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A NEW COMMON FIXED POINT THEOREM IN ORDERED b-METRIC-LIKE SPACES

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Abstract. In this paper, we establish the existence of common fixed points in partially ordered complete b-metric-like spaces. An example is provided to support our results. The results obtained in this paper improve and extend the corresponding results announced recently.

Keywords: Fixed point; b-metric-like space; Partially ordered set; Altering distance function.

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1. Introduction-Preliminaries

Fixed point theory, which is an important branch modern mathematics, has been extensively studied in different framework of spaces. It is know that fixed point problems of nonlinear operators, which find a lot of applications in applied sciences, such as, signal processing, image reconstruction, nuclear magnetic resonance, include variational inequalities, saddle point problems, equilibrium problems and inclusion problems as special cases; see [1-5] and the references therein. However, most of the results obtained in metric spaces. Recently, different generalizations of the metric spaces have been introduced; see [6-10] the references therein. In

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1989, The concept of b-metric spaces was introduced and studied by Bakhtin [11] and Czerwik [12]. Then, Amini-Harandi [13,14] introduced the notion of metric-like spaces. In 2013, Alghamdi [15] introduced the notion of b-metric-like spaces. Since the b-metric-like space is a natural generalization of metric spaces and metric-like spaces, we will focus on the results in b-metric-like spaces in this paper. The organization of this article is as following. In Section 1, we provide some necessary preliminaries and definitions which play an important role in this article. In Section 2, common fixed point theorems are established in the framework of ordered b-metric-like spaces. Moreover, an example is provided to illustrate the obtained results.

Next, we recall the following definitions.

Definition 1.1. [11] A b-metric on a nonempty set X is a function $d : X \times X \to [0,\infty)$ such that for all $x,y,z \in X$ and a constant $b \geq 1$ the following three conditions hold:

(d_1) if $d(x,y) = 0 \Leftrightarrow x = y$;

(d_2) $d(x,y) = d(y,x)$;

(d_3) $d(x,y) \leq b(d(x,z)+d(z,y))$.

The pair (X,d) is called a b-metric space.

Definition 1.2. [13] A metric-like on a nonempty set X is a function $d : X \times X \to [0,\infty)$ such that for all $x,y,z \in X$ the following three conditions hold:

(d_1) if $d(x,y) = 0 \Rightarrow x = y$;

(d_2) $d(x,y) = d(y,x)$;

(d_3) $d(x,y) \leq d(x,z)+d(z,y)$.

The pair (X,d) is called a metric-like space.

Definition 1.3. [15] A b-metric-like on a nonempty set X is a function $d : X \times X \to [0,\infty)$ such that for all $x,y,z \in X$ and a constant $b \geq 1$ the following three conditions hold:

(d_1) if $d(x,y) = 0 \Rightarrow x = y$;

(d_2) $d(x,y) = d(y,x)$;

(d_3) $d(x,y) \leq b(d(x,z)+d(z,y))$.

The pair (X,d) is called a b-metric-like space.

Definition 1.4. [15] Let (X,d) be a b-metric-like space, and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n\to\infty} d(x,x_n) = d(x,x)$ and we say that the sequence $\{x_n\}$ is convergent to x and denote it by $x_n \to x$ as $n \to \infty$.

Definition 1.5. [15] Let (X,d) be a b-metric-like space.

(S_1) A sequence $\{x_n\}$ is said to be Cauchy if and only if $\lim_{m,n\to\infty} d(x_m,x_n)$ exists and is finite.

(S_2) A b-metric-like space (X,d) is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$, so that

$$\lim_{m,n\to\infty} d(x_n,x_m) = d(x,x) = \lim_{n\to\infty} d(x_n,x).$$

Proposition 1.6. [15] *Let (X,d,b) be a b-metric-like space, and let $\{x_n\}$ be a sequence in X such that* $\lim_{n\to\infty} d(x_n,x) = 0$. *Then*

(A) x is unique;

(B) $\frac{1}{b}d(x,y) \leq \lim_{n\to\infty} d(x_n,y) \leq bd(x,y)$, *for all $y \in X$.*

Definition 1.7. [16] The function $\varphi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function, if the following properties hold:

1. φ is continuous and non-decreasing.
2. $\varphi(t) = 0$ if and only if $t = 0$.

Definition 1.8. [17] Let $(X,\preceq)$ be a partially ordered set. Then two mappings $f,g : X \to X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$, for all $x \in X$.

In order to prove the main results result, we also need the following.

Let(X,d,b) be a b-metric-like space. Define [15] $d^s : X^2 \to [0,\infty)$ by

$$d^s(x,y) = |2d(x,y) - d(x,x) - d(y,y)|. \tag{1.1}$$

Clearly, $d^s(x,x) = 0$ for all $x \in X$.

2. Main results

Theorem 2.1. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a b-metric-like d on X such that (X,d) is a b-complete b-metric-like space and let $f,g : X \to X$ be two weakly* *increasing mappings with respect to $\preceq$. Suppose f satisfies $d(x,fx) \geq d(x,x)$ and g satisfies $d(x,gx) \geq d(x,x)$, $\forall x \to X$, and*

$$\psi(b^4 d(fx,gy)) \leq \psi(M_b(x,y)) - \varphi(M_b(x,y)) + L\psi(N(x,y)), \tag{2.1}$$

where $M_b(x,y) = \max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{6b}\}$ and $N(x,y) = \min\{d^s(x,fx), d^s(y,fx), d^s(x,gy)\}$, where d^s is defined in (1.1) for all comparable elements $x,y \in X$, $L \geq 0$, ψ and φ are altering distance functions. If either f or g continuous, then f and g have a common fixed point.

Proof. Let us divide the proof into two parts as follows.

Part I. We prove that u is a fixed point of f if and only if u is a fixed point of g.

Suppose that u is a fixed point of f, that is, $fu = u$. As $u \preceq u$, by (2.1), we have

$$\begin{aligned}\psi(b^4 d(u,gu)) &= \psi(b^4 d(fu,gu)) \\ &\leq \psi\left(\max\left\{d(u,u), d(u,fu), d(u,gu), \frac{d(u,gu)+d(u,fu)}{6b}\right\}\right) \\ &\quad - \varphi\left(\max\left\{d(u,u), d(u,fu), d(u,gu), \frac{d(u,gu)+d(u,fu)}{6b}\right\}\right) \\ &\quad + L\min\{d^s(u,fu), d^s(u,gu)\}.\end{aligned} \tag{2.2}$$

Since $d(u,fu) = d(u,u) \leq d(u,gu)$, we have

$$\frac{d(u,gu)+d(u,fu)}{6b} \leq \frac{d(u,gu)}{3b} \leq d(u,gu),$$

In view of $\min\{d^s(u,u), d^s(u,gu)\} = 0$, we find from (2.2) that

$$\begin{aligned}\psi(b^4 d(u,gu)) &= \psi(b^4 d(fu,gu)) \\ &\leq \psi(\max\{d(u,u), d(u,gu)\}) \\ &\quad - \varphi(\max\{d(u,u), d(u,gu)\}) \\ &\quad + L\min\{d^s(u,u), d^s(u,gu)\} \\ &= \psi(d(u,gu)) - \varphi(d(u,gu)) \\ &\leq \psi(b^4 d(u,gu)) - \varphi(d(u,gu)).\end{aligned}$$

Thus, we have $\varphi(d(u,gu)) = 0$. Therefore, $d(u,gu) = 0$ and hence $gu = u$. Similarly, we can show that if u is a fixed point of g, then u is a fixed point of f.

Part II. Letting $x_0 \in X$, we construct a sequence $\{x_n\}$ in X, such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all non-negative integers. As f and g are weakly increasing with respect to $\preceq$. It follows that

$$\begin{aligned} x_1 = fx_0 &\preceq gfx_0 = x_2 = gx_1 \\ &\preceq fgx_1 = x_3 = fx_2 \\ &\preceq \cdots x_{2n+1} = fx_{2n} \\ &\preceq gfx_{2n} = x_{2n+2} \\ &\preceq \cdots . \end{aligned}$$

If $x_{2n} = x_{2n+1}$, for some $n \in N$, then $x_{2n} = fx_{2n}$. Thus, x_{2n} is a fixed of f. By the first part, we conclude that x_{2n} is also a fixed point of g. If $x_{2n+1} = x_{2n+2}$, for some $n \in N$, then $x_{2n+1} = gx_{2n+1}$. Thus, x_{2n+1} is a fixed of g. By the first part, we conclude that x_{2n+1} is also a fixed point of f. Therefore, we assume that $x_n \neq x_{n+1}$, for all $n \in N$. Now, we complete the proof in the following steps.

Step 1. Prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

As x_{2n} and x_{2n+1} are comparable, by (2.1), we have

$$\begin{aligned} \psi(d(x_{2n+1},x_{2n+2})) &\leq \psi(b^4 d(x_{2n+1},x_{2n+2})) \\ &= \psi(b^4 d(fx_{2n},gx_{2n+1})) \\ &\leq \psi(M_b(x_{2n},x_{2n+1})) - \varphi(M_b(x_{2n},x_{2n+1})) + L\psi(N(x_{2n},x_{2n+1})), \end{aligned}$$

where

$$\begin{aligned} M_b(x_{2n},x_{2n+1}) &= \max\{d(x_{2n},x_{2n+1}), d(x_{2n},fx_{2n}), d(x_{2n+1},gx_{2n+1}), \\ &\qquad \frac{d(fx_{2n},x_{2n+1}) + d(x_{2n},gx_{2n+1})}{6b}\} \\ &= \max\{d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2}), \frac{d(x_{2n+1},x_{2n+1}) + d(x_{2n},x_{2n+2})}{6b}\}. \end{aligned}$$

It follows that

$$d(x_{2n},x_{2n+2}) \le bd(x_{2n},x_{2n+1})+bd(x_{2n+1},x_{2n+2})$$

$$d(x_{2n+1},x_{2n+1}) \le bd(x_{2n},x_{2n+1})+bd(x_{2n+1},x_{2n}) = 2bd(x_{2n},x_{2n+1}).$$

So, we have

$$\begin{aligned} M_b(x_{2n},x_{2n+1}) &= \max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),\frac{d(x_{2n+1},x_{2n+1})+d(x_{2n},x_{2n+2})}{6b}\} \\ &\le \max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),\frac{3bd(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}{6b}\} \\ &\le \max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}), \\ &\quad \frac{3bd(x_{2n},x_{2n+1})+bd(x_{2n+1},x_{2n+2})+2bd(x_{2n+1},x_{2n+2})}{6b}\} \\ &= \max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),\frac{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}{2}\} \\ &= \max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\} \end{aligned}$$

and

$$\begin{aligned} N(x_{2n},x_{2n+1}) &= \min\{d^s(x_{2n},fx_{2n}),d^s(x_{2n+1},fx_{2n}),d^s(x_{2n},gx_{2n+1})\} \\ &= \min\{d^s(x_{2n},x_{2n+1}),d^s(x_{2n+1},x_{2n+1}),d^s(x_{2n},x_{2n+2})\} \\ &= 0. \end{aligned}$$

Hence, we have

$$\begin{aligned} \psi(d(x_{2n+1},x_{2n+2})) \le &\psi(\max\{d(x_{2n+1},x_{2n+2}),d(x_{2n},x_{2n+1})\}) \\ &-\varphi(\max\{d(x_{2n+1},x_{2n+2}),d(x_{2n},x_{2n+1})\}). \end{aligned} \tag{2.3}$$

If $\max\{d(x_{2n+1},x_{2n+2}),d(x_{2n},x_{2n+1})\} = d(x_{2n+1},x_{2n+2})$, then (2.3) becomes

$$\psi(d(x_{2n+1},x_{2n+2})) \le \psi(d(x_{2n+1},x_{2n+2}))-\varphi(d(x_{2n+1},x_{2n+2})) < \psi(d(x_{2n+1},x_{2n+2})),$$

which yields a contradiction. So,

$$\max\{d(x_{2n+1},x_{2n+2}),d(x_{2n},x_{2n+1})\} = d(x_{2n},x_{2n+1}).$$

It follows (2.3) that

$$\psi(d(x_{2n+1},x_{2n+2})) \le \psi(d(x_{2n},x_{2n+1}))-\varphi(d(x_{2n},x_{2n+1})) < \psi(d(x_{2n},x_{2n+1})). \tag{2.4}$$

Similarly, we can show that

$$\psi(d(x_{2n+1},x_{2n})) \le \psi(d(x_{2n-1},x_{2n})) - \varphi(d(x_{2n-1},x_{2n})) < \psi(d(x_{2n-1},x_{2n})). \tag{2.5}$$

By (2.4) and (2.5), we get that $d(x_n,x_{n+1})$ is a non-increasing sequence of positive numbers. Hence, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = r.$$

Letting $n \to \infty$ in (2.4), we get

$$\psi(r) \le \psi(r) - \varphi(r) \le \psi(r),$$

which implies that $\varphi(r) = 0$ and hence $r = 0$. So, we have

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.6}$$

Step 2. Prove that $\{x_n\}$ is a Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose the contrary, that is, $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$, for which we can find two subsequences of positive integers $\{x_{2m_i}\}$ and $\{x_{2n_i}\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, d(x_{2m_i},x_{2n_i}) \ge \varepsilon. \tag{2.7}$$

This means that

$$d(x_{2m_i},x_{2n_i-2}) < \varepsilon. \tag{2.8}$$

From (2.7), (2.8), we get

$$\begin{aligned}
&d(x_{2m_i},x_{2n_i+1})\\
&\quad\le bd(x_{2n_i},x_{2n_i+1}) + bd(x_{2m_i},x_{2n_i})\\
&\quad< bd(x_{2n_i},x_{2n_i+1}) + b^2 d(x_{2n_i},x_{2n_i-1}) + b^2 d(x_{2m_i},x_{2n_i-1})\\
&\quad\le bd(x_{2n_i},x_{2n_i+1}) + b^2 d(x_{2n_i},x_{2n_i-1}) + b^3 d(x_{2n_i-1},x_{2n_i-2}) + b^3 d(x_{2m_i},x_{2n_i-2})\\
&\quad< bd(x_{2n_i},x_{2n_i+1}) + b^2 d(x_{2n_i},x_{2n_i-1}) + b^3 d(x_{2n_i-1},x_{2n_i-2}) + \varepsilon b^3.
\end{aligned}$$

Using (2.6), we have $\limsup_{i\to\infty} d(x_{2m_i}, x_{2n_i+1}) \leq \varepsilon b^3$. Again, from (2.7), we get $\varepsilon \leq d(x_{2m_i}, x_{2n_i}) \leq bd(x_{2m_i}, x_{2n_i+1}) + bd(x_{2n_i}, x_{2n_i+1})$. Using (2.6), we have $\frac{\varepsilon}{b} \leq \limsup_{i\to\infty} d(x_{2m_i}, x_{2n_i+1})$. So, we obtain

$$\frac{\varepsilon}{b} \leq \limsup_{i\to\infty} d(x_{2m_i}, x_{2n_i+1}) \leq \varepsilon b^3. \tag{2.9}$$

Similarly, we can obtain

$$\begin{aligned} &\frac{\varepsilon}{b} \leq \limsup_{i\to\infty} d(x_{2m_i-1}, x_{2n_i}) \leq \varepsilon b^3, \\ &\varepsilon \leq \limsup_{i\to\infty} d(x_{2m_i}, x_{2n_i}) \leq \varepsilon b^4, \\ &\frac{\varepsilon}{b^2} \leq \limsup_{i\to\infty} d(x_{2m_i-1}, x_{2n_i+1}) \leq \varepsilon b^4. \end{aligned} \tag{2.10}$$

Since x_{2n_i} and x_{2m_i-1} are comparable, using (2.1) we have

$$\begin{aligned} \psi(b^4 d(x_{2n_i+1}, x_{2m_i})) &= \psi(b^4 d(fx_{2n_i}, gx_{2m_i-1})) \\ &\leq \psi(M_b(x_{2n_i}, x_{2m_i-1})) - \varphi(M_b(x_{2n_i}, x_{2m_i-1})) + L\psi(N(x_{2n_i}, x_{2m_i-1})), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} M_b(x_{2n_i}, x_{2m_i-1}) &= \max\{d(x_{2n_i}, x_{2m_i-1}), d(x_{2n_i}, fx_{2n_i}), d(x_{2m_i-1}, gx_{2m_i-1}), \\ &\qquad \frac{d(x_{2n_i}, gx_{2m_i-1}) + d(x_{2m_i-1}, fx_{2n_i})}{6b}\} \\ &= \max\{d(x_{2n_i}, x_{2m_i-1}), d(x_{2n_i}, x_{2n_i+1}), d(x_{2m_i-1}, x_{2m_i}), \\ &\qquad \frac{d(x_{2n_i}, x_{2m_i}) + d(x_{2m_i-1}, x_{2n_i+1})}{6b}\} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} N(x_{2n_i}, x_{2m_i-1}) &= \min\{d^s(x_{2n_i}, fx_{2n_i}), d^s(x_{2m_i-1}, fx_{2n_i}), d^s(x_{2n_i}, gx_{2m_i-1})\} \\ &= \min\{d^s(x_{2n_i}, fx_{2n_i+1}), d^s(x_{2m_i-1}, x_{2n_i+1}), d^s(x_{2n_i}, x_{2m_i})\}. \end{aligned} \tag{2.13}$$

From (2.6), (2.10), (2.11) and (2.13), we get

$$\begin{aligned}
\frac{\varepsilon}{6b}+\frac{\varepsilon}{6b^3} &= \min\{\frac{\varepsilon}{b},\frac{\varepsilon}{6b}+\frac{\varepsilon}{6b^3}\} \\
&\leq \limsup_{i\to\infty} M_b(x_{2n_i},x_{2m_i-1}) \\
&= \max\{\limsup_{i\to\infty} d(x_{2n_i},x_{2m_i-1}),0,0, \\
&\quad \frac{\limsup_{i\to\infty} d(x_{2n_i},x_{2m_i})+\limsup_{i\to\infty} d(x_{2n_i+1},x_{2m_i-1})}{6b}\} \\
&\leq \min\{\varepsilon b^3,\frac{\varepsilon b^2}{6b}+\frac{\varepsilon b^4}{6b^3}\} = \varepsilon b^3.
\end{aligned}$$

So, we have

$$\frac{\varepsilon}{6b}+\frac{\varepsilon}{6b^3} \leq \limsup_{i\to\infty} M_b(x_{2n_i},x_{2m_i-1}) \leq \varepsilon b^3 \tag{2.14}$$

and

$$\limsup_{i\to\infty} N(x_{2n_i},x_{2m_i-1}) = 0. \tag{2.15}$$

Similarly, we can obtain

$$\frac{\varepsilon}{6b}+\frac{\varepsilon}{6b^3} \leq \liminf_{i\to\infty} M_b(x_{2n_i},x_{2m_i-1}) \leq \varepsilon b^3. \tag{2.16}$$

Now, from (2.11), (2.14), (2.15) and (2.16), we have

$$\begin{aligned}
\psi(\varepsilon b^3) = \psi(b^4\frac{\varepsilon}{b}) &\leq \psi(b^4 \limsup_{i\to\infty} d(x_{2n_i+1},x_{2m_i})) \\
&\leq \psi(\limsup_{i\to\infty} M_b(x_{2n_i+},x_{2m_i-1})) - \varphi(\liminf_{i\to\infty} M_b(x_{2n_i+},x_{2m_i-1})) \\
&\leq \psi(\varepsilon b^3) - \varphi(\liminf_{i\to\infty} M_b(x_{2n_i+},x_{2m_i-1})),
\end{aligned}$$

which implies that

$$\varphi(\liminf_{i\to\infty} M_b(x_{2n_i+},x_{2m_i-1})) = 0.$$

So $\liminf_{i\to\infty} M_b(x_{2n_i+},x_{2m_i-1}) = 0$. This is a contradiction to (2.16). Hence $\{x_n\}$ is a Cauchy sequence in X.

Step 3. Prove that

$$\lim_{m,n\to\infty} d(x_n,x_m) = 0.$$

Suppose that there is $c \geq 0$, such that $\lim_{m,n\to\infty} d(x_n,x_m) = c$. From (2.1), we have

$$\begin{aligned}\psi(d(x_{n+1},x_{m+1})) &\leq \psi(b^4 d(x_{n+1},x_{m+1})) \\ &= \psi(b^4 d(fx_n, gx_m)) \\ &\leq \psi(M_b(x_n,x_m)) - \varphi(M_b(x_n,x_m)) + L\psi(N(x_n,x_m)),\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}M_b(x_n,x_m) &= \max\{d(x_n,x_m), d(x_n,fx_n), d(x_m,gx_m), \\ &\qquad \frac{d(fx_n,x_m)+d(x_n,gx_m)}{6b}\} \\ &= \max\{d(x_n,x_m), d(x_n,x_{n+1}), d(x_m,x_{m+1}), \\ &\qquad \frac{d(x_{n+1},x_m)+d(x_n,x_{m+1})}{6b}\}.\end{aligned} \tag{2.18}$$

From (2.17) and (2.18), we have $\psi(c) \leq \psi(c) - \varphi(c) \leq \psi(c)$, which implies that $\varphi(c) = 0$ and hence $c = 0$. So, we have

$$\lim_{n\to\infty} d(x_n,x_m) = 0. \tag{2.19}$$

Step 4. As $\{x_n\}$ is a Cauchy sequence in X, which is a complete b-metric-like space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. So, from Definition 1.3, have

$$\lim_{m,n\to\infty} d(x_n,u) = d(u,u) = \lim_{m,n\to\infty} d(x_n,x_m) = 0$$

and $\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n} = u$. Without any loss of generality, we may assume that f is continuous. So $\lim_{n\to\infty} fx_{2n} = fu$. From Proposition 1.6, we get $\lim_{n\to\infty} fx_{2n} = fu = u$. So, we have $fu = u$. Thus, u is a fixed point of f. By the first part, we conclude that u is also a fixed point of g.

Theorem 2.2. *Under the hypotheses of Theorem 2.1, without the continuity assumption of one of the functions f or g, for any non-decreasing sequence x_n in X such that $x_n \to x \in X$, let us have $x_n \preceq x$, for all $n \in N$. Then, f and g have a common fixed point in x.*

Proof. From Theorem 2.1, we construct an increasing sequence $\{x_n\} \in X$ such that $x_n \to u$, for some $u \in X$. Using the assumption on X, we have $x_n \preceq u$, for all $n \in N$. Now, we show that $fu = gu = u$. By (2.1), we have

$$\begin{aligned} \psi(b^4 d(x_{2n+1}, gu)) &= \psi(b^4 d(fx_{2n}, gu)) \\ &\leq \psi(M_b d(x_{2n}, u)) - \varphi(M_b d(x_{2n}, u)) + L\psi(N(x_{2n}, u)), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} M_b(x_{2n}, u) &= \max\{d(x_{2n}, u), d(x_{2n}, fx_{2n}), d(u, gu) \frac{d(x_{2n}, gu) + d(fx_{2n}, u)}{6b}\} \\ &= \max\{d(x_{2n}, u), d(x_{2n}, x_{2n+1}), d(u, gu) \frac{d(x_{2n}, gu) + d(x_{2n+1}, u)}{6b}\} \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} N(x_{2n}, u) &= \min\{d^s(x_{2n}, fx_{2n}), d^s(u, fx_{2n}), d^s(x_{2n}, gu)\} \\ &= \min\{d^s(x_{2n}, x_{2n+1}), d^s(u, x_{2n+1}), d^s(x_{2n}, gu)\}. \end{aligned} \tag{2.22}$$

Letting $n \to \infty$ in (2.21) and (2.22) and using Proposition 1.6, we get

$$\begin{aligned} \frac{d(u, gu)}{6b^2} &= \min\{\frac{d(u, gu)}{b}, \frac{d(u, gu)}{6b^2}\} \\ &\leq \limsup_{n\to\infty} M_b(x_{2n}, u) \leq \max\{d(u, gu), \frac{bd(u, gu)}{6b}\} = d(u, gu) \end{aligned} \tag{2.23}$$

and $N(x_{2n}, u) \to 0$. Similarly, we obtain

$$\frac{d(u, gu)}{6b^2} \leq \liminf_{n\to\infty} M_b(x_{2n}, u) \leq d(u, gu). \tag{2.24}$$

Using (2.20), (2.23) and Proposition 1.6, we get

$$\begin{aligned} \psi(b^3 d(u, gu)) = \psi(b^4 \frac{1}{b} d(u, gu)) &\leq \psi(b^4 \limsup_{n\to\infty} d(x_{2n+1}, gu)) \\ &\leq \psi(\limsup_{n\to\infty} M_b(x_{2n}, u)) - \varphi(\liminf_{n\to\infty} M_b(x_{2n}, u)) \\ &\leq \psi(d(u, gu)) - \varphi(\liminf_{n\to\infty} M_b(x_{2n}, u)) \\ &\leq \psi(b^3 d(u, gu)) - \varphi(\liminf_{n\to\infty} M_b(x_{2n}, u)). \end{aligned}$$

Therefore, $\varphi(\liminf_{n\to\infty} M_b(x_{2n}, u)) \leq 0$, equivalently, $\liminf_{n\to\infty} M_b(x_{2n}, u) = 0$. Thus, from (2.20) we get $u = gu$ and hence u is a fixed point of g. On the other hand, similar to the first part of the proof of Theorem 2.1, we can show that $fu = u$. Hence, u is a common fixed point of f and g.

Example 2.3. *Let $X = [0,\infty)$ be equipped with the b-metric-like $d(x,y) = (x+y)^2$ for all $x,y \in X$, where $b = 2$. Define a relation $\preceq$ on X by $x \preceq y$ iff $y \leq x$, the functions $f,g : X \to X$ by $fx = \ln(1+\frac{x}{13})$ and $fx = \ln(1+\frac{x}{9})$, and the altering distance functions $\psi,\varphi : [0,+\infty)$ by $\psi(t) = qt$ and $\varphi(t) = (q-1)t$, where $1 \leq q \leq \frac{36}{16}$. Then, we have the following:*

(1) $(X,\preceq)$ is a partially ordered set having the b-metric-like d, where the b-metric-like space (X,d) is complete.

(2) f and g are weakly increasing mappings with respect to $\preceq$.

(3) f and g are continuous.

(4) f,g satisfies:

$$\psi(b^4 d(fx,gy)) \leq \psi(M_b(x,y)) - \varphi(M_b(x,y)) + L\psi(N(x,y)),$$

where

$$M_b(x,y) = \max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{6b}\}$$

and

$$N(x,y) = \min\{d^s(x,fx), d^s(y,fx), d^s(x,gy)\}$$

Proof. The proof of (1) is clear. To prove (2), for each $x \in X$, we know that $1+\frac{x}{13} \leq e^{\frac{x}{13}}$ and $1+\frac{x}{9} \leq e^{\frac{x}{9}}$. So, $fx = \ln(1+\frac{x}{13}) \leq x$ and $gx = \ln(1+\frac{x}{9}) \leq x$. Hence, $fgx = \ln(1+\frac{gx}{13}) \leq gx$ and $gfx = \ln(1+\frac{fx}{9}) \leq fx$, for each $x \in X$.Therefore, f and g are weakly increasing mappings with respect to $\preceq$. It is easy to see that f and g are continuous. To prove (4), let $x,y \in X$ with $x \preceq y$. So, $y \leq X$. Thus, we have the following cases.

Case 1. If $\frac{y}{9} \leq \frac{x}{13} \leq \frac{x}{9}$, then we have

$$(1+\frac{x}{13})(1+\frac{y}{9}) \leq (1+\frac{x}{9})(1+\frac{y}{9}) \Rightarrow \ln(1+\frac{x}{13})(1+\frac{y}{9}) \leq \ln(1+\frac{x}{9})(1+\frac{y}{9}).$$

Now, using the mean value theorem for function $\ln(1+t)$, for $t \in [\frac{y}{9}, \frac{x}{9}]$, we have

$$\begin{aligned}\psi(b^4 d(fx,gy)) &= 16qd(fx,gy)\\ &= 16q(\ln(1+\frac{x}{13})+\ln(1+\frac{y}{9}))^2 = 16q(\ln(1+\frac{x}{13})(1+\frac{y}{9}))^2\\ &\le 16q(\ln(1+\frac{x}{9})(1+\frac{y}{9}))^2 = 16q(\ln(1+\frac{x}{9})+\ln(1+\frac{y}{9}))^2\\ &\le 16q(\frac{x}{9}+\frac{y}{9})^2 \le \frac{36}{81}(x+y)^2\\ &\le d(x,y) \le M_2(x,y) = \psi(M_2(x,y)) - \varphi(M_2(x,y)),\end{aligned}$$

that is, we have $\psi(b^4 d(fx,gy)) \le \psi(M_b(x,y)) - \varphi(M_b(x,y)) + L\psi(N(x,y))$, for each $L \ge 0$.

Case 2. If $\frac{x}{13} \le \frac{y}{9} \le \frac{x}{9}$, then we have $\psi(b^4 d(fx,gy)) \le \psi(M_b(x,y)) - \varphi(M_b(x,y)) + L\psi(N(x,y))$, for each $L \ge 0$. The proof is the same to Case 1.

Thus, all the hypotheses of Theorem 2.1 are satisfied and hence f and g have a common fixed point. Indeed, 0 is the unique common fixed point of f and g.

Finally, Let us finish this paper with the following remarks.

Remark 2.4. Theorems 2.1 and 2.2 not only improve and extend the corresponding results of Alghamdi [15], but also improve and extend the corresponding results of Roshan [18] and others.

Remark 2.5. A b-metric-like is a metric-like if $b = 1$, so our results can be viewed as a generalization and extension of corresponding results and several other comparable results.

Conflict of Interests

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