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TWISTTED ξ-(α,β) EXPANSIVE MAPPINGS IN METRIC SPACES

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Abstract. In this paper, we introduce a pair of twisted ζ-(α,β) expansive mappings in metric spaces and prove fixed point theorems for these mappings. Some examples are also provided to support our main results.

Keywords: Twisted ζ-(α,β) expansive mapping; Twisted (α,β)-admissible mapping; (E.A) property; CLR property.

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1. Introduction

Fixed point theory has fascinated applications with the evaluation of the Banach Contraction Principle. Moreover, fixed point theory provides very important tools for providing the existence and uniqueness of solutions of various mathematical models. Fixed point theorems have remarkable influence on various applications such as the theory of differential and integral equations [15], the game theory, military, sports and medicine as well as in economics [2].

In 1984, Wang *et al.* [13] presented some interesting work on expansion mappings in metric spaces which correspond to some contractive mappings, see [7] and the references therein.

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Further, Khan *et al.* [5] generalized the result of [13] by using control functions. Rhoades [8] and Taniguchi [14] generalized the results of Wang [13] for a pair of mappings. Further, Kang [4] generalized the results of Khan *et al.* [6], Rhoades [8] and Taniguchi [14] for expansion mappings.

Recently, Shahi *et al.* [11] introduced the notion of $(\xi-\alpha)$ expansive for a map and proved fixed point results for such type of a map. For the sake of completeness, we recall some basic definitions and fundamental results.

Let χ denote all the functions $\zeta:[0,\infty)\to[0,\infty)$, that satisfies the following properties:

(i) ζ is non decreasing;

(ii) $\sum_{n=1}^{+\infty}\zeta^n(a)<+\infty$ for each $a>0$, where ζ^n is the nth iterate of ζ;

(iii) $\zeta(a+b)=\zeta(a)+\zeta(b)$ for all $a,b\in[0,\infty)$.

Definition 1.1. [11] Let (X,d) be a metric space and $T:X\to X$ be a given mapping. T is said to be $(\xi-\alpha)$ expansive mapping if there exist two functions $\xi\in\chi$ and $\alpha:X\times X\to[0,+\infty)$ such that

$$\xi(d(Tx,Ty))\geq\alpha(x,y)d(x,y)\quad \text{for all } x,y\in X.$$

In 2011, Samet *et al.* [9] introduced the notion of α-admissible mappings as follows:

Definition 1.2. [9] Let $T:X\to X$ and $\alpha:X\times X\to[0,+\infty)$. T is said to be α-admissible if

$$x,y\in X,\ \ \alpha(x,y)\geq 1 \text{ implies } \alpha(Tx,Ty)\geq 1.$$

In 2013, Salimi *et al.* introduced twisted (α,β) mapping as follows:

Definition 1.3. Let $f:X\to X$ and $\alpha,\beta:X\times X\to[0,\infty)$. We say that f is twisted (α,β) admissible mapping if $x,y\in X$,

$$\begin{cases}\alpha(x,y)\geq 1,\\ \beta(x,y)\geq 1,\end{cases}\quad \text{implies}\quad \begin{cases}\alpha(fy,fx)\geq 1,\\ \beta(fy,fx)\geq 1.\end{cases}$$

Example 1.4. Let $X=R$ be endowed with usual metric $d(x,y)=|x-y|$ for all $x,y\in X$ and $f:X\to X$ be defined by $f(x)=x^2$.

Also define $\alpha,\beta : X\times X \to [0,\infty)$ by

$$\alpha(x,y)=\beta(x,y)=\begin{cases}1 & \text{if } x,y\in[0,1],\\ 0 & \text{otherwise.}\end{cases}$$

Then clearly f is twisted (α,β) admissible.

In 2013, Salimi also introduce the notion of twisted (α,β)-ψ-contractive mappings as follows:

Denote with Ψ the family of non decreasing function $\psi:[0,\infty)\to[0,\infty)$ such that $\sum_{n=1}^{\infty}\psi^n(t)<+\infty$ for all $t>0$, where ψ^n is the nth iterate of ψ.

Definition 1.5. [10] Let (X,d) be a metric space and $f : X\to X$ be a twisted (α,β)-admissible mapping. Then f is said to be a

(a) twisted (α,β)-ψ-contractive mapping of type (I), if

$$\alpha(x,y)\beta(x,y)d(fx,fy)\le\psi(d(x,y))$$

holds for all $x,y\in X$, where $\psi\in\Psi$.

(b) twisted (α,β)-ψ-contractive mapping of type (II), if there is $0<l\le 1$ such that

$$(\alpha(x,y)\beta(x,y)+l)^{d(fx,fy)}\le(1+l)^{\psi(d(x,y))}$$

holds for all $x,y\in X$, where $\psi\in\Psi$.

(c) twisted (α,β)-ψ-contractive mapping of type (III), if there is $l\ge 1$ such that

$$(d(fx,fy)+l)^{\alpha(x,y)\beta(x,y)}\le\psi(d(x,y)+l$$

holds for all $x,y\in X$, where $\psi\in\Psi$.

Recently, Kang *et al.* [6] introduce twisted ξ-(α,β) expansive mappings in metric spaces as follows:

Definition 1.6. Let (X,d) be a metric space and $T : X\to X$ be a twisted (α,β) admissible mapping then T is said to be a

(a) twisted ξ-(α,β) expansive mapping of type I, if

$$\xi(d(Tx,Ty)) \geq \alpha(x,y)\beta(x,y)d(x,y) \tag{1.1}$$

holds for all $x,y \in X$, where $\xi \in \chi$.

(b) twisted ξ-(α,β) expansive mapping of type II, if

$$(1+l)^(\xi(d(Tx,Ty))) \geq (\alpha(x,y)\beta(x,y)+l)^{d(x,y)} \tag{1.2}$$

holds for all $x,y \in X$, where $\xi \in \chi$ and $0 < l \leq 1$.

(c) twisted ξ-(α,β) expansive mapping of type III, if

$$\xi(d(Tx,Ty)) + l \geq (d(x,y)+l)^{\alpha(x,y)\beta(x,y)} \tag{1.3}$$

holds for all $x,y \in X$, where $\xi \in \chi$, for some $l \geq 1$.

Now we define twisted (α,β) admissible mappings for a pair of mapping as follows:

Definition 1.7. Let $f,g : X \to X$ and $\alpha,\beta : X \times X \to [0,\infty)$, we say that f is twisted (α,β) admissible mapping with respect to g if $x,y \in X$,

$$\begin{cases} \alpha(gx,gy) \geq 1, \\ \beta(gx,gy) \geq 1, \end{cases} \quad \text{implies} \quad \begin{cases} \alpha(fy,fx) \geq 1, \\ \beta(fy,fx) \geq 1. \end{cases}$$

Example 1.8. Let $X = R$ be endowed with usual metric $d(x,y) = |x-y|$ for all $x,y \in X$. Define $f,g : X \to X$ by $f(x) = 1-x$ and $g(x) = x^2 - 1$. Also define $\alpha,\beta : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1 & \text{if } x \in [0,1],\ y \in [-1,0], \\ 0, & \text{otherwise.} \end{cases}$$

Now, $\alpha(gx,gy) \geq 1$ and $gx \in [0,1]$, $gy \in [-1,0]$. If $gx \in [0,1]$ implies $0 \leq x^2 - 1 \leq 1$, this implies $-1 < 1-x < 0$ i.e., $fx \in [-1,0]$. If $gy \in [-1,0]$ implies $-1 \leq y^2 - 1 \leq 0$, implies $0 < 1-y < 1$ i.e., $fy \in [0,1]$. Therefore, $\alpha(fy,fx) \geq 1$. Hence f is twisted (α,β) admissible map with respect to g.

Now we define a pair of twisted ξ-(α,β) expansive mapping of type I or II or III as follows:

Definition 1.9. Let (X,d) be a metric space and $f,g : X \to X$ be two mappings. f is said to be a

(a) twisted ξ-(α,β) expansive mapping with respect to g of type I, if

$$\xi(d(fx,fy)) \geq \alpha(gx,gy)\beta(gx,gy)d(gx,gy) \tag{1.4}$$

holds for all $x,y \in X$ and $\xi \in \chi$.

(b) twisted ξ-(α,β) expansive mapping with respect to g of type II, if

$$(1+l)^{\xi(d(fx,fy))} \geq (\alpha(gx,gy)\beta(gx,gy)+l)^{d(gx,gy)} \tag{1.5}$$

holds for all $x,y \in X$ and $\xi \in \chi$, $0 < l \leq 1$.

(c) twisted ξ-(α,β) expansive mapping with respect to g of type III, if

$$\xi(d(fx,fy))+l \geq (d(gx,gy)+l)^{\alpha(gx,gy)\beta(gx,gy)} \tag{1.6}$$

holds for all $x,y \in X$ and $\xi \in \chi$, $l \geq 1$.

Example 1.10 Let $X = R$ be endowed with usual metric $d(x,y) = |x-y|$ for all $x,y \in X$. Define $f,g : X \to X$ by $f(x) = 3x-2$ and $g(x) = 2x-1$. Also define $\alpha,\beta : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Now, $\alpha(gx,gy) \geq 1$ implies $\alpha(fy,fx) \geq 1$. Then clearly f is twisted (α,β) admissible with respect to g. Define $\zeta : [0,\infty) \to [0,\infty)$ by $\zeta(t) = \frac{2t}{3}$. Then clearly, ζ is non decreasing and $\sum_{n=1}^{+\infty} \zeta^n(a) < +\infty$ for each $a > 0$, where ζ^n is the nth iterate of ζ. Also, $\zeta(a+b) = \zeta(a)+\zeta(b)$ for all $a,b \in [0,\infty)$. Clearly, $\xi(d(fx,fy)) \geq \alpha(gx,gy)\beta(gx,gy)d(gx,gy)$ holds for all $x,y \in X$, therefore f is twisted ξ-(α,β) expansive map with respect to g of type I.

Example 1.11. Let $X = R$ be endowed with usual metric $d(x,y) = |x-y|$ for all $x,y \in X$. Define $f,g : X \to X$ by $f(x) = 2x+1$ and $g(x) = x$. Also define $\alpha,\beta : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1 & \text{if } x,y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now, $\alpha(gx,gy) \geq 1$ implies $\alpha(fy,fx) \geq 1$. Then clearly f is twisted (α,β) admissible with respect to g. Define $\zeta : [0,\infty) \to [0,\infty)$ by $\zeta(t) = \frac{t}{2}$. Then clearly, ζ is non decreasing and

$\sum_{n=1}^{+\infty} \zeta^n(a) < +\infty$ for each $a > 0$, where ζ^n is the nth iterate of ζ. Also, $\zeta(a+b) = \zeta(a) + \zeta(b)$ for all $a, b \in [0, \infty)$. Clearly,

$$(1+l)^{\xi(d(fx,fy))} \geq (\alpha(gx,gy)\beta(gx,gy)+l)^{d(gx,gy)}$$

$$\xi(d(fx,fy))+l \geq (d(gx,gy)+l)^{\alpha(gx,gy)\beta(gx,gy)}$$

holds for all $x, y \in X$, therefore f is twisted ξ-(α,β) expansive map with respect to g of type II. Similarly f is twisted ξ-(α,β) expansive map with respect to g of type III.

2. Weakly compatible mappings

In 1996, Jungck [3] introduce the notion of weakly compatible mappings as follows:

Definition 2.1. Two maps f and g are said to be weakly compatible if they commute at coincidence points.

Theorem 2.2. *Let (X,d) be a complete metric space. Let $f, g : X \to X$ be bijective twisted ξ-(α,β) expansive maps of type I or type II or type III satisfying the following conditions:*

(2.1) f^{-1} is twisted (α,β)-admissible with respect to g^{-1},

(2.2) there exists $x_0 \in X$ such that $\alpha(g^{-1}x_0, f^{-1}x_0) \geq 1$ and $\beta(g^{-1}x_0, f^{-1}x_0) \geq 1$,

(2.3) if $\{x_n\}$ be a sequence in X such that if $\alpha(g^{-1}x_{2n}, g^{-1}x_{2n+1}) \geq 1$, $\alpha(g^{-1}x_{2n}, g^{-1}x_{2n-1}) \geq 1$ and $\beta(g^{-1}x_{2n}, g^{-1}x_{2n+1}) \geq 1$, $\beta(g^{-1}x_{2n}, g^{-1}x_{2n-1}) \geq 1$ for all n implies $\alpha(x_{2n}, x_{2n+1}) \geq 1$, $\alpha(x_{2n}, x_{2n-1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$, $\beta(x_{2n}, x_{2n-1}) \geq 1$,

(2.4) if $\{gx_n\}$ be a sequence in X such that $\alpha(gx_{2n-1}, gx_{2n}) \geq 1$, $\alpha(gx_{2n-1}, gx_{2n-2}) \geq 1$ and $\beta(gx_{2n-1}, gx_{2n}) \geq 1$, $\beta(gx_{2n-1}, gx_{2n-2}) \geq 1$ for all n and if $gx_n \to z \in g(X)$ as $n \to \infty$, then there exists, sub sequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gf^{-1}z) \geq 1$ and $\beta(gx_{n(k)}, gf^{-1}z) \geq 1$ for all k.

Then f and g have a coincidence point.

Proof. From (2.2), we see that there exists $x_0 \in X$ such that $\alpha(g^{-1}x_0, f^{-1}x_0) \geq 1$ and $\beta(g^{-1}x_0, f^{-1}x_0) \geq 1$ and since f and g are bijective, so there exists, $x_1 \in X$ such that $g^{-1}x_1 = f^{-1}x_0 = y_0$.

Continuing in this way, we get

$$g^{-1}x_{n+1} = f^{-1}x_n = y_n, \tag{2.5}$$

that is,

$$x_n = gy_{n-1} = fy_n, \quad \text{for all } n \in N. \tag{2.6}$$

Now $\alpha(g^{-1}x_0, f^{-1}x_0) \geq 1$ implies $\alpha(g^{-1}x_0, g^{-1}x_1) \geq 1$ and f^{-1} is twisted (α,β)-admissible with respect to g^{-1}. Therefore,

$$\alpha(f^{-1}x_1, f^{-1}x_0) \geq 1 \quad \text{implies} \quad \alpha(g^{-1}x_2, g^{-1}x_1) \geq 1 \text{ i.e., } \alpha(y_1, y_0) \geq 1.$$

Again $\alpha(g^{-1}x_2, g^{-1}x_1) \geq 1$ implies $\alpha(f^{-1}x_1, f^{-1}x_2) \geq 1$, i.e., $\alpha(y_1, y_2) \geq 1$. Continuing in this way, we get

$$\alpha(y_{2n+1}, y_{2n}) \geq 1 \quad \text{and} \quad \alpha(y_{2n+1}, y_{2n+2}) \geq 1.$$

Similarly, we have $\beta(y_{2n+1}, y_{2n}) \geq 1$ and $\beta(y_{2n+1}, y_{2n+2}) \geq 1$. Using (2.3), we get

$$\alpha(x_{2n}, x_{2n-1}) \geq 1 \quad \text{and} \quad \alpha(x_{2n}, x_{2n+1}) \geq 1.$$

Similarly, we have

$$\beta(x_{2n}, x_{2n-1}) \geq 1 \quad \text{and} \quad \beta(x_{2n}, x_{2n+1}) \geq 1.$$

Therefore, we obtain

$$\alpha(gy_{2n-1}, gy_{2n-2}) \geq 1 \quad \text{and} \quad \alpha(gy_{2n-1}, gy_{2n}) \geq 1,$$

and

$$\beta(gy_{2n-1}, gy_{2n-2}) \geq 1 \quad \text{and} \quad \beta(gy_{2n-1}, gy_{2n}) \geq 1.$$

Case I. If f is twisted ξ-(α,β) expansive mapping of type I with respect to g and using (1.4), then we have

$$\begin{aligned} d(fy_{2n}, fy_{2n-1}) &\leq \alpha(gy_{2n-1}, gy_{2n-2}) \cdot \beta(gy_{2n-1}, gy_{2n-2}) \cdot d(gy_{2n-1}, gy_{2n-2}) \\ &\leq \xi(d(fy_{2n-1}, fy_{2n-2})). \end{aligned}$$

Again from (1.4), we have

$$d(fy_{2n}, fy_{2n+1}) \leq \alpha(gy_{2n-1}, gy_{2n}) \cdot \beta(gy_{2n-1}, gy_{2n}) \cdot d(gy_{2n-1}, gy_{2n})$$

$$\leq \xi(d(fy_{2n-1}, fy_{2n})).$$

Continuing this process, we get

$$d(fy_n, fy_{n+1}) \leq \xi^n(d(fy_0, fy_1)).$$

Case II. If f is twisted ξ-(α,β) expansive mapping of type II with respect to g and using (1.5)

$$(1+l)^{d(fy_{2n},fy_{2n-1})} \leq (\alpha(gy_{2n-1}, gy_{2n-2}) \cdot \beta(gy_{2n-1}, gy_{2n-2}) + l)^{d(gy_{2n-1},gy_{2n-2})}$$
$$\leq (1+l)^{\xi(d(fy_{2n-1},fy_{2n-2}))},$$

that is,

$$d(fy_{2n}, fy_{2n-1}) \leq \xi(d(fy_{2n-1}, fy_{2n-2})).$$

From (1.5), we have

$$(1+l)^{d(fy_{2n},fy_{2n+1})} \leq (\alpha(gy_{2n-1}, gy_{2n}) \cdot \beta(gy_{2n-1}, gy_{2n}) + l)^{d(gy_{2n-1},gy_{2n})}$$
$$\leq (1+l)^{\xi(d(fy_{2n-1},fy_{2n}))},$$

that is,

$$d(fy_{2n}, fy_{2n+1}) \leq \xi(d(fy_{2n-1}, fy_{2n})).$$

Continuing this process, we have

$$d(fy_n, fy_{n+1}) \leq \xi^n(d(fy_0, fy_1)).$$

Case III. If f is twisted ξ-(α,β) expansive of type III with respect to g, we find from (1.6) that

$$d(fy_{2n}, fy_{2n-1}) + l \leq (d(gy_{2n-1}, gy_{2n-2}) + l)^{\alpha(gy_{2n-1},gy_{2n-2})\cdot\beta(gy_{2n-1},gy_{2n-2})}$$
$$\leq \xi(d(fy_{2n-1}, fy_{2n-2})) + l,$$

that is,

$$d(fy_{2n+2}, fy_{2n+1}) \leq \xi(d(fy_{2n+1}, fy_{2n})).$$

It follows from (1.6) that

$$d(fy_{2n}, fy_{2n+1}) + l \leq (d(gy_{2n-1}, gy_{2n}) + l)^{\alpha(gy_{2n-1},gy_{2n})\cdot\beta(gy_{2n-1},gy_{2n})}$$
$$\leq \xi(d(fy_{2n-1}, fy_{2n})) + l,$$

that is,

$$d(fy_{2n}, fy_{2n+1}) \leq \xi(d(fy_{2n-1}, fy_{2n})).$$

Continuing this process, we get

$$d(fy_n, fy_{n+1}) \leq \xi^n(d(fy_0, fy_1)).$$

Thus in all cases we have

$$d(fy_n, fy_{n+1}) \leq \xi^n(d(fy_0, fy_1)).$$

Now for any positive integer k, we consider

$$\begin{aligned} d(fy_n, fy_{n+k}) &\leq d(fy_n, fy_{n+1}) + d(fy_{n+1}, fy_{n+2}) + \ldots + d(fy_{n+k-1}, fy_{n+k}) \\ &\leq \xi^n(d(fy_0, fy_1)) + \xi^{n+1}(d(fy_0, fy_1)) + \ldots + \xi^{n+k+1}(d(fy_0, fy_1)). \end{aligned}$$

Letting $n \to \infty$, we have $\{fy_n\}$ or $\{gy_n\}$ is a Cauchy sequence in (X,d). Since X is complete and f is bijective. So there exists $z \in X$ such that $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = fz$.

Now we are in a position to show z is coincidence point of f and g. From (2.4) there exists subsequence $\{gy_{n(k)}\}$ of $\{gy_n\}$ such that $\alpha(gy_{n(k)}, gf^{-1}fz) \geq 1$ and $\beta(gy_{n(k)}, gf^{-1}fz) \geq 1$ for all k, implies $\alpha(gy_{n(k)}, gz) \geq 1$ and $\beta(gy_{n(k)}, gz) \geq 1$.

Case i. If f is twisted ξ-(α,β) expansive of type I with respect to g, then we consider

$$\begin{aligned} d(gz, fz) &\leq d(gz, fy_{n(k)+1}) + d(fy_{n(k)+1}, fz) \\ &\leq d(fz, fy_{n(k)+1}) + \alpha(gy_{n(k)}, gz)\beta(gy_{n(k)}, gz)d(fy_{n(k)+1}, gz) \\ &= d(fz, fy_{n(k)+1}) + \alpha(gy_{n(k)}, gz)\beta(gy_{n(k)}, gz)d(gy_{n(k)}, gz) \\ &\leq d(fz, fy_{n(k)+1}) + \xi(d(fy_{n(k)}, fz)). \end{aligned}$$

Letting $n \to \infty$, we get $d(gz, fz) = 0$, a contradiction. Hence $gz = fz$, i.e., that z is coincidence point of f and g.

Case ii. If f is twisted ξ-(α,β) expansive mapping of type II with respect to g, then consider

$$\begin{aligned} (1+l)^{d(gz,fz)} &\leq (1+l)^{d(gz,gy_n(k)} + d(gy_n(k), fz)) \\ &\leq (1+l)^{d(gy_n(k),fz)}(\alpha(gy_n(k), gz)\beta(gy_n(k), gz) + l)^{d(gz,gy_n(k))} \end{aligned}$$

$$\leq (1+l)^{d(gy_n(k),fz)}(1+l)^{\xi(d(fz,fy_n(k)))}$$
$$\leq (1+l)^{(d(gy_n(k),fz)+\xi(d(fz,fy_n(k))))},$$

which implies that $d(gz,fz) \leq d(gy_n(k),fz)+\xi(d(fz,fy_n(k)))$. Letting $n\to\infty$, we get $gz = fz$.

Case iii. If f is twisted ξ-(α,β) expansive mapping of type III with respect to g, then we consider

$$\begin{aligned} d(gz,fz)+l &\leq d(gz,gy_n(k))+d(gy_n(k),fz)+l \\ &\leq d(gy_n(k),fz)+(d(gy_n(k),gz)+l)^{\alpha(gy_n(k),gz)\beta(gy_n(k),gz)} \\ &\leq d(gy_n(k),fz)+\xi(d(fy_n(k),fz))+l, \end{aligned}$$

which implies that $d(gz,fz) \leq d(gy_n(k),fz)+\xi(d(fy_n(k),fz))$. Letting $n\to\infty$, we get $gz = fz$. Hence in all cases we get that z is coincidence point of f and g.

Example 2.3. Let $X = R$ be endowed with usual metric $d(x,y) = |x-y|$ for all $x,y \in X$. Define $f,g : X \to X$ by $f(x) = 2x-1$ and $g(x) = 1-x$. Clearly g is bijective. Then $f^{-1}(x) = \frac{x+1}{2}$ and $g^{-1}(x) = 1-x$. Also define $\alpha,\beta : X\times X \to [0,\infty)$ by

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Now, $\alpha(g^{-1}x,g^{-1}y) \geq 1$ implies $\alpha(f^{-1}y,f^{-1}x) \geq 1$. Then clearly f^{-1} is twisted (α,β) admissible map with respect to g^{-1}. Define $\zeta : [0,\infty) \to [0,\infty)$ by $\zeta(t) = \frac{t}{2}$. Then ζ is non decreasing and $\sum_{n=1}^{+\infty} \zeta^n(a) < +\infty$ for each $a > 0$, where ζ^n is the nth iterate of ζ. Also, $\zeta(a+b) = \zeta(a)+\zeta(b)$ for all $a,b \in [0,\infty)$. Clearly, $\xi(d(fx,fy)) \geq \alpha(gx,gy)\beta(gx,gy)d(gx,gy)$ holds for all $x,y \in X$, therefore f is twisted ξ-(α,β) expansive map with respect to g of type I. For $x_0 = 1$, we have

$$\alpha(g^{-1}x_0,f^{-1}x_0) \geq 1 \quad \text{and} \quad \beta(g^{-1}x_0,f^{-1}x_0) \geq 1.$$

All the condition of Theorem 2.2 are satisfied, therefore, f and g have a coincidence point 1.

Theorem 2.4. *In addition of previous theorem, we assume that*

(2.7) Suppose for all $u,v\in C(g^{-1},f^{-1})$*, there exists* $w\in X$*, such that*

$$\alpha(g^{-1}u,g^{-1}w)\geq 1 \quad \textit{and} \quad \alpha(g^{-1}v,g^{-1}w)\geq 1$$

and

$$\beta(g^{-1}u,g^{-1}w)\geq 1 \quad \textit{and} \quad \beta(g^{-1}v,g^{-1}w)\geq 1.$$

(2.8) If $\{g^{-1}x_n\}$ *be a sequence such that* $\alpha(g^{-1}s,g^{-1}x_{2n})\geq 1,\ \alpha(g^{-1}x_{2n+1},g^{-1}s)\geq 1$ *and* $\beta(g^{-1}s,g^{-1}x_{2n})\geq 1,\ \beta(g^{-1}x_{2n+1},g^{-1}s)\geq 1$*, implies that* $\alpha(s,x_{2n})\geq 1,\ \alpha(x_{2n+1},s)\geq 1$ *and* $\beta(s,x_{2n})\geq 1,\beta(x_{2n+1},s)\geq 1$ *for all* $n\in N$ *and* $s\in C(g^{-1},f^{-1})$*.*

(2.9) f *and* g *are weakly compatible maps.*

Then f *and* g *have unique common fixed point.*

Proof. Let u and v be coincidence points of f and g, which implies that $gu=fu=s$ and $gv=fv=t$. Also, $g^{-1}s=f^{-1}s=u$ and $g^{-1}t=f^{-1}t=v$. So $t,s\in C(g^{-1},f^{-1})$. From hypothesis there exists $w\in X$ such that

$$\alpha(g^{-1}s,f^{-1}w)\geq 1 \quad \text{and} \quad \alpha(g^{-1}t,f^{-1}w)\geq 1$$

and

$$\beta(g^{-1}s,f^{-1}w)\geq 1 \quad \text{and} \quad \beta(g^{-1}t,f^{-1}w)\geq 1.$$

Let $\{w_n\}$ be a sequence such that $g^{-1}w_{n+1}=f^{-1}w_n=p_n$, where $w_0=w$.

$$g^{-1}w_{n+1}=f^{-1}w_n=p_n \ \text{ implies that } w_n=fp_n=gp_{n-1}.$$

Since f^{-1} is twisted (α,β) admissible with respect to g^{-1} therefore

$$\alpha(g^{-1}s,g^{-1}w_0)\geq 1 \quad \text{and} \quad \beta(g^{-1}s,g^{-1}w_0)\geq 1,$$

which implies

$$\alpha(f^{-1}w_0,f^{-1}s)\geq 1 \quad \text{and} \quad \beta(f^{-1}w_0,f^{-1}s)\geq 1$$

or

$$\alpha(g^{-1}w_1,g^{-1}s)\geq 1 \quad \text{and} \quad \beta(g^{-1}w_1,g^{-1}s)\geq 1.$$

Continuing the process, we get

$$\alpha(g^{-1}w_{2n+1},g^{-1}s)\geq 1,\ \ \beta(g^{-1}w_{2n+1},g^{-1}s)\geq 1$$

and

$$\alpha(g^{-1}s, g^{-1}w_{2n}) \geq 1 \quad \text{and} \quad \beta(g^{-1}s, g^{-1}w_{2n}) \geq 1, \text{ for all } n \in N.$$

From (2.8), we get

$$\alpha(w_{2n+1}, s) \geq 1, \ \ \beta(w_{2n+1}, s) \geq 1$$

and

$$\alpha(s, w_{2n}) \geq 1, \quad \beta(s, w_{2n}) \geq 1.$$

If f is twisted ξ-(α,β) of type I with respect to g, then consider

$$\begin{aligned} d(gu, gp_{2n-1}) &\leq \alpha(gu, gp_{2n-1})\beta(gu, gp_{2n-1})d(gu, gp_{2n-1}) \\ &\leq \xi(d(fu, fp_{2n-1}) = \xi(d(gu, gp_{2n-2}). \end{aligned}$$

Again

$$\begin{aligned} d(gp_{2n}, gu) &\leq \alpha(gp_{2n}, gu)\beta(gp_{2n}, gu)d(gp_{2n}, gu) \\ &\leq \xi(d(fp_{2n}, fu)) = \xi(d(gp_{2n-1}, gu). \end{aligned}$$

Continuing this process, we have

$$d(gp_{2n}, gu) \leq \xi^n(d(gp_0, gu)).$$

Hence, we get the same result. Letting $n \to \infty$, we get $\lim_{n\to\infty} d(gp_{2n}, gu) = 0$. Similarly, we have $\lim_{n\to\infty} d(gp_{2n}, gv) = 0$. Hence $gu = gv$. Now u is coincidence point of f and g. Since f and g are weakly compatible, therefore. $g^2u = fgu = gfu$. This implies $gz = fz$, where $gu = fu = z$.

Now, we have u and z are two coincidence points of f and g. So $gu = gz = fz = fu = z$, i.e., z is fixed point of f and g. Uniqueness follows easily.

Example 2.5. Proceeding Example 2.3, we note that f and g commutes at their coincidence points, i.e., f and g are weakly compatible. Also, all the conditions of Theorem 2.4 are satisfied, therefore f and g have the fixed point 1.

3. Weakly compatible with property (E.A)

Aamri and Moutawakil [1] introduced property (E.A) in metric spaces as follows:

Definition 3.1. Let f and g be two self maps of a metric space (X,d). The pair (f,g) is said to satisfy property (E.A), if there exists a sequence $\{x_n\}$ in X such that

$$\lim fx_n = \lim gx_n = t.$$

Example 3.2. Let $X = [0,1]$. Define $f,g : X \to X$ as

$$fx = x^3 \text{ and } gx = \frac{x}{5}, \quad \text{for all } x \in X.$$

Consider $x_n = \frac{1}{n}$. Clearly, $\lim fx_n = \lim gx_n = 0$.

This shows that f and g satisfy property (E.A).

Theorem 3.3. *Let (X,d) be a metric space. Let $f,g : X \to X$ be bijective twisted ξ-(α,β) expansive maps of type I or type II or type III satisfying the following conditions:*

(3.1) $f(X)$ is closed subspace of X,

(3.2) f and g satisfy the property (E.A),

(3.3) if $\{x_n\}$ be a sequence in X such that $\lim fx_n = \lim gx_n = fz$, then there exists a subsequence $\{x_{n(k)}\}$ such that $fx_{n(k)} = gx_{n(k)-1}$, $\alpha(gx_{n(k)},gz) \geq 1$ and $\beta(gx_{n(k)},gz) \geq 1$ for all k.

Then f and g have a coincidence point.

Proof. Since f and g satisfy property (E.A), we see that there exists a sequence $\{x_n\}$ in X such that $\lim fx_n = \lim gx_n = t$. Since $f(X)$ is closed subspace of X, we have $\lim fx_n = \lim gx_n = t = fz$ for some $z \in X$.

Now we show z is coincidence point of f and g. From the hypothesis, there exists a subsequence $\{x_{n(k)}\}$ such that $fx_{n(k)} = gx_{n(k)-1}$, $\alpha(gx_{n(k)},gz) \geq 1$ and $\beta(gx_{n(k)},gz) \geq 1$ for all k.

Case 1. If f is twisted ξ-(α,β) expansive maps of type I with respect to g, then we consider

$$\begin{aligned}
d(fz,gz) &\leq d(gz,gx_{n(k)}) + d(gx_{n(k)},fz) \\
&\leq d(gx_{n(k)},fz) + \alpha(gx_{n(k)},gz)\beta(gx_{n(k)},gz)d(gx_{n(k)},gz) \\
&\leq d(gx_{n(k)},fz) + \xi(d(fx_{n(k)},fz)).
\end{aligned}$$

Letting $n \to \infty$, we get

$$d(gz,fz) = 0, \quad \text{implies} \quad gz = fz.$$

Hence z is coincidence point of f and g.

Case 2. If f is twisted ξ-(α,β) expansive maps of type II with respect to g, we consider

$$\begin{aligned}(1+l)^{d(gz,fz)} &\leq (1+l)^{d(gz,gx_n(k)} + d(gx_n(k),fz))\\ &\leq (1+l)^{d(gx_n(k),fz)}(\alpha(gx_n(k),gz)\beta(gx_n(k),gz)+l)^{d(gx_n(k),gz)}\\ &\leq (1+l)^{d(gx_n(k),fz)}(1+l)^{\xi(d(fx_n(k),fz))}\\ &\leq (1+l)^{d(gx_n(k),fz} + \xi(d(fz,fx_n(k))),\end{aligned}$$

which implies that $d(gz,fz) \leq d(gx_n(k),fz) + \xi(d(fz,fx_n(k)))$. Letting $n\to\infty$, we get $gz = fz$.

Case 3. If f is twisted ξ-(α,β) expansive mapping of type III with respect to g, then we consider

$$\begin{aligned}d(gz,fz)+l &\leq d(gz,gx_n(k)) + d(gx_n(k),fz) + l\\ &\leq d(gx_n(k),fz) + (d(gx_n(k),gz)+l)^{(\alpha(gx_n(k),gz)\beta(gx_n(k),gz))}\\ &\leq d(gx_n(k),fz) + \xi(d(fx_n(k),fz)) + l,\end{aligned}$$

which implies that $d(gz,fz) \leq d(gx_n(k),fz) + \xi(d(fx_n(k),fz))$. Letting $n\to\infty$, we get $gz = fz$. Hence in all cases we get that z is a coincidence point of f and g.

Theorem 3.4. *In addition of previous theorem, we assume that*

(3.4) for all $u,v \in C(f,g)$ there exists $w \in X$ such that $\alpha(gu,gw) \geq 1$, $\beta(gu,gw) \geq 1$ and $\alpha(gv,gw) \geq 1$, $\beta(gv,gw) \geq 1$,

(3.5) f and g are weakly compatible,

(3.6) f is twisted (α,β) admissible with respect to g.

Then f and g have a unique common fixed point.

Proof. We claim that $u,v \in C(g,f)$, which yields $gu = gv$. By (3.4), there exists $w\in X$ such that $\alpha(gu,gw)\geq 1$, $\beta(gu,gw)\geq 1$, $\alpha(gv,gw)\geq 1$, and $\beta(gv,gw)\geq 1$. Let us define the sequence $\{w_n\}$ in X by $fw_n = gw_{n-1}$ for $n \geq 0$ and $w_0 = w$. Since f is twisted (α,β) admissible with respect to g so $\alpha(gu,gw) \geq 1, \beta(gu,gw) \geq 1$, implies $\alpha(fw,fu) \geq 1$, $\beta(fw,fu) \geq 1$ or $\alpha(gw_1,gu) \geq 1$,

$\beta(gw_1, gu) \geq 1$. Continuing this process, we get

$$\alpha(gu, gw_{2n}) \geq 1,\ \beta(gu, gw_{2n}) \geq 1 \text{ and } \alpha(gw_{2n+1}, gu) \geq 1,\ \beta(gw_{2n+1}, gu) \geq 1.$$

If f is twisted ξ-(α,β) mapping of type I with respect to g, then we consider

$$\begin{aligned} d(gu, gw_{2n}) &\leq \alpha(gu, gw_{2n})\beta(gu, gw_{2n})d(gu, gw_{2n}) \\ &\leq \xi(d(fu, fw_{2n})) \\ &= \xi(d(gu, gw_{2n-1})) \end{aligned}$$

and

$$\begin{aligned} d(gw_{2n+1}, gu) &\leq \alpha(gw_{2n+1}, gu)\beta(gw_{2n+1}, gu)d(gw_{2n+1}, gu) \\ &\leq \xi(d(fw_{2n+1}, fu)) = \xi(d(gw_{2n}, gu)). \end{aligned}$$

Continuing this process, we get

$$d(gu, gw_n) \leq \xi^n(d(gu, gw_0)).$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} d(gu, gw_n) = 0.$$

Similarly, we find $\lim d(gv, gw_n) = 0$. This implies that $gu = gv$. Similarly, we get the same result if f is twisted ξ-(α,β) mapping of type II or type III with respect to g. Now u is coincidence point of f and g. Also, f and g are weakly compatible. Therefore, $fgu = gfu$ implies $gz = fz$ where $gu = fu = z$. Hence u and z are two coincidence points of f and g implies $gu = gz$. Now $gu = gz = fz = fu = z$.

Hence we get z is a fixed point of f and g. Uniqueness follows easily.

4. Weakly compatible with CLR property

Sintunavarat and Kumam [12] introduced a new property called "common limit in the range of g property" (i.e., (CLRg) property) as follows:

Definition 4.1. Let (X,d) be a metric space and $f,g : X \to X$ two mappings. The maps f and g are said to be satisfy the common limit in the range of g property if there exists a sequence $\{x_n\}$ in X such that $\lim fx_n = \lim gx_n = gx$ for some $x \in X$.

The common limit in the range of g property will be denoted by (CLRg) property.

Example 4.2. Let $X = [0,1]$. Define $f,g : X \to X$ as

$$fx = x^3 \text{ and } gx = \frac{x}{5}, \quad \text{for all} x \in X.$$

Consider $x_n = \frac{1}{n}$. Clearly, $\lim fx_n = \lim gx_n = g(0)$.
This shows that f and g satisfy the (CLRg) property.

Theorem 4.3. *Let (X,d) be a metric space and f and g be two self twisted ξ-(α,β) expansive maps satisfying the condition (3.4) and the following condition*

(4.1) f and g satisfy the CLRf property

Then f and g have a common coincidence point.

Proof. Since f and g satisfy the CLRf property, we see that there exists a sequence $\{x_n\}$ in X such that $\lim fx_n = \lim gx_n = fz$. From Theorem 4.1, we find the desired conclusion immediately.

Conflict of Interests

The authors declare that there is no conflict of interests.

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