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FIXED POINTS FOR GENERALISED α-ψ CONTRACTIVE MAPPINGS IN CONE METRIC SPACES

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Abstract. The purpose of this paper is to introduce a new mapping called generalized α-ψ contractive mapping in cone metric space and then we extend and generalize some fixed point theorem for such mappings.

Keywords: Fixed points; Cone metric spaces; Generalized α-ψ contractive mapping; α-admissible mapping; Normal cone.

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1. Introduction

Fixed point theory is one of the most important research field in non-linear analysis and the study of fixed point of mapping satisfying certain contractive conditions has been at the center of strong research activity.

In 2007, Huang and Zhang [1] introduce the concept of cone metric space which is a generalization of metric space. They have proved some fixed point theorem of contractive mapping

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in cone metric space. Rezapour and Hamlbarani [3] showed that there are no normal cones with normal constant $K < 1$, and for each $k > 1$ there are cones with a normal constant $K > k$ Further some authors [3, 4, 6] generalized some definitions and results in cone metric spaces. Fore more recent fixed point theorems in cone metric spaces we refer to [2, 5, 7, 8].

In the last decade, in [9] J. Gornicke, B.E. Rhoades used generalized contractive mapping to obtain common fixed point. In this paper we proved a fixed point theorem. Example is also provided to demonstrate the main result. Moreover from our main result we have introduced some additional condition to find unique fixed point.

2. Preliminaries

First we introduce some notations and definitions (see [1]) that will be used subsequently.

Definition 2.1. Let E be the real Banach space with a given norm $\|\cdot\|_E$ and 0_E is zero vector of E. Then a non empty subset P of E is called a cone if and only if

(1) P is non-empty and $P \neq \{0_E\}$.
(2) P is closed.
(3) $ax+by \in P$ for all $x,y \in P$ and $a,b \in R$ with $a,b \geq 0$ that is, P is convex.
(4) $P \cap (-P) = \{0_E\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y-x \in P$.

We write $x \leq y$ to indicate $x \leq y$ but $x \neq y$ and $x \ll y$ will stand for $y-x \in \mathrm{Int}(P)(\mathrm{Int}(P) \cong$ interior of $P)$.

Definition 2.2. The cone $P \subset E$ is called normal if there is number K such that for all $x,y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$
where K is least positive number satisfying the above inequality and called normal constant of P.

Definition 2.3. The cone $P \subset E$ is called regular if every increasing sequence which is bounded above is convergent. That is if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq x_3 \leq \ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently the cone P is regular if and only if decreasing sequence which is bounded below is convergent.

Now for the following discussion assume that E is Banach space, P is a cone in E with $\mathrm{Int}\,P \neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.4. Let X be non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies

A$_1$. $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ iff $x = y$.

A$_2$. $d(x,y) = d(y,x)$ for all $x,y \in X$.

A$_3$. $d(x,y) \leq d(x,z) + d(y,z)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

Example 2.5. Let $E = R^2$, $P = \{(x,y) \in E | x,y \geq 0\} \subset R^2$, $X = R$ and $d : X \times X \to E$ such that

$$d(x,y) = (|x-y|, a|x-y|)$$

where $a \geq 0$ is constant. Then (X,d) is a cone metric space.

Now refer to [7] for further details.

Definition 2.6. Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) then $\{x_n\}$ is said to be convergent to x if for every $c \in E$ with $0 \ll c$ there exist N such that $d(x_n,x) \ll c$ for all $n \geq N$.

We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$

(2) If for every $c \in E$ with $0 \ll c$, there is a positive integer N such that for all $n,m > N$, $d(x_n, x_{m)} \ll c$. Then the sequence $\{x_n\}$ is called a cauchy sequence in X.

(3) If every Cauchy sequence in X is convergent then (X,d) is called a complete cone metric space.

Lemma 2.7. *Let (X,d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ converges to x if and only if $d(x_n,x) \to 0$ $(n \to \infty)$.*

Lemma 2.8. *Let (X,d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a cauchy sequence if and only if $d(x_n,x_m) \to \infty$ $(n,m \to \infty)$.*

Lemma 2.9. *Let (X,d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then limit of $\{x_n\}$ is unique. That is if $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, Then $x = y$.*

Lemma 2.10. *Let (X,d) be a cone metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x then $\{x_n\}$ is a Cauchy sequence in X.*

Lemma 2.11. *Let (X,d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequence in X with $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n,y_n) \to d(x,y)$ as $n \to \infty$.*

Recently Samet et al. [2] introduced the notion of α-ψ contractive mappings and α-admissible mappings in metric spaces as follows:

Definition 2.12. Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$, we say that T is α-admissible if $x,y \in X$, $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$.

Denote with Ψ the family of non-decreasing function $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{\infty} \psi_n < +\infty$ for each $t > 0$, where ψ^n is nth iteration of ψ.

Lemma 2.13. *For every function $\psi : [0,+\infty) \to [0,+\infty)$ the following holds:*
If ψ is non decreasing, then for each $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$ implies $\psi(t) < t$ and $\psi(0) = 0$.

Definition 2.14. Let (X,d) be a metric space and $T : X \times X$ be a mapping, then T is said to be an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)) \quad \text{for all } x,y \in X$$

Further Kang et al. [10] introduce the notion of this mapping in cone metric space as follows:

Definition 2.15. Let (X,d) be a cone metric space and P be a normal cone with normal constant K. Let $T : X \to X$ be a mapping. Then T is said to be an α-ψ contractive mapping if there exist two functions

$\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)) \quad \text{for all } x,y \in X$$

Now we present new notion of generalized α-ψ-contractive mappings in cone metric spaces and derive fixed point results for these mappings in cone metric space.

Definition 2.16. Let (X,d) be a cone metric space and P be a normal cone with normal constant K. Let $T : X \to X$ be a mapping. Then T is said to be generalized α-ψ contractive mapping if there exist two functions
$\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y)) \quad \text{for all} x,y \in X \tag{2.1}$$

where $M(x,y) = \max\left\{d(x,y), \dfrac{d(x,Tx)+d(y,Ty)}{2}, \dfrac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

3. Main result

Samet [2] proved the following theorem.

Theorem 3.1. *Let (X,d) be a complete metric space and $T : X \to X$ be an α-ψ contractive mapping satisfying the following conditions:*

(3.1) T is α-admissible;

(3.2) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;

(3.3) T is continuous

If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n,x) \geq 1$ for all n, then T has a fixed point.

Recently, Kang et al. [10] proved the following theorem in Cone metric space.

Theorem 3.2. *Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. $T : X \to X$ be an α-ψ contractive mapping satisfying the following conditions:*

(C1) T is α-admissible;

(C2) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;

(C3) T is continuous

If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n,x) \geq 1$ for all n, then T has a fixed point.

Now we prove Theorem 3.2 in setting of generalized α-ψ contractive mapping in cone metric spaces as follows:

Theorem 3.3. *Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K and $T: X \to X$ be any generalized α-ψ contractive mapping satisfying the following conditions:*

(3.1) T is α-admissible.

(3.2) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(3.3) T is continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ for all n. Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X such that

$$Tx_n = x_{n+1} \quad \text{for some } n \in N. \tag{3.4}$$

In particular if $x_n = x_{n+1}$ for some $n \in N$ then x_n is a fixed point for T. Assume that $x_n \neq x_{n+1}$ for all $n \in N$. Since T is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \text{ implies } \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in N. \tag{3.5}$$

Now applying inequality (2.1) and (3.8), we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \\ &\leq \psi(M(x_{n-1}, x_n)), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{ d(x_{n-1}, Tx_{n-1}), \frac{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)}{2}, \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2} \right\} \\ &= \max\left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}, \frac{d(x_{n-1}, x_{n+1})}{2} \right\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

Owing to monotonicity of the function ψ, and using the inequalities (3.4) and (3.6), we have for all $n \geq 1$

$$d(x_n, x_{n+1}) \leq \psi \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \tag{3.7}$$

If for some $n \geq 1$, we have $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$. Then (3.7) becomes

$$d(x_n, x_{n+1}) \leq \psi d(x_n, x_{n+1}),$$

which implies

$$\|d(x_n, x_{n+1})\| \leq \|\psi d(x_n, x_{n+1})\| < \|d(x_n, x_{n+1})\|.$$

This is a contradiction. Thus for all $n \geq 1$, we have

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n). \tag{3.8}$$

In view of (3.7) and (3.8), we get for all $n \geq 1$

$$d(x_n, x_{n+1}) \leq \psi d(x_{n-1}, x_n). \tag{3.9}$$

Continuing this process inductively, we obtain

$$d(x_n, x_{n+1}) \leq \psi^n(x_0, x_1) \quad \text{for all } n \in N. \tag{3.10}$$

Now for $n > m$, using (3.10) and triangular inequality , we obtain

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \ldots + d(x_{m+1}, x_m) \\ &\leq \psi^{n-1}(d(x_0, x_1)) + \psi^{n-2}(d(x_0, x_1)) + \ldots \psi^m(d(x_0, x_1)) \\ &\leq (\psi^{n-1} + \psi^{n-2} + \ldots + \psi^m)(d(x_0, x_1)) \\ &\leq \frac{\psi^m}{1-\psi}(d(x_0, x_1)). \end{aligned}$$

Since P is normal cone with normal constant K, we find that

$$\|d(x_n, x_m)\| \leq K \left\| \frac{\psi^m}{1-\psi}(d(x_0, x_1)) \right\|,$$

which implies $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a cauchy sequence in cone metric space (X, d). Since (X, d) is complete. So there exist $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Case 1: If T is continuous, then we have $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$.

By uniqueness of limit, $Tx^* = x^*$. Hence x^* is a fixed point of T.

Case 2: If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.

Now we show that $\|d(Tx^*, x^*)\| \geq 0$ as $n \to \infty$. On contrary, assume $\|d(Tx^*, x^*)\| > 0$. We have

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx_n, Tx^*) + d(Tx_n, x^*) \\ &\leq \alpha(x_n, x^*) d(Tx_n, Tx^{*)} + d(Tx_n, x^*) \\ &\leq \psi M(x_n, x^*) + d(Tx_n, x^*). \end{aligned}$$

Since P is normal cone with normal constant K, we have

$$\|d(Tx^*, x^*)\| \leq K(\|\psi(M(x_n, x^*))\| + \|d(x_{n+1}, x^*)\|, \tag{3.11}$$

where

$$M(x_n, x^*) = \max\left\{ d(x_n, x^*), \frac{d(x_n, Tx_n) + d(x^*, Tx^*)}{2}, \frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2} \right\}.$$

Letting $n \to \infty$, we have

$$M(x_n, x^*) = \frac{d(x^*, Tx^*)}{2}.$$

Using in (3.11) and taking $n \to \infty$, We have

$$\begin{aligned} \|d(Tx^*, x^*)\| &\leq K \left\| \psi\left(\frac{d(Tx^*, x^*)}{2} \right) \right\| \\ &< \frac{K}{2} \|d(Tx^*, x^*)\|, \end{aligned}$$

which is not true for all $K > 0$. So we get a contradiction. Therefore $\|d(Tx^*, x^*)\| \to 0$ as $n \to \infty$. It implies $Tx^* = x^*$ and hence x^* is a fixed point of T. This completes the proof.

Example 3.4. Let us consider $X = [0, \infty)$ and $E = R^2$, $P = \{(x, y) \in E | x, y > 0\} \subset R^2$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, a|x - y|)$ where $a \geq 0$ is a constant. Then (X, d) is cone metric space.

Define $T : X \to X$ by

$$Tx = \begin{cases} 2x - \dfrac{13}{7} & \text{if } x > 1, \\ \dfrac{x}{7} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x < 0. \end{cases}$$

We observe that here T is continuous and Banach contraction principle in setting of cone metric space cannot be applied.

Now we define a mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly T is generalized α-ψ contractive mapping with $\psi(t) = \dfrac{3t}{5}$ for all $t \geq 0$. Infact for all $x, y \in X$, we have

$$\begin{aligned} \alpha(x,y)d(Tx,Ty) &= 1 \cdot (|Tx - Ty|, a|Tx - Ty|) \\ &= \left(\left| \frac{x}{7} - \frac{y}{7} \right|, a \left| \frac{x}{7} - \frac{y}{7} \right| \right) \\ &= \frac{(|x-y|, a|x-y|)}{7} \\ &= \frac{1}{7} d(x,y) \\ &\leq \frac{3}{5} d(x,y) \leq \frac{3}{5} M(x,y) = \psi(M(x,y)). \end{aligned}$$

More over there exists $x_0 \in X$, such that $\alpha(x_0, Tx_0) \geq 1$. For $x_0 = 1$, we have

$$\alpha(1, T_1) = \alpha\left(1, \frac{1}{7}\right) = 1.$$

Now it remains to show that T is α-admissible. Let $x, y \in X$, such that $\alpha(x,y) \geq 1$.

Therefore we have $x, y \in [0,1]$. By definition of T and α we have

$$Tx = \frac{x}{7} \in [0,1], \quad Ty = \frac{y}{7} \in [0,1] \text{ and } \alpha(Tx, Ty) = 1.$$

So, T is α-admissible.

Now all the hypothesis of Theorem 3.3 are satisfied. Consequently T has a fixed point. Note that Theorem 3.3 guarantees only the existence of fixed point but not uniqueness. In this example, 0 and $\frac{13}{7}$ are two fixed points of T.

Example 3.5. Let us consider $X = [0,\infty)$ and $E = R^2$, $P = \{(x,y) \in E | x,y > 0\} \subset R^2$ and $d : X \times X \to E$ such that $d(x,y) = (|x-y|, a|x-y|)$ where $a \geq 0$ is a constant. Then (X,d) is cone metric space.

Define $T : X \to X$ by

$$Tx = \begin{cases} 5x - \frac{7}{2} & \text{if } x > 2, \\ \frac{x}{5} & \text{if } 0 \leq x \leq 2. \end{cases}$$

We observe that T is not continuous at 2. The Banach contraction principle in setting of cone metric space cannot be applied in this case.

Now we define a mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly T is generalized α-ψ contractive mapping with $\psi(t) = \frac{t}{4}$ for all $t \geq 0$, infact for all $x,y \in X$, we have

$$\begin{aligned} \alpha(x,y)d(Tx,Ty) &= 1 \cdot (|Tx-Ty|, a|Tx-Ty|) \\ &= \left(\left| \frac{x}{5} - \frac{y}{5} \right|, a \left| \frac{x}{5} - \frac{y}{5} \right| \right) \\ &= \frac{1}{5}(|x-y|, a|x-y|) \\ &= \frac{1}{5} d(x,y) \\ &\leq \frac{1}{4} d(x,y) \leq \frac{1}{4} M(x,y) = \psi(M(x,y)). \end{aligned}$$

More over there exists $x_0 \in X$, such that $\alpha(x_0, Tx_0) \geq 1$. For $x_0 = 1$, we have

$$\alpha(1, T1) = \alpha\left(1, \frac{1}{5}\right) = 1.$$

Now it remains to show that T is α-admissible. Let $x,y \in X$, such that $\alpha(x,y) \geq 1$. Therefore we have $x,y \in [0,1]$. By definition of T and α, we have

$$Tx = \frac{x}{5} \in [0,1], \;\; Ty = \frac{y}{5} \in [0,1] \;\text{ and }\; \alpha(Tx,Ty) = 1.$$

So T is α-admissible.

Finally let $\{x_n\}$ be a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to \infty$. Since $\alpha(x_n,x_{n+1}) \geq 1$ for all n, by definition of α, we have $x_n \in [0,1]$ for all n and $x \in [0,1]$. Then $\alpha(x_n,x) = 1$.

Now all the hypothesis of Theorem 3.3 are satisfied. Consequently T has a fixed point. Note that Theorem 3.3 guarantees and existence of a fixed point but not uniqueness. In this example, 0 and $\frac{7}{8}$ are two fixed points of T.

To assure the uniqueness of fixed point we will consider the following hypothesis.

$(*)$ For all $x,y \in X$ there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.

Theorem 3.6. *Theorem 3.3 yields a unique fixed point after adding hypothesis $(*)$ to it.*

Proof. Suppose that x, y are two fixed points of T. From $(*)$, there exists $z \in X$ such that

$$\alpha(x,z) \geq 1 \quad \text{and} \quad \alpha(y,z) \geq 1. \tag{3.12}$$

Define a sequence $\{z_n\}$ in X by $Tz_n = Z_{n+1}$ for all $n \geq 0$ and $z_0 = z$. Since T is α-admissible, therefore from (11), we get

$$\alpha(x,z_n) \geq 1 \text{ and } \alpha(y,z_n) \geq 1 \quad \text{for all } n \in N. \tag{3.13}$$

Using inequalities (2.1) and (3.13), we obtain

$$\begin{aligned} d(x,z_{n+1}) &= d(Tx,Tz_n) \\ &\leq \alpha(x,z_n)d(Tx,Tz_n) \\ &\leq \psi M(x,z_n). \end{aligned} \tag{3.14}$$

On the other hand, we have

$$\begin{aligned} M(x,z_n) &= \max\left\{d(x,z_n), \frac{d(x,Tx)+d(z_n,Tz_n)}{2}, \frac{d(x,Tz_n)+d(z_n,Tx)}{2}\right\} \\ &\leq \max\{d(x,z_n), d(x,z_{n+1})\}. \end{aligned} \tag{3.15}$$

Now owing to the monotonicity of ψ and using inequality (3.14), we get

$$d(x,z_{n+1}) \leq \psi \max\{d(x,z_n), d(x,z_{n+1})\} \quad \text{for all } n. \tag{3.16}$$

Without loss of generality, suppose that $d(x,z_n) > 0$ for all n. If

$$\max\{d(x,z_n), d(x,z_{n+1})\} = d(x,z_{n+1}),$$

then (3.19) becomes

$$d(x,z_{n+1}) \leq \psi d(x,z_{n+1}),$$

$$\|d(x,z_{n+1})\| \leq \|\psi d(x,z_{n+1})\| < \|d(x,z_{n+1})\|,$$

which is a contradiction. Thus we have

$$\max\{d(x,z_n), d(x,z_{n+1})\} = d(x,z_n). \tag{3.17}$$

In view of (3.16) and (3.17), we get for all $n \geq 1$

$$d(x,z_{n+1}) \leq \psi d(x,z_n).$$

Continuing the process inductively, we get

$$d(x,z_n) \leq \psi^n d(x,z_0) \quad \text{for all } n \geq 1. \tag{3.18}$$

Since P be a normal cone with normal constant K, we have

$$\|d(x,z_n)\| \leq K\|\psi^n d(x,z_0)\|.$$

Letting $n \to \infty$, we get $\|d(x,z_n)\| \to 0$ as $n \to \infty$. This implies that

$$z_n \to x \ \text{ as } \ n \to \infty. \tag{3.19}$$

Similarly we find that $z_n \to x$ as $n \to \infty$. Hence, we get the uniqueness in y.

Conflict of Interests

The authors declare that there is no conflict of interests.

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