows that from the above inequality that

$$\begin{aligned}
\Psi_c(F(B)) &\leq \Bigg[\frac{4}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(m_1(s)+2Tm_1(s)m_2(s)\Big)ds \\
&\quad +\frac{4|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\Big(m_1(s)+2Tm_1(s)m_2(s)\Big)ds\Bigg]\Psi_c\big(B\big) \\
&\leq \Bigg[\frac{4}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_0^t \big(m_1(s)+2Tm_1(s)m_2(s)\big)^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1} \\
&\quad +\frac{4|b|}{|a+b|\Gamma(\alpha)}\left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right)^{1-\alpha_1} \\
&\quad \times\left(\int_0^T \big(m_1(s)+2Tm_1(s)m_2(s)\big)^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1}\Bigg]\Psi_c\big(B\big) \\
&\leq \left[\frac{4M}{\Gamma(\alpha)}\left(\frac{t^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}}\right)^{1-\alpha_1}+\frac{4M|b|}{|a+b|\Gamma(\alpha)}\left(\frac{T^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}}\right)^{1-\alpha_1}\right]\Psi_c\big(B\big) \\
&\leq \left[1+\frac{|b|}{|a+b|}\right]\frac{4M}{\Gamma(\alpha)}\left(\frac{T^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}}\right)^{1-\alpha_1}\Psi_c\big(B\big).
\end{aligned}$$

Using the condition (3.1), we claim that F is a Ψ_c-contraction on $\mathbb{W}_r$. By Lemma 2.4, there is a fixed point x of F on $\mathbb{W}_r$, which is a solution of (1.1). This completes the proof.

In the following theorem, we replace the hypothesis (H3) by

• $(\widehat{H3})$ For $\alpha_2 \in (0,\alpha)$, there are real-valued functions $\varphi_1(t), \varphi_2(t) \in L^{\frac{1}{\alpha_2}}(J,\mathbb{R}_+)$ and there exist a L^1-integrable and nondecreasing functions $\psi_1,\psi_2 : [0,\infty)\to[0,\infty)$ such that

$$\|f\big(t,x(t),(Sx)(t)\big)\| \leq \varphi_1(t)\,\psi_1\big(\|x(t)\|\big)+\|Sx(t)\|,$$

$$\|k\big(t,s,x(s)\big)\| \leq \varphi_2(t)\,\psi_2\big(\|x(s)\|\big),$$

for each $s \in [0,t]$, $t \in J$ and all $x,y \in X$.

For brevity, let $\Phi_1 = \Big(\int_J \big(\varphi_1(s)\big)^{\frac{1}{\alpha_2}}\Big)^{\alpha_2}$, $\Phi_2 = \Big(\int_J \big(\varphi_2(s)\big)^{\frac{1}{\alpha_2}}\Big)^{\alpha_2}$.

Theorem 3.2. *Assume that (H1), (H2), ($\widehat{H3}$) and (H4) hold. If the condition* (3.1) *satisfied. Then the fractional BVP* (1.1) *has at least one solution on J.*

Proof. The continuous operator $F : C(J,X) \to C(J,X)$ is defined as in Theorem 3.1. Let $\mathbb{W}_r = \{x \in C(J,X), \|x\|_\infty \leq r\}$, where

(3.5)
$$\left[\frac{\Phi_1\psi_1(r)}{\Gamma(\alpha)} + \frac{T\Phi_2\psi_2(r)}{\Gamma(\alpha)} + \frac{|b|\Phi_1\psi_1(r)}{|a+b|\Gamma(\alpha)} + \frac{T|b|\Phi_2\psi_2(r)}{|a+b|\Gamma(\alpha)}\right]\left(\frac{T^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2} + \frac{|c|}{|a+b|} \leq r.$$

Then $\mathbb{W}_r \subset C(J,X)$ is bounded and convex. For each $x \in \mathbb{W}_r$ and $t \in J$, by using ($\widehat{H3}$), (3.5) and Hölder inequality, we have

$$\begin{aligned}
&\|(F(x))(t)\| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f\big(s,x(s),(Sx)(s)\big)\|ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|f\big(s,x(s),(Sx)(s)\big)\|ds + \frac{|c|}{|a+b|} \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\varphi_1(s)\,\psi_1\big(\|x(s)\|\big) + \|Sx(s)\|\Big)ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\Big(\varphi_1(s)\,\psi_1\big(\|x(s)\|\big) + \|Sx(s)\|\Big)ds + \frac{|c|}{|a+b|} \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\varphi_1(s)\,\psi_1\big(\|x\|_\infty\big) + T\varphi_2(s)\,\psi_2\big(\|x\|_\infty\big)\Big)ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\Big(\varphi_1(s)\,\psi_1\big(\|x\|_\infty\big) + T\varphi_2(s)\,\psi_2\big(\|x\|_\infty\big)\Big)ds + \frac{|c|}{|a+b|} \\
&\leq \frac{\psi_1(r)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_1(s)ds + \frac{T\psi_2(r)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_2(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|b|\psi_1(r)}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\varphi_1(s)ds+\frac{T|b|\psi_2(r)}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\varphi_2(s)ds+\frac{|c|}{|a+b|}\\
\leq\ & \frac{\Phi_1\psi_1(r)}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}+\frac{T\Phi_2\psi_2(r)}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\\
&+\frac{|b|\Phi_1\psi_1(r)}{|a+b|\Gamma(\alpha)}\left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}+\frac{T|b|\Phi_2\psi_2(r)}{|a+b|\Gamma(\alpha)}\left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\\
&+\frac{|c|}{|a+b|}\\
\leq\ & \frac{\Phi_1\psi_1(r)}{\Gamma(\alpha)}\left(\frac{T^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}+\frac{T\Phi_2\psi_2(r)}{\Gamma(\alpha)}\left(\frac{T^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}\\
&+\frac{|b|\Phi_1\psi_1(r)}{|a+b|\Gamma(\alpha)}\left(\frac{T^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}+\frac{T|b|\Phi_2\psi_2(r)}{|a+b|\Gamma(\alpha)}\left(\frac{T^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}+\frac{|c|}{|a+b|}\\
\leq\ & \left[\frac{\Phi_1\psi_1(r)}{\Gamma(\alpha)}+\frac{T\Phi_2\psi_2(r)}{\Gamma(\alpha)}+\frac{|b|\Phi_1\psi_1(r)}{|a+b|\Gamma(\alpha)}+\frac{T|b|\Phi_2\psi_2(r)}{|a+b|\Gamma(\alpha)}\right]\left(\frac{T^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}+\frac{|c|}{|a+b|}\\
\leq\ & r.
\end{aligned}
$$

Thus, $\|F(x)\|_\infty \leq r$ and we conclude that for all $x\in\mathbb{W}_r$, $F(\mathbb{W}_r)\subseteq\mathbb{W}_r$, that is, $F:\mathbb{W}_r\to\mathbb{W}_r$.

Next, let B bounded subset of $\mathbb{W}_r$, $x\in B$ and $0\leq t_1\leq t_2\leq T$; then by using $(\widehat{H}3)$ and Hölder inequality, again we have

$$
\begin{aligned}
&\|(F(x))(t_2)-(F(x))(t_1)\|\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\left\|f\big(s,x(s),(Sx)(s)\big)\right\|ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left\|f\big(s,x(s),(Sx)(s)\big)\right\|ds\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\Big(\varphi_1(s)\,\psi_1\big(\|x(s)\|\big)+\|Sx(s)\|\Big)ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Big(\varphi_1(s)\,\psi_1\big(\|x(s)\|\big)+\|Sx(s)\|\Big)ds\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\Big(\varphi_1(s)\,\psi_1\big(\|x\|_\infty\big)+T\varphi_2(s)\,\psi_2\big(\|x\|_\infty\big)\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Big(\varphi_1(s)\,\psi_1\big(\|x\|_\infty\big)+T\varphi_2(s)\,\psi_2\big(\|x\|_\infty\big)\Big)ds\\
\le\;&\frac{\psi_1(r)}{\Gamma(\alpha)}\int_0^{t_1}(t_2-s)^{\alpha-1}\varphi_1(s)ds-\frac{\psi_1(r)}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\varphi_1(s)ds\\
&+\frac{T\psi_2(r)}{\Gamma(\alpha)}\int_0^{t_1}(t_2-s)^{\alpha-1}\varphi_2(s)ds-\frac{T\psi_2(r)}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\varphi_2(s)ds\\
&+\frac{\psi_1(r)}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\varphi_1(s)ds+\frac{T\psi_2(r)}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\varphi_2(s)ds\\
\le\;&\frac{\Phi_1\psi_1(r)}{\Gamma(\alpha)}\Big(\int_0^{t_1}(t_2-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\Big)^{1-\alpha_2}-\frac{\Phi_1\psi_1(r)}{\Gamma(\alpha)}\Big(\int_0^{t_1}(t_1-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\Big)^{1-\alpha_2}\\
&+\frac{T\Phi_2\psi_2(r)}{\Gamma(\alpha)}\Big(\int_0^{t_1}(t_2-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\Big)^{1-\alpha_2}-\frac{T\Phi_2\psi_2(r)}{\Gamma(\alpha)}\Big(\int_0^{t_1}(t_1-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\Big)^{1-\alpha_2}\\
&+\frac{\Phi_1\psi_1(r)}{\Gamma(\alpha)}\Big(\int_{t_1}^{t_2}(t_2-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\Big)^{1-\alpha_2}+\frac{T\Phi_2\psi_2(r)}{\Gamma(\alpha)}\Big(\int_{t_1}^{t_2}(t_2-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\Big)^{1-\alpha_2}\\
\le\;&\frac{\Phi_1\psi_1(r)}{\Gamma(\alpha)}\left(\frac{(t_2-t_1)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}-\frac{t_2^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}-\frac{\Phi_1\psi_1(r)}{\Gamma(\alpha)}\left(\frac{-t_1^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}\\
&+\frac{T\Phi_2\psi_2(r)}{\Gamma(\alpha)}\left(\frac{(t_2-t_1)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}-\frac{t_2^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}-\frac{T\Phi_2\psi_2(r)}{\Gamma(\alpha)}\left(\frac{-t_1^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}\\
&+\frac{\Phi_1\psi_1(r)}{\Gamma(\alpha)}\left(\frac{-(t_2-t_1)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}+\frac{T\Phi_2\psi_2(r)}{\Gamma(\alpha)}\left(\frac{-(t_2-t_1)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}.
\end{aligned}
$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero and since x is an arbitrary in B, we conclude that $F(B) \subseteq C(J,X)$ is bounded and equicontinuous. The remaining proof can be completed as in the proof of previous Theorem 3.1 and hence we omit the details.

4. **Stability of solutions**

In this section, we study the uniform stability of solution of the fractional BVP (1.1). For more details one can see [5, 7, 9, 23].

Let $\overline{x}(t)$ a solution of the fractional BVP

$$\begin{cases} {}^cD^{\alpha}x(t) = f\big(t, x(t), (Sx)(t)\big), t \in J = [0,T], \alpha \in (0,1], \\ ax(0) + bx(T) = \overline{c}. \end{cases} \tag{4.1}$$

Definition 4.1. *The solution of the fractional BVP* (1.1) *is called uniformly stable, if for any* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$, *such that for any two solutions* $x(t)$ *and* $\overline{x}(t)$ *of fractional BVP* (1.1) *and* (4.1) *respectively, one has* $|c - \overline{c}| < \delta(\varepsilon)$, *then* $\|x(t) - \overline{x}(t)\| < \varepsilon$, *for all* $t \in J$.

Theorem 4.1. *Assume that (H1)-(H5) hold. Also suppose that*

- *(H6) There exists a constant* $\alpha_3 \in (0, \alpha)$ *and real-valued functions* $n_1(t), n_2(t) \in L^{\frac{1}{\alpha_3}}(J, \mathbb{R})$, *such that*

$$\|f\big(t, x_1, x_2\big) - f\big(t, y_1, y_2\big)\| \leq n_1(t)\big(\|x_1 - y_1\| + \|x_2 - y_2\|\big),$$

$$\|k\big(t, s, x_1\big) - k\big(t, s, y_1\big)\| \leq n_2(t)\|x_1 - y_1\|,$$

for each $s \in [0,t]$, $t \in J$ *and all* $x_1, x_2, y_1, y_2 \in X$.

- *(H7)*

$$\left[1 + \frac{|b|}{|a+b|}\right] \frac{N}{\Gamma(\alpha)} \left(\frac{T^{\frac{\alpha-\alpha_3}{1-\alpha_3}}}{\frac{\alpha-\alpha_3}{1-\alpha_3}} \right)^{1-\alpha_3} < 1, \tag{4.2}$$

where, $N = \|n_1 + Tn_1n_2\|_{L^{\frac{1}{\alpha_3}}(J,\mathbb{R})}$.

Then, the solution of fractional BVP (1.1) *is uniformly stable.*

Proof. Let $x(t)$ and $\overline{x}(t)$ be the solutions of the fractional BVP (1.1) and (4.1) respectively. Then for any $t \in J$, from (2.3), (4.2) and Hölder inequality, we have

$$\begin{aligned}
&\|x(t) - \overline{x}(t)\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f\big(s, x(s), (Sx)(s)\big) - f\big(s, \overline{x}(s), (S\overline{x})(s)\big) \right\| ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left\| f\big(s, x(s), (Sx)(s)\big) - f\big(s, \overline{x}(s), (S\overline{x})(s)\big) \right\| ds + \frac{|c - \overline{c}|}{|a+b|} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} n_1(s) \Big(\|x(s) - \overline{x}(s)\| + \|(Sx)(s) - (S\overline{x}(s))(s)\| \Big) ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}n_1(s)\Big(\|x(s)-\overline{x}(s)\|+\|(Sx)(s)-(S\overline{x}(s))(s)\|\Big)ds\\
&+\frac{|c-\overline{c}|}{|a+b|}\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}n_1(s)\Big(\|x(s)-\overline{x}(s)\|+Tn_2(s)\|x(s)-\overline{x}(s)\|\Big)ds\\
&+\frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}n_1(s)\Big(\|x(s)-\overline{x}(s)\|+Tn_2(s)\|x(s)-\overline{x}(s)\|\Big)ds\\
&+\frac{|c-\overline{c}|}{|a+b|}\\
&\leq \frac{\|x-\overline{x}\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(n_1(s)+Tn_1(s)n_2(s)\Big)ds\\
&+\frac{|b|\,\|x-\overline{x}\|_\infty}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\Big(n_1(s)+Tn_1(s)n_2(s)\Big)ds+\frac{|c-\overline{c}|}{|a+b|}\\
&\leq \frac{\|x-\overline{x}\|_\infty}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_3}}ds\right)^{1-\alpha_3}\left(\int_0^t \Big(n_1(s)+Tn_1(s)n_2(s)\Big)^{\frac{1}{\alpha_3}}ds\right)^{\alpha_3}\\
&+\frac{|b|\,\|x-\overline{x}\|_\infty}{|a+b|\Gamma(\alpha)}\left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_3}}ds\right)^{1-\alpha_3}\left(\int_0^t \Big(n_1(s)+Tn_1(s)n_2(s)\Big)^{\frac{1}{\alpha_3}}ds\right)^{\alpha_3}\\
&+\frac{|c-\overline{c}|}{|a+b|}\\
&\leq \frac{N\|x-\overline{x}\|_\infty}{\Gamma(\alpha)}\left(\frac{t^{\frac{\alpha-\alpha_3}{1-\alpha_3}}}{\frac{\alpha-\alpha_3}{1-\alpha_3}}\right)^{1-\alpha_3}+\frac{N|b|\,\|x-\overline{x}\|_\infty}{|a+b|\Gamma(\alpha)}\left(\frac{T^{\frac{\alpha-\alpha_3}{1-\alpha_3}}}{\frac{\alpha-\alpha_3}{1-\alpha_3}}\right)^{1-\alpha_3}+\frac{|c-\overline{c}|}{|a+b|}\\
&\leq \left[1+\frac{|b|}{|a+b|}\right]\frac{N\|x-\overline{x}\|_\infty}{\Gamma(\alpha)}\left(\frac{T^{\frac{\alpha-\alpha_3}{1-\alpha_3}}}{\frac{\alpha-\alpha_3}{1-\alpha_3}}\right)^{1-\alpha_3}+\frac{|c-\overline{c}|}{|a+b|}.
\end{aligned}
$$

Thus, we get

$$
\|x-\overline{x}\|_\infty\left(1-\left[1+\frac{|b|}{|a+b|}\right]\frac{N}{\Gamma(\alpha)}\left(\frac{T^{\frac{\alpha-\alpha_3}{1-\alpha_3}}}{\frac{\alpha-\alpha_3}{1-\alpha_3}}\right)^{1-\alpha_3}\right)\leq\frac{|c-\overline{c}|}{|a+b|},
$$

and

$$
\|x-\overline{x}\|_\infty\leq\left(1-\left[1+\frac{|b|}{|a+b|}\right]\frac{N}{\Gamma(\alpha)}\left(\frac{T^{\frac{\alpha-\alpha_3}{1-\alpha_3}}}{\frac{\alpha-\alpha_3}{1-\alpha_3}}\right)^{1-\alpha_3}\right)^{-1}\frac{1}{|a+b|}|c-\overline{c}|.
$$

Thus, for each $\varepsilon > 0$, we can find

$$\delta(\varepsilon) = \left(1 - \left[1 + \frac{|b|}{|a+b|}\right] \frac{N}{\Gamma(\alpha)} \left(\frac{T^{\frac{\alpha-\alpha_3}{1-\alpha_3}}}{\frac{\alpha-\alpha_3}{1-\alpha_3}} \right)^{1-\alpha_3} \right) \left(\frac{1}{|a+b|} \right)^{-1} \varepsilon,$$

such that $\|x - \bar{x}\|_\infty < \varepsilon$ whenever $|c - \bar{c}| < \delta(\varepsilon)$. This proves that the solution $x(t)$ is uniformly stable.

Remark 4.1. *The above Theorem 4.1 proves not only uniformly stable solution but also uniqueness of solution.*

Conflict of Interests

The authors declare that there is no conflict interests.

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