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FIXED POINT THEOREMS OF CONTRACTION MAPPINGS ON ZERO AT INFINITY VARIETIES

GHORBAN KHALILZADEH*, TOORAJ AMIRI

Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran

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Abstract. There are a great number of concepts about fixed points of contraction mappings on metric spaces. In this paper, we have tried to show that the set of all zero at infinity varieties is a complete metric space and some results and examples about fixed points of contraction mappings on this space have been provided.

Keywords: variety; zero at infinity variety; contraction map; Lipschitzian map and fixed point.

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1. Introduction

1.1. **preliminaries.** P. G. Dixon in [4] introduced varieties of Banach algebras and proved an analoque of Birkhoff's theorem [1] for varieties of banach algebras. In this section, some important concepts about this subject have been mentioned.

Definition 1.1. *For each Banach algebra $\mathscr{A}$. and $\delta > 0$, we define*

$$\|p\|_{\mathscr{A},\delta} = \sup\{\|p(x_1,\ldots,x_n)\| : x_i \in \mathscr{A}, \|x_i\| \leq \delta, 1 \leq i \leq n\}$$

*Corresponding author

E-mail address: gh_khalilzadeh@yahoo.com

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We shall denote $\|p\|_{\mathscr{A},1} = \|p\|_{\mathscr{A}}$, where $p = p(X_1,...,X_n)$ is a polynomial. Throughout this paper, a polynomial is a non-commuting polynomial without constant term.

As a law, we mean a formal expression $\|p\| \leq K$, where $K \in \mathbb{R}$ and p is a polynomial. We say that $\mathscr{A}$ satisfies the above law, if $\|p\|_{\mathscr{A}} \leq K$. Also, the law $\|p\| \leq K$ is homogeneous if p is a homogeneous polynomial.

When we talk about algebras, a variety is a non-empty class v of complex associative algebras which is closed under taking subalgebras, quotient algebras, direct sum and isomorphic images. Birkhoff has proved that a non-empty class of complex associative algebras v is a variety if and only if there is a set L of polynomials such that,

$$v = \{\mathscr{A} : p(x_1,...,x_n) = 0, (x_1,...,x_n \in \mathscr{A}), \forall p \in L\}.$$

Definition 1.2. *A non-empty class v of Banach algebras is said to be a variety if there exists a non-negative real-valued function,*

$$p \longmapsto f(p)$$

on the set of all polynomials, such that v is precisely a class of Banach algebras $\mathscr{A}$ for which,

$$\|p\|_{\mathscr{A}} \leq f(p)$$

for each

$$p = p(X_1,...,X_n).$$

Remark 1.1. *If a variety defined by a family of homogeneous laws then, it is called H-variety and we denote the set of all H-varieties by L_H.*

In[4], Dixon defined varieties of Banach algebras and proved an analogue of Birkhoff's theorem based on varieties of universal algebras, for Banach algebras.

Theorem 1.1. *(* [4] *theorem 2.3) For each non-empty class v of Banach algebras, the followings are equivalent,*

(i)v is closed under taking closed subalgebras, quotient algebras, products(direct sums) and images under isometric isomorphisms.

(ii)v is a variety.

Definition 1.3. *Let C be a class of Banach algebras and $v(C)$ be the intersection of all varieties containing C. Then, $v(C)$ is a variety called the variety generated by C. If C consists of a single Banach algebra $\mathscr{A}$, then $v(C)$ is written as $v(\mathscr{A})$ and it is said to be singly generated.*

Definition 1.4. *Let v is a variety and $\{L_\alpha\}_{\alpha\in I}$ to be the families of laws which determine v. We define*

$$|p|_v = \inf\{K : \exists\alpha \in I; (\|p\| \leq K) \in L_\alpha\}$$

where p is a polynomial. The family $\{|p|_v\}_p$ is a family of laws which determine v.

Each variety is determined by a family of laws. But among such families one is particular noteworthy; namely, the family of laws with minimal right-hand sides K. The function giving these right-hand sides is as the following. For each variety v and polynomial p,

$$|p|_v = \sup\{\|p\|_{\mathscr{A}} : \mathscr{A} \in v\}.$$

The following theorem shows that this supremum is always obtained.

Theorem 1.2. *([2] theorem 2.4) For each variety v , there exists a $\mathscr{A} \in v$ such that for all polynomial p, we have*

$$|p|_v = \|p\|_{\mathscr{A}}.$$

This theorem shows that this supremum is always obtaind.

Corollary 1.3. *([2] corollary 2.5) Each variety of Banach algebras is singly generated.*

Corollary 1.4. *([2] corollary 2.6) Let v_1, v_2 be two varietirs. Then, $v_1 \subseteq v_2$ if and only if for all polynomials p, we have*

$$|p|_{v_1} \leq |p|_{v_2}.$$

We note that, partially ordered by inclusion, the class of all varieties is a complete lattice.

Definition 1.5. *Let L be the lattice of all varieties, and L_H be the lattice of all H-varieties. Let P be the set of all polynomials, and P_H be the set of all homogeneous polynomials also, P_{NH} be the set of all non homogeneous polynomials. We define*

$$P_1 = \{p \in P : |p|_1 < 1\}$$

$$P_{H1} = \{p \in P_H : |p| < 1\}$$

$$P_{NH1} = \{p \in P_{NH} : |p| < 1\}.$$

1.2. **Zero at infinity varieties.**

Definition 1.6. *Suppose that v is a variety. Then, it is zero at infinity if for each $\varepsilon > 0$ there exists $N > 0$ such that for all $p \in P_{H1}$, if $deg(p) > N$ then $|p|_v \leq \varepsilon$.*
We show the set of all zero at infinity varieties by L^0.

Definition 1.7. *If $x \in \mathbb{R}$ with $0 < x < 1$ and $v \neq N_2$ be a variety, then v_x is the variety that is determined by following lows*

$$\|p\| < x^{i-1}|p|_v$$

where p is a homogeneous polynomial with $deg(p) = i$.

The next theorem has been proved in [3] but we have done some reforms in its proof for specification.

Theorem 1.5. *The set of all varieties is a complete metric space.*

Proof. Let $v \in L$. Define $\phi_v : P \to R^+$ as

$$\phi_v(p) = |p|_v.$$

However, the mapping $\psi : L \to L^\infty(P_1)$ with $\psi(v) = \phi_v$ is well defined and one to one. Because $v = w$ if and only if $|p|_v = |p|_w$ for all polynomials $p \in P_1$. Therefore, $v = w$ if and only if $\phi_v(p) = \phi_w(p)$ for all polynomials $p \in P_1$. Consequently, $v = w$ if and only if $\psi(v) = \psi(w)$. Hence $L^\infty(P_1)$ induces the followin metric to L:

$$d_L(v,w) = d(\phi_v, \phi_w) = \|\phi_v - \phi_w\|_\infty.$$

Now, we want to show that ψ is continuous. Let $\{v_n\}_{n=1}^\infty$ be a sequence in L such that $v_n \longrightarrow v$. So, for each $\varepsilon > 0$ there exists $N > 0$ such that for any $n > N$ we have $d_L(v_n, v) < \varepsilon$. Hence, $\|\phi_{v_n} - \phi_v\|_\infty < \varepsilon$ and $\|\psi(v_n) - \psi(v)\|_\infty < \varepsilon$.
We define, $\Phi = \{\phi_v : v \in L\}$ and show that (Φ, d) is a closed subset of $L^\infty(P_1)$ metric space. Let

$\{\phi_{v_n}\}_{n=1}^{\infty}$ is a cauchy sequence in Φ. For all $\varepsilon > 0$ there exists an $N > 0$ such that for $n,m > N$ we have

$$\|\phi_{v_n} - \phi_{v_m}\|_\infty < \varepsilon$$

Therefore, $\sup_{p\in P_1} |\phi_{v_n}(p) - \phi_{v_m}(p)| < \varepsilon$ and consequently, $\sup_{p\in P_1} \big||p|_{v_n} - |p|_{v_m}\big| < \varepsilon$. Then, for all $p \in P_1$ it is concluded that, $\big||p|_{v_n} - |p|_{v_m}\big| < \varepsilon$ and it means that the sequence $\{|p|_{v_n}\}_{n=1}^{\infty}$ is a cauchy sequence in $\mathbb{R}$. So, it converges to an $\alpha \in \mathbb{R}$ and there is a variety $v \in L$ such that $\alpha = |P|_v$ where $p \in P_1$. Therefore, we have $|p|_{v_n} \longrightarrow |p|_v$ and it means that for all $\varepsilon > 0$ there exists an $N > 0$ such that, for all $n > N$ we have

$$\big||p|_{v_n} - |p|_v\big| < \varepsilon \qquad (\forall p \in P_1).$$

So, we have $\sup_{p\in P_1} \big||p|_{v_n} - |p|_v\big| < \varepsilon$ and it is clear that $\|\phi_{v_n} - \phi_{v_m}\|_\infty < \varepsilon$. Then, $\phi_{v_n} \longrightarrow \phi_v$ with $\|.\|_\infty$. Hence, Φ is a closed subset of $L^\infty(P_1)$. □

By previous theorem, (L, d_L) Similarly, (L_H, d_H) is also a metric space with

$$d_H(V,W) = \sup_{p\in P_{H1}} \big||p|_V - |p|_W\big|.$$

Also, (L, d_{NH}) is a metric space with

$$d_{NH}(V,W) = \sup_{p\in P_{NH1}} \big||p|_V - |p|_W\big|.$$

Theorem 1.6. *If v is a variety such that $v \neq N_2$ then*

(i) The mapping $x \to v_x$ from $[0,1)$ into (L_H, d_H) is a strictly increasing injective continuous mapping.

(ii) If v is zero at infinity, then previous mapping is also continuous at 1.

Proof. Take $v = v(\mathscr{A})$ for some Banach algebra $\mathscr{A}$. Consider the algebra $\mathscr{A}_x$, consisting of the algebra $\mathscr{A}$ with the norm x^{-1} times the $\mathscr{A}$-norm. For all homogeneous polynomial p of degree i, we have

$$\begin{aligned}\|p\|_{\mathscr{A}_x} &= x^{-1}|p|_{\mathscr{A},x}\\ &= x^{i-1}|p|_{\mathscr{A}}.\end{aligned}$$

Thus for any homogeneous polynomial p of degree i, we have

$$|p|_{v_x} = x^{i-1}|p|_v.$$

Moreover, if $0 < x_1 < x_2 < 1$ then, for any homogeneous p with $|p|_v \neq 0$ and $deg(p) > 1$, we have

$$|p|_{v_{x_1}} < |p|_{v_{x_2}}.$$

So, the mapping $x \to v_x$ of $[0,1)$ into the H-varieties is an injective. Now, take $0 < a < b < 1, x_n \in (0,b)$ and $x_n \to a$. Then

$$\begin{aligned} d_H(v_{x_n}, v_a) &= \sup_{p \in P_{H1}} | \ |p|_{v_{x_n}} - |p|_{v_a}| \\ &= \sup_{p \in P_{H1}} |x_n^{i-1} - a^{i-1}||p|_v \\ &\leq \sup_{p \in P_{H1}} |x_n - a||x_n^{i-2} + ... + a^{i-2}| \\ &< |x_n - a| \sup_{p \in P_{H1}} (i-1)b^{i-2}. \end{aligned}$$

Since $(i-1)b^{i-2}$ is convergent, it is bounded. Thus, $v_{x_n} \to v_a$. Now, take $x_n \in [0,1)$ and $x_n \to 0$. Put $p \in P_{H1}$ and $deg(p) > 1$, then

$$\begin{aligned} ||p|_{v_{x_n}} - |p|_{N_2}| &= |p|_{v_{x_n}} - |p|_{N_2} \\ &\leq |p|_{v_{x_n}} \\ &= x_n^{i-1}|p|_v \\ &\leq x_n^{i-1} \\ &\leq x_n. \end{aligned}$$

Since $v_0 = N_2$, the mapping is continuous.

Let $x_n \in [0,1]$ and $x_n \to 1$ and v be a zero at infinity variety. Also, if $\varepsilon > 0$ then, there exists $N' > 0$ such that, if $p \in P_{H1}$ and $deg(p) > N'$, then $|p|_v < \varepsilon/2$.

We have

$$||p|_{v_{x_n}} - |p|_{H(v)}| = |p|_v|x_n^{i-1} - 1|.$$

Now, if $i > N'$ then

$$|p|_v|x_n^{i-1} - 1| \leq 2|p|_v < \varepsilon.$$

Also, for each $1 < i \leq N$, $x_n^{i-1} \to 1$. So, there exists $N > 0$ such that for all $n > N$, $|x_n^{i-1} - 1| < \varepsilon$. Thus, for all $n > N$,

$$|p|_v |x_n^{i-1} - 1| \leq |x_n^{i-1}| < \varepsilon.$$

Thus, for all $p \in P_{H1}$ and all $n > N$, we have $||p|_{v_{x_n}} - 1| < \varepsilon$. Hence, $v_{x_n} \to H(v)$, because $H(v) = v_1$ □

Corollary 1.7. *Let v be a zero at ifinity variety. Then, $\{v_x : 0 < x < 1\}$ is a connected, complete and compact subspace of (L_H, d_H).*

Theorem 1.8. *The subspace (L_H^0, d_H) of zero at infinity H-varieties is closed in (L, d_L).*

Proof. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of zero at infinity varieties. Take $v_n \to v$. Thus, for each $\varepsilon > 0$, there exists $N > 0$, such that for all $n \in \mathbb{N}$, if $n \geq N$, then

$$\sup_{p \in P_1} \left| |p|_{v_n} - |p|_v \right| < \varepsilon$$

Since the metric spaces (L, d_L) and (L_{NH}, d_{NH}) are equivalent, we have

$$\sup_{q \in P_{NH1}} \left| |q|_{v_n} - |q|_v \right| < \varepsilon.$$

Let p be a homogeneous polynomial, where $q_i = p$ and $(i = 1, 2, \ldots)$. Since

$$\sup_{q \in P_{NH1}} \left| |q|_{v_n} - |q|_v \right| < \varepsilon.$$

for all q, such that $q_i = p$, we have

$$\left| |q|_{v_n} - |q|_v \right| < \varepsilon.$$

Thus, $\left| \inf_q |q|_{v_n} - |q|_v \right| < \varepsilon$, $\left| |p|_{v_n} - |p|_v \right| < \varepsilon$ and $\left| |p|_{v_N} - |p|_v \right| < \varepsilon$. Since v_N is a zero at infinity variety, for each $\varepsilon > 0$, there exists $N_1 > 0$ such that for all polynomials $p \in P_{H1}$, if $\deg(p) > N_1$, then $|p|_{v_N} < \varepsilon$. Thus, $|p|_v < \varepsilon$, and v is a zero at infinity. □

Definition 1.8. *Let v be a zero at infinity variety. We define*

$$[N_2, v] = \{w | N_2 \subseteq w \subseteq v, w \textit{ is a variety}\}.$$

It is obvious that the members of $[N_2, v]$ are zero at infinity varieties.

Theorem 1.9. *Let v be a zero at infinity variety. Then, $[N_2, v]$ is a closed set.*

Proof. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of zero at infinity varieties such that $v_n \in [N_2, v]$ for all $n \in \mathbb{N}$. So, $N_2 \subseteq v_n \subseteq v$. If $v' \nsubseteq v$, then there exists $c \in v'$ such that $c \notin v$. So, there exists $p_0 \in P_{H1}$ such that $\|p_0\|_c > |p_0|_v$, and $|p_0|_{v'} \geq \|p_0\|_c > |p_0|_v > |p_0|_{v_n}$. By $v_n \to v'$, for each $\varepsilon > 0$, there exists $N > 0$ such that for all $n \geq N$

$$\sup_{p \in P_1} \big||p|_{v_n} - |p|_{v'}\big| < \varepsilon.$$

Thus, for all polynomials $p \in P_{H1}$, we will have $\big||p|_{v_n} - |p|_{v'}\big| < \varepsilon$. Now, if $\varepsilon = |p_0|_{v'} - |p_0|_v$, then $-(|p_0|_{v'} - |p_0|_v) < |p_0|_{v_n} - |p_0|_{v'} < |p_0|_{v'} - |p_0|_v$. Thus, for all $n \geq N$, $|p_0|_v < |p_0|_{v_n}$ and this is a contradiction. □

Corollary 1.10. *Let v be a zero at infinity variety. Then, $[N_n, v]$ for all $n \in \mathbb{N}$ and $n > 2$ are complete metric subspaces.*

Corollary 1.11. *Let v be a zero at infinity variety. Then, the closed intervals $[N_2, v]$ is complete metric subspace.*

Theorem 1.12. *Let v be zero at infinity variety. Then, $\{v_x | 0 < x < 1\}$ is a path-connected subspace of metric space (L_H, d_H).*

Proof. Take $0 \leq a < a' \leq 1$. Also v_a and $v_{a'}$ are two zero at infinity varieties. We define the mapping $f : [0,1] \to \{v_a | 0 \leq a \leq 1\}$ as follows

$$f(t) = v_{a+t(a'-a)}.$$

Then, we have $f(0) = v_a$ and $f(1) = v_{a'}$.

Now, we prove that f is a continuous mapping of $[0,1]$ onto $\{v_a | 0 \leq a \leq 1\}$. For all $n \in \mathbb{N}$, if $t_n, t \in [0,1]$ and $t_n \to t$, Then

$$\begin{aligned} d(f(t_n), f(t)) &= \sup_{p \in P_{H1}} \big||p|_{v_{a+t_n(a'-a)}} - |p|_{v_{a+t(a'-a)}}\big| \\ &= \sup_p \big|[a + t_n(a'-a)]^{i-1}|p|_v - [a + t(a'-a)]^{i-1}|p|_v\big| \\ &= \sup_p |p|_v \big|(a + t_n(a'-a))^{i-1} - (a + t(a'-a))^{i-1}\big| \end{aligned}$$

$$= \sup_{p\in P_{H1}} |p|_v|(a+t_n(a'-a)-a-t(a'-a))[(a+t_n(a'-a))^{i-2}+ (a+t_n(a'-a))^{i-3}(a+t(a'-a))+...+(a+t(a'-a))^{i-2}]|$$

$$= \sup_{p\in P_{H1}} |p|_v|(t_n-t)(a'-a)[(a+t_n(a'-a))^{i-2}+ (a+t_n(a'-a))^{i-3}(a+t(a'-a))+...+(a+t(a'-a))^{i-2}]|$$

$$< |t_n-t| \sup_{p\in P_{H1}} (i-2)(a')^{i-2}.$$

Since $(i-2)(a')^{i-2}$ is bounded and $\{|t_n-t|\}_{n=1}^{\infty}$ is convergent to 0,

$$d(f(t_n),f(t)) \to 0.$$

□

Corollary 1.13. *Let L_H^0 be the set of all zero at infinity H-varieties. Let $\alpha \in I$ and $v_a^{\alpha} \in L_H^0$ be a zero at infinity variety. If $C_\alpha = \{v_a^{\alpha} | 0 \le a \le 1\}$, then $\cup_{\alpha\in I} C_\alpha$ is connected.*

Corollary 1.14. *If $\{x_n\}$ is a convergent sequence in $[0,1]$ with $x_n \to x$ and v_{x_n} is a zero at infinity variety for all $n \in \mathbb{N}$, then the sequence $\{v_{x_n}\}$ is convergent to v_x.*

1.3. **Fixed point of zero at infinity varieties.**

Proposition 1.15. *Let (L^0, d_L) be the complete metric space of zero at infinity varieties. Suppose that $\{x_n\} \subset [0,1]$ is a sequence and $T : L^0 \to L^0$ is defined as $T(v_{x_n}) = v_{x_{n+1}}$ for all $v_{x_n}, v_{x_{n+1}} \in L^0$, then T is a Lipschitzian map.*

Proof. suppose that $v_{x_n}, v_{y_n} \in L^0$ where $\{x_n\}, \{y_n\}$ are sequences in $[0,1]$. Then, we have

$$\begin{aligned} d_L(T^k v_{x_n}, T^k v_{y_n}) &= d_L(v_{x_{n+k}}, v_{y_{n+k}}) \\ &= \sup_p |x_{n+k}^{i-1} - y_{n+k}^{i-1}||p|_v \\ &\le L_k \sup |x_n^{i-1} - y_n^{i-1}||p|_v \\ &= L_k d_L(v_{x_n}, v_{y_n}) \end{aligned}$$

By Archimedean property, L_k exists for any $k \in \mathbb{N}$. So, it's done. □

Proposition 1.16. *Let* (L^0, d_L) *be the complete metric space of zero at infinity varieties and* $T : L^0 \to L^0$ *be a Lipschitzian mapping with constant* $k \neq 1$*, defined as above. Also, let* $\psi : L^0 \to (0, \infty)$ *be defined as* $\psi(v_x) = \frac{1}{1-k} d_L(v_x, Tv_x)$ *for all* $v_x \in L^0$*. If* $\{v_{x_n}\}$ *is a sequence in* L^0 *such that* $v_{x_{n+1}} = Tv_{x_n}$ *and* $\{x_n\} \in [0,1]$*, then we have,*

i) ψ *is continuous.*

ii) $d_L(v_{x_n}, v_{x_{n+1}}) = \psi(v_{x_n}) - \psi(v_{x_{n+1}})$ *for all* $n \in \mathbb{N}_0$.

iii) For all $n \in \mathbb{N}_0$,

$$d_L(v_{x_n}, v_x) = \psi(v_{x_n}) - \psi(v_x).$$

Proof. i)Suppose $\{v_{x_n}\}$ is a sequence in L^0 such that $v_{x_n} \to v_x$, so $d_L(v_{x_n}, v_x) \to 0$. Then we have,

$$\begin{aligned}
|\psi(v_{x_n}) - \psi(v_x)| &= |\frac{1}{1-k} d_L(v_{x_n}, Tv_{x_n}) - \frac{1}{1-k} d_L(v_x, Tv_x)| \\
&= |\frac{1}{1-k}| \Big| \sup_p \big| |p|_{v_{x_n}} - |p|_{Tv_{x_n}} \big| - \sup_p \big| |p|_{v_x} - |p|_{Tv_x} \big| \Big| \\
&\leq |\frac{1}{1-k}| (\sup \big| |p|_{v_{x_n}} - |p|_{Tv_{x_n}} - |p|_{v_x} + |p|_{Tv_x} \big|) \\
&\leq |\frac{1}{1-k}| (\sup \big| |p|_{v_{x_n}} - |p|_{v_x} \big| + \sup \big| |p|_{Tv_{x_n}} - |p|_{Tv_x} \big|) \\
&= |\frac{1}{1-k}| \big(d_L(v_{x_n}, v_x) + d_L(Tv_{x_n}, Tv_x) \big) \\
&= |\frac{1+k}{1-k}| d_L(v_{x_n}, v_x)
\end{aligned}$$

Therefore, $\psi(v_{x_n}) \to \psi(v_x)$ and consequently ψ is continuous.

ii)If $v_{x_n}, v_{x_{n+1}} \in L^0$, then we have,

$$\begin{aligned}
\psi(v_{x_n}) - \psi(v_{x_{n+1}}) &= \frac{1}{1-k} d_L(v_{x_n}, Tv_{x_N}) - \frac{1}{1-k} d_L(v_{x_{n+1}}, Tv_{x_{n+1}}) \\
&= \frac{1}{1-k} \big(d_L(v_{x_n}, Tv_{x_n}) - d_L(v_{x_{n+1}}, Tv_{x_{n+1}}) \big) \\
&= \frac{1}{1-k} \big(d_L(v_{x_n}, Tv_{x_n}) - d_L(Tv_{x_n}, T(Tv_{x_{n+1}})) \big) \\
&= \frac{1}{1-k} \big(d_L(v_{x_n}, Tv_{x_n}) - k d_L(v_{x_n}, Tv_{x_n}) \big)
\end{aligned}$$

$$= d_L(v_{x_n}, Tv_{x_n})$$

iii) According to the corollary(2.9), it is concluded that there exists a $v_x \in L^0$ such that $v_{x_n} \to v_x$. In the first part, it is proved that ψ is continuous, therefore, $\psi(v_{x_n}) \to \psi(v_x)$. Also, we have

$$d_L(v_{x_n}, v_x) = \psi(v_{x_n}) - \lim_{n\to\infty} \psi(v_{x_m})$$

$$= \psi(v_{x_n}) - \psi(v_x)$$

□

Theorem 1.17. *Let L^0 be the complete metric space of zero at infinity varieties and $\psi : L^0 \to (-\infty, \infty)$ a proper, bounded below and lower semicontinuous function. Suppose that for each $v_u \in L^0$ with $\inf_{v_x \in L^0} \psi(v_x) < \psi(v_u)$ there exists a $v_w \in L^0$ such that $v_w \neq v_u$ and*

$$d_L(v_u, v_w) \leq \psi(v_u) - \psi(v_w).$$

Then, there is a $v_{x_0} \in L^0$ such that $\psi(v_{x_0}) = \inf_{v_x \in L^0} \psi(v_x)$

Proof. It is similar to proof of theorem(4.1.2) in [5]. □

Corollary 1.18. *Let L^0 be the complete metric space of zero at infinity varieties and $T : L^0 \to L^0$ a Lipschitzian map with constant $k \neq 1$. Let $\psi : L^0 \to (-\infty, \infty]$ be defined as $\psi(v_x) = \frac{1}{1-k} d_L(v_x, Tv_x)$ for all $v_x \in L^0$. Then we have $\psi(v_{x_0}) = \inf_{v_x \in L^0} \psi(v_x)$ where $\psi(v_{x_0}) = \lim_{n\to\infty} \psi\big(T^n(V_{x_0})\big)$*

Theorem 1.19. *Let L^0 be the complete metric space of zero at infinity varieties and $\psi : L^0 \to (-\infty, \infty]$ be a proper bounded below and lower semicontinuous function. Let $T : L^0 \to L^0$ be a mapping such that, $d_L(v_x, Tv_x) \leq \psi(v_x) - \psi(Tv_x)$ for all $v_x \in L^0$. Then, there exists a $v_y \in L^0$ such that $v_y = Tv_y$ and $\psi(v_y) < \infty$.*

Proof. It is like the proof of theorem(4.1.3)[5]. □

Theorem 1.20. *Let L^0 be the complete metric space of zero at infinity varieties and $T : L^0 \to L^0$ be defined as $Tv_x = v_{\alpha(x)}$ where $\alpha : [0,1] \to [0,1]$ is a function. Then, T has a fixed point if and only if α has a fixed point. Also, if $x_0 \in [0,1]$ is a fixed point for α then v_x is a fixed point for T.*

Proof. Suppose that T has a fixed point. There exists $v_x \in L^0$ such that $Tv_x = v_x$. Therefore, $v_x = v_{\alpha(x)}$ and we have $v_x \subset v_{\alpha(x)}$, then,

$$|p|_{v_x} \leq |p|_{v_{\alpha(x)}}$$

$$\sup_p x^{i-1}|p|_v \leq \sup_p \alpha(x)^{i-1}|p|_v$$

$$x^{i-1} \leq \alpha(x)^{i-1}$$

Previous inquality is correct for any $i \in \mathbb{N}$, so it is concluded that $x \leq \alpha(x)$. Similarly, it is obtained that $\alpha(x) \leq x$, so we will have $\alpha(x) = x$. Therefore, α has a fixed point. Inverse is obvious. □

Theorem 1.21. *Suppose L^0 is the complete metric space of zero at infinity varieties and $\psi : L^0 \to (-\infty, \infty]$ is defined as $\psi(v_x) = f(x)$ where $f : [0,1] \to (-\infty, \infty]$ is a continuous bijection. For $v_x \neq N_2$, let $T : L^0 \to L^0$ be defined as $Tv_x = v_y$ such that $|x - y| < \frac{f(x)-f(y)}{(i-1)\sup_p |p|_v}$ where $i = \deg(p)$ and $y \in [0,1]$. Then, T has a fixed point.*

Proof.

$$\begin{aligned}
d_L(v_x, v_y) &= \sup_p |x^{i-1} - y^{i-1}||p|_v \\
&\leq |x-y| \sup_p |x^{i-2} + x^{i-3}y + ... + xy^{i-3} + y^{i-2}||p|_v \\
&\leq |x-y| \sup_p (i-1)|p|_v \\
&\leq \frac{f(x) - f(y)}{(i-1)\sup_p |p|_v} \sup_p (i-1)|p|_v \\
&\leq f(x) - f(y) \\
&= \psi(v_x) - \psi(v_y) \\
&= \psi(v_x) - \psi(Tv_x)
\end{aligned}$$

by previous theorem, ther is $v_{x_0} \in L^0$ such that $Tv_{x_0} = v_{x_0}$. □

Remark 1.2. *The fixed point of the mapping T in previous theorem need not to be unique.*

Theorem 1.22. *Let L^0 be the complete metric space of zero at infinity varieties and $T : L^0 \to L^0$ a contaction mapping with Lipschitzian constant $k \in (0,1)$. Then, we have the following,*
i) There exists a unique fixed point $v_x \in L^0$ for T.
ii) For arbitrary $v_x \in L^0$ the picard iteration process is defined by

$$v_{x_{n+1}} = T v_{x_n} \qquad \forall n \in \mathbb{N}$$

converge to v_x.
iii) For all $n \in \mathbb{N}$ we have $d_L(v_{x_n}, v_{x_0}) \leq (1-k)^{-1} k^n d_L(v_{x_1}, v_{x_0})$.

Proof. Similar to the proof of the Theorem 4.1.5 in [5]. □

Lemma 1.23. *Let $\{x_n\}$ be an increasing (decreasing) sequence in $\mathbb{R}$ and for each $n \in \mathbb{N}$ we have $x_n > \frac{x_{n+1}+x_{n-1}}{2}$ $(x_n < \frac{x_{n+1}+x_{n-1}}{2})$ Then, for each $m,l \in \mathbb{N}$ we have,*

$$|x_{m+n} - x_{l+n}| < |x_m - x_l|$$

such that $n \in \mathbb{N}$.

Proof. Suppose $\{x_n\}$ is an increasing sequence in $\mathbb{R}$ and $x_n > \frac{x_{n+1}+x_{n-1}}{2}$ for all $n \in \mathbb{N}$.
So we have

$$x_{n+1} - x_n < x_n - x_{n-1}.$$

If $m,l \in \mathbb{N}$ and $m > l$ then,

$$x_{m+1} - x_m < x_m - x_{m-1} < \ldots < x_{l+1} - x_l.$$

Therefore, $x_{m+1} - x_m < x_{l+1} - x_l$ and it conclude that,

$$x_{m+1} - x_{l+1} < x_m - x_l. \qquad (*)$$

Suppose $k \in \mathbb{N}$ and,

$$x_{m+k} - x_{l+k} < x_m - x_l. \qquad (**)$$

By $(*)$ we have

$$x_{m+K+1} - x_{l+k+1} < x_{m+k} - x_{l+k}$$

and by $(**)$ we have

$$x_{m+k} - x_{l+k} < x_m - x_l.$$

Therefore,

$$x_{m+K+1} - x_{l+k+1} < x_m - x_l.$$

By induction, it has done. □

Corollary 1.24. *Let L^0 be the complete metric space of zero at infinity varieties and $\{x_n\} \subset [0,1]$ a decreasing sequence such that $x_n < \frac{x_{n+1}+x_{n-1}}{2}$ for each $n \in \mathbb{N}$. If $T : L^0 \to L^0$ is defined as $Tv_{x_n} = v_{x_{n+1}}$ where $\{v_{x_n}\}$ is a sequnce in L^0, then we have*
i) T is a contraction.
ii) There exists a unique fixed point $v_x \in L^0$ for T where $x = \lim_{n\to\infty} x_n$.
iii) For arbitrary $v_{x_0} \in L^0$ the picard iteration process is convergent to v_x.
iv) For all $n \in \mathbb{N}$ it is proved that $d_L(v_{x_n}, v_x) \leq (1-k)^{-1} k^n d_L(v_{x_0}, v_{x_1})$ where $k \in (0,1)$ is the Lipschitz constant of T.

Proof. Part (i) must be proved, and the rest of the cases according to the previous theorem is obvious. Take$n \in \mathbb{N}$ and $v_{x_m}, v_{x_l} \in L^0$ where m,l are natural numbers. Then,

$$\begin{aligned}
d_L(T^n v_{x_m}, T^n v_{x_l}) &= d_L(v_{x_{m+n}}, v_{x_{l+n}}) \\
&= \sup_p \big| |p|_{v_{x_{m+n}}} - |p|_{v_{x_{l+n}}} \big| \\
&= \sup_p \big| x_{m+n}^{i-1} - x_{l+n}^{i-1} \big| |p|_v \\
&= \sup_p \big| x_{m+n} - x_{l+n} \big| \big| x_{m+n}^{i-2} + x_{m+n}^{i-3} x_{l+n} + \dots + x_{l+n}^{i-2} \big| |p|_v \\
&< \sup_p \big| x_m - x_l \big| \big| x_{m+n}^{i-2} + x_{m+n}^{i-3} x_{l+n} + \dots + x_{l+n}^{i-2} \big| |p|_v \\
&\leq \sup_p \big| x_m - x_l \big| \big| x_m^{i-2} + x_m^{i-3} x_l + \dots + x_l^{i-2} \big| |p|_v \\
&= \sup_p \big| |p|_{v_{x_m}} - |p|_{v_{x_l}} \big| \\
&= d_L(v_{x_m}, v_{x_l}) \quad .
\end{aligned}$$

So, there exists $k \in (0,1)$ such that

$$d_L(T^n v_{x_m}, T^n v_{x_l}]) \leq k d_L(v_{x_m}, v_{x_l}).$$

Therefore, T is a contraction mapping. □

Example 1.9. *Let $X = L^0$ and $T : X \to X$ be a mapping defined as*

$$Tv_x = v_{\frac{x}{2}}.$$

Then, T is a contraction, because

$$\begin{aligned} d_L(Tv_x, Tv_y) &= d_L(v_{\frac{x}{2}}, v_{\frac{y}{2}}) \\ &= \sup_p |\frac{x^{i-1} - y^{i-1}}{2^{i-1}}||p|_v \\ &\leq \frac{1}{2} \sup_p \left|x^{i-1}|p|_v - y^{i-1}|p|_v\right| \\ &= \frac{1}{2} d_L(v_x, v_y). \end{aligned}$$

Also, T is a uniformely Lipschitzian mapping. Hence, by previous theorem it has a fixed point in X.

Example 1.10. *Let $X = L^0$ and $T : X \to X$ be a mapping defined as*

$$Tv_x = v_{1-x}$$

for all $x \in [0,1]$. Then, T is non contraction but it has a fixed point, because we have

$$\begin{aligned} d_L(Tv_x, Tv_y) &= d_L(v_{1-x}, v_{1-y}) \\ &= \sup_p |(1-x)^{i-1} - (1-y)^{i-1}||p|_v \\ &= \sup_p |x-y||(1-x)^{i-1} + (1-x)^{i-2}(1-y) + ... + (1-y)^{i-1}||p|_v. \end{aligned}$$

If $x < \frac{1}{2}, y < \frac{1}{2}$ then we have $1-x > x, 1-y > y$ so

$$\begin{aligned} d_L(Tv_x, Tv_y) &= \sup_p |x-y||(1-x)^{i-1} + (1-x)^{i-2}(1-y) + ... + (1-y)^{i-1}||p|_v \\ &> \sup_p |x-y||x^{i-1} + x^{i-2}y + ... + y^{i-1}||p|_v \end{aligned}$$

by previous relations, it is concluded that

$$d_L(Tv_x, Tv_y) > d_L(v_x, v_y).$$

Therefore, T is non contraction but it has a fixed point, because

$$Tv_{\frac{1}{2}} = v_{\frac{1}{2}}.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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